Perfect Competition as the Blueprint for Efficiency and Incentive Compatibility

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Abstract

For exchange environments with a finite number of individuals, we show that efficient, dominant strategy incentive compatible mechanisms must be perfectly competitive, i.e., each individual must be unable to influence prices or anyone's wealth. The characterization applies whether preferences are ordinal or quasi-linear. We also prove an asymptotic result. Perfectly competitive incentive compatible mechanisms are shown to be non-generic (although non-vacuous) in finite economies, while they are generic (but non-universal) in continuum economies. They are also shown to be equivalent to mechanisms in which each individual *fully appropriates* his social contribution. We use these results to provide bridges to related work.

KEYWORDS: Perfect competition, full appropriation, incentive compatible mechanism

JEL CLASSIFICATIONS: C72 (Noncooperative Games); D51 (Exchange and Production Economies)

1 Introduction

We give a characterization of efficient, dominant strategy incentive compatible mechanisms for exchange environments with a finite number of individuals. The most important feature is its canonical form: such mechanisms must be perfectly competitive, in the sense that each individual must be unable to influence prices or anyone's wealth. The central property of a perfectly competitive mechanism permitting the achievement of efficiency and incentive compatibility we call "full appropriation."

It is easy to see that a perfectly competitive mechanism suffices for efficiency and incentive compatibility: Efficiency is guaranteed by the well-known optimality of Walrasian equilibrium, whether or not individuals can influence prices; while incentive compatibility follows from the fact that a Walrasian equilibrium yields each individual his most desirable outcome, assuming the individual cannot influence prices or his wealth, a valid hypothesis under perfect competition. The characterization therefore turns on the issue of necessity.

A perfectly competitive mechanism can only exist on a perfectly competitive environment; hence, the characterization implicitly specifies the environment required for efficiency and incentive compatibility. Since the typical view is that perfect competition is synonymous with large numbers, the characterization raises questions of existence. Even though it may be true as a rule that finite economies are not perfectly competitive whereas continuum economies are, it will be important to recognize that there are exceptions to *both* of these statements. The fact that there are perfectly competitive finite economies is especially significant for our analysis. It allows us to establish our main result, a non-vacuous characterization for environments with a finite number of individuals and ordinal preferences.

The key to the main result is the construction of a domain restriction — a richness condition — under which a possibility theorem can be proved. The domain restriction describes a "neighborhood" around any given population. Under the richness condition, we show that perfect competition is not only sufficient but also necessary for efficiency and incentive compatibility. Examples illustrating our main result (see Section 5) show that, although they are sparse, it is easy to find finite populations with neighborhoods which are perfectly competitive. The geometry of these neighborhoods mimics conditions typically holding in the continuum. Thus the characterization in *finite* economies shows why the continuum provides the natural environment both for incentive compatibility and perfect competition.

Many results in the literature on mechanism design trace a pattern suggesting the characterization above, but some point in apparently orthogonal or even opposite directions. We divide the literature on mechanism design for exchange environments into the following three categories.

FINITE NUMBERS AND ORDINAL PREFERENCES: Hurwicz (1972) demonstrated that for 2-person, pure-exchange economies the Walrasian mechanism is manipulable. He then shows the same holds for any 2-person, individually rational allocation mechanism. A related, 2-person impossibility result, without the individual rationality assumption, is proved by Dasgupta, Hammond, and Maskin (1979) and by Zhou (1991). Other impossibility results are demonstrated by Satterthwaite and Sonnenschein (1981). While we characterize possibility, we show as a corollary that when the domain is sufficiently rich to permit the exercise of monopoly power, i.e., most domains for finite-agent economies, impossibility follows.

FINITE NUMBERS AND QUASI-LINEAR PREFERENCES: For models with quasi-linear preferences, there is a general characterization of all incentive compatible mechanisms satisfying a qualified notion of efficiency in which transfers of the money commodity need not balance: they must be in the Vickrey-Clarke-Groves family (Vickrey (1961), Clarke (1971), Groves and Loeb (1975), Green and Laffont (1977), Holmström (1979)). The defining feature is that each individual appropriates the whole gains from trade minus a lump sum. Such a *full appropriation principle* does not appear to be related to the indispensability-of-perfect-competition theme highlighted here. One of our goals is to show (in Section 4) that there is an intimate connection. Once one insists on full efficiency (i.e., budget balancing), the Groves characterization implies that for incentive compatibility in quasi-linear economies, no individual should be able to influence the terms-of-trade, that is, full appropriation with budget balancing and perfect competition are equivalent.[1]

There has been no characterization of efficient incentive compatible mechanisms for finite ordinal economies. Does the "full appropriation principle" also characterize efficient implementation in the ordinal case? A corollary of our main characterization result provides a positive answer. Combined with the above analogous result for quasi-linear economies shows that the full appropriation principle and the uniqueness of perfect competition (for

[1]The result was shown assuming individual rationality in Makowski and Ostroy (1987). Here we show the result holds even without an individual rationality constraint.

successful implementation) are really two sides of the same coin; each helps illuminate the significance of the other.

Continuum Economies: For continuum economies, Hammond (1979), Kleinberg (1980), Champsaur and Laroque (1981), McLennan (1981), Mas-Colell (1982) and others have shown that the only "no-envy" allocations are Walrasian equilibria — either Walrasian equilibria that exclude transfer payments when there are initial endowments or, when the specification of individual characteristics does not include initial endowments, equal wealth Walrasian equilibria. Under the assumption that the mechanism is continuous in the distribution of agents' characteristics, infinitesimal individuals cannot influence prices. Hence, these contributions show that efficiency and the no-envy version of incentive compatibility imply that individuals cannot influence the wealths of others. Our main result confirms these conclusions and supplements them by showing that the inability to influence prices that is taken for granted in the continuum is also essential for the incentive compatibility conclusion in the finite model.

Our treatment of incentive compatibility in the continuum is based on limiting results for finite economies. (See Roberts and Postlewaite (1976).) Unlike finite economies where we show (along with Hurwicz and Walker (1990) for the quasi-linear case) that efficient incentive compatible mechanisms are generically impossible, we also show that there exist mechanisms that are asymptotically efficient and incentive compatible for a generic set of continuum economies.

Two related contributions establish what appear to be rather different conclusions. Hurwicz and Walker (1990) give a characterization of efficient incentive compatible mechanisms for exchange economies with finite numbers and quasi-linear preferences that is only remotely related to perfect competition. Barberà and Jackson (1995) characterize all incentive compatible mechanisms for exchange economies with finite numbers and ordinal preferences; they find that none of them comes close to being efficient, even as the number of individuals increases. We shall explain the apparent discrepancies after the presentation of our results.

In Section 2, we describe a finite numbers exchange environment consistent with either ordinal or quasi-linear preferences. In Section 3, we give the main result (Theorem 1) for the ordinal model; and, in Section 4, we give the analogous result for the model with quasi-linear preferences. In Section 5, the main result is illustrated by example. In Section 6, we demonstrate the sense in which most large economies exhibit efficiency and incentive compatibility and explain the need for qualifications even in the continuum. Section 7 is devoted to a proof of Theorem 1. Section 8 concludes with a discussion of the connections between this paper and the work of others. An Appendix contains proofs not included in the previous sections.

2 The Model

There are n individuals, indexed by i or j, and ℓ commodities. The set of all individuals is denoted by I. Each individual's consumption set is $\Omega \subset \mathbf{R}^\ell$. $\mathcal{U}$ denotes the set of admissible utility functions u for each individual, where $u : \Omega \to \mathbf{R}$. A *population* is a vector of utility functions $\mathbf{u} = (u_1, \dots, u_i, \dots, u_n)$. The economy's aggregate endowment is $\omega \in \mathbf{R}^\ell_+$ and there is no production; so the attainable allocations for any population $\mathbf{u}$ is given by

$$X = \{\mathbf{x} = (x_1, \dots, x_i, \dots x_n) \in \Omega^n : \sum_i x_i = \omega\}.$$

Let $\mathcal{D} \subseteq \mathcal{U}^n$, with typical element $\mathbf{u}$. A *mechanism* is a mapping $f : \mathcal{D} \to X$. Throughout we assume f is efficient and anonymous on $\mathcal{D}$:

Pareto efficiency For all $\mathbf{u} \in \mathcal{D}$, there is no $\mathbf{x} \in X$ such that $u_i(x_i) \geq u_i(f_i(\mathbf{u}))$ for each i and $u_i(x_i) > u_i(f_i(\mathbf{u}))$ for some i.

Anonymity For all $\mathbf{u} \in \mathcal{D}$, $u_i = u_j$ implies $u_i(f_i(\mathbf{u})) = u_j(f_j(\mathbf{u}))$.

For any population $\mathbf{u}$, let $(\mathbf{u}, v, i)$ denote the same population except the utility function of individual i is replaced by v. The mechanism f is *incentive compatible* (IC) at $\mathbf{u}$ if, for each i

$$u_i(f_i(\mathbf{u})) \geq u_i(f_i(\mathbf{u}, v, i)) \text{ for all } (\mathbf{u}, v, i) \in \mathcal{D}.$$

f is *incentive compatible* if it is incentive compatible at each $\mathbf{u} \in \mathcal{D}$.

Let $A(x, u) = \{y \in \Omega : u(y) \geq u(x)\}$; it represents the at-least-as-good-as-x set for an individual with preferences u. A *price vector* is a $p \in \mathbf{R}^\ell_{++}$, with $p_\ell = 1$ (a normalization). The price vector p *supports* the allocation $f(\mathbf{u})$ at $\mathbf{u}$ if $p\omega \leq p \sum_i A(f_i(\mathbf{u}), u_i)$. Let $\mathcal{P}(\mathbf{u})$ be the set of all price vectors that support $f(\mathbf{u})$ at $\mathbf{u}$. Notice we restrict p to be strictly positive. This is without loss of generality since, for both the ordinal and quasi-linear cases, we will impose a strict monotonicity assumption on preferences.

(The definition does not require allocations that i prefers to x_i to cost strictly more than $p\bar{\omega}$. But, for both the ordinal and quasi-linear cases, it will be the case that $p\bar{\omega} > \inf p\Omega$, which implies this requirement via a standard argument.)

3 Ordinal Economies

Now assume the consumption set Ω equals $\mathbf{R}_+^\ell$ and the aggregate endowment ω is strictly positive. Restrict preferences to the set $\mathcal{U}$ consisting of all continuous, quasi-concave functions on $\mathbf{R}_+^\ell$ that are strictly increasing and continuously differentiable on $\mathbf{R}_{++}^\ell$ and satisfy the following interiority assumption:

Interiority $u \in \mathcal{U}$ implies $u(x) = 0$ unless $x \gg 0$. Further, for any given $p \gg 0$ and $u \in \mathcal{U}$, there is a closed convex cone C in $\mathbf{R}_{++}^\ell \cup \{0\}$ such that $\{x : \nabla u(x) \propto p\} \subset C$.

For example, all Cobb-Douglas functions satisfy the interiority assumption. It guarantees that any efficient allocation will be in $\mathbf{R}_{++}^\ell$ for all individuals with non-zero consumption. Thus $f(\mathbf{u})$ will have only one price support at $\mathbf{u}$, denoted by $p(\mathbf{u})$, where $p(\mathbf{u}) \propto \nabla u_i(f_i(\mathbf{u}))$ for all i with $f_i(\mathbf{u}) \gg 0$.

We also will assume that the mechanism is continuous, at least in utilities. More precisely, let $u^k \to u_i$ mean the sequence of utility functions $\{u^k\}_{k=1,2,\ldots}$ converges to u_i uniformly on compacta. It is straightforward to verify that any incentive compatible mechanism f will be continuous in the following limited sense: Suppose $\mathbf{u}$, $\mathbf{u}^k \in \mathcal{D}$, where $\mathbf{u}^k = (\mathbf{u}, u^k, i)$. Then $u^k \to u$ implies $u^k(f_i(\mathbf{u}^k)) \to u_i(f_i(\mathbf{u}))$.[2] We will make a stronger assumption.

Continuity Suppose $\mathbf{u}$, $\mathbf{u}^k \in \mathcal{D}$, where $\mathbf{u}^k = (\mathbf{u}, u^k, i)$. Then $u^k \to u$ implies $u^k(f_i(\mathbf{u}^k)) \to u_i(f_i(\mathbf{u}))$ and $u_j(f_j(\mathbf{u}^k)) \to u_j(f_j(\mathbf{u}))$ for all individuals $j \neq i$.

3.1 Main Result

Let $\mathcal{D}$ be a neighborhood of $\mathbf{u}$, the domain of the mechanism. The smaller is $\mathcal{D}$, the greater the number of mechanisms consistent with incentive compatibility at $\mathbf{u}$; for example, if

[2]Here is a sketch of the proof. Let $x^k = f_i(\mathbf{u}^k)$ and $x = f_i(\mathbf{u})$. Suppose $u^k(x^k) \not\to u_i(x)$. Then on a subsequence $u^k(x^k) \to a$, where $a \neq u_i(x)$. Suppose $a < u_i(x)$. Then for some sufficiently large k, $u^k(x) > u^k(x^k)$. This is a contradiction to the incentive compatability assumption, because consumer i will be better off if he claims he is of type u_i rather than u^k. Similarly, $a > u_i(x)$ leads to a contradiction.

$\mathcal{D} = \{\mathbf{u}\}$, a singleton, then IC is trivial. We will show that if $\mathcal{D}$ is sufficiently large then only one mechanism is IC, a perfectly competitive mechanism. Notice that perfect competition always suffices for IC. Therefore, if we can establish that perfect competition is necessary for $\mathcal{D}$, then it is necessary and sufficient for any larger domain; in particular, the characterization also holds for $\mathcal{D} = \mathcal{U}^n$. On the opposite side, the larger is $\mathcal{D}$, the more difficult it is to achieve a characterization that is non-vacuous; in particular, on the universal domain, $\mathcal{U}^n$, we shall see in Section 3.3 that no IC mechanism exists (see also Barberà and Jackson (1995)). Thus, the desideratum that the characterization be non-vacuous implies that the domain must be carefully chosen so that $\mathcal{D}$ is not *too* rich.

We will now describe a richness condition on $\mathcal{D}$ that yields our (non-vacuous) characterization. (The reader may skim over the details and return to them when reading the proof of Theorem 1 in Section 7.) Elements of $\mathcal{D}$ will be classified into movements from $\mathbf{u}$

- one person at a time
- by any number of persons
- that are first-order similar.

One person changes will be unrestricted, i.e, the set $\{\mathbf{u}, \mathcal{U}, i\} := \{(\mathbf{u}, v, i) : v \in \mathcal{U}\}$ will be permitted in $\mathcal{D}$. Changes by any number of persons may only be required to exhibit a first-order similarity to $\mathbf{u}$.

Suppose $f_i(\mathbf{u}) \gg 0$. The set of *first-order similar* preferences for i at $\mathbf{u}$ is

$$N_i(\mathbf{u}) = \{v \in \mathcal{U} : \nabla v(f_i(\mathbf{u})) = \nabla u_i(f_i(\mathbf{u}))\}.$$

Observe that first-order similarity only restricts the shape of i's indifference curve through $f_i(\mathbf{u})$. In particular, $p(\mathbf{u})$ supports $f_i(\mathbf{u})$ for any preferences in $N_i(\mathbf{u})$. It will be notationally convenient to define $N_i(\mathbf{u})$ for any i; so if $f_i(\mathbf{u}) = 0$, let $N_i(\mathbf{u}) = \{u_i\}$, a singleton.

Linear preferences are an important example of first-order similarity. While linear preferences cannot be included in $\mathcal{D}$ because of the interiority assumption, individuals will be permitted to claim preferences that are linear inside large cones contained in Ω. In particular, let $u_{\mathbf{u}}$ be any preferences in $\mathcal{U}$ satisfying

- $u_{\mathbf{u}}(x) = p(\mathbf{u}) \cdot x$ for all $x \in C(\mathbf{u})$
- $u_{\mathbf{u}}$ is strictly quasi-concave in $\mathbf{R}^{\ell}_{++} \backslash C(\mathbf{u})$,

where $C(\mathbf{u})$ is any given closed convex cone in $\mathbf{R}^{\ell}_{++} \cup \{0\}$ containing ω in its interior and also containing the "income expansion paths" for the n individuals, $\{x : \nabla u_i(x) \propto p(\mathbf{u})$ for some $i\}$. So, inside $C(\mathbf{u})$—where all individuals' consumptions $f_i(\mathbf{u})$ reside—the indifference curves of $u_{\mathbf{u}}$ are linear with slope $p(\mathbf{u})$. Given the interiority assumption, such preferences exist for any $\mathbf{u} \in \mathcal{U}^n$.

For any pair of preferences u and v, let $[u, v]$ represent the set of all convex combinations of u and v, that is, $\{u' : u' = \alpha u + (1-\alpha)v, \alpha \in [0,1]\}$. Define the set of *flattenings* of $\mathbf{u}$ as those $\mathbf{v}$ where each $v_i \in [u_i, u_{\mathbf{u}}]$:

$$\mathcal{L}^0(\mathbf{u}) = \{\mathbf{v} = (v_1, \ldots, v_i, \ldots, v_n) : v_i \in [u_i, u_{\mathbf{u}}] \text{ for each } i\}.$$

$\mathcal{L}^0$ permits any number of persons to change from $\mathbf{u}$ *provided* that each is a flattening. For the characterization, such flattenings must be permitted in $\mathcal{D}$.

Let

$$\mathcal{L}(\mathbf{u}) = \mathcal{L}^0(\mathbf{u}) \cup \{(\mathbf{v}, v, i) : \mathbf{v} \in \mathcal{L}^0(\mathbf{u}),\ v \in N_i(\mathbf{v}),\ i \in I\}.$$

The set $\mathcal{L}(\mathbf{u})$ is a hybrid: it consists of changes from $\mathbf{u}$ by any number of persons provided they are flattenings, plus any one person changing to some first-order similar preferences. In other words, $\mathbf{v}' \in \mathcal{L}(\mathbf{u})$ means that $\mathbf{v}'$ is first-order similar to $\mathbf{u}$ and at most one individual's preferences in $\mathbf{v}'$ involve a *steepening* relative to his preferences in $\mathbf{u}$.

Finally we consider $\mathcal{L}(\mathbf{v})$, but only for those $\mathbf{v} \in \{\mathbf{u}, \mathcal{U}, i\}$. Hence, at $\mathbf{v}$ there is at most one individual with MRS at $x_i = f_i(\mathbf{u})$ that is not proportional to $p = p(\mathbf{u})$. Let $q = q(\mathbf{v})$. Observe, if $p \neq q$ then $\mathcal{L}(\mathbf{v})$ includes all flattenings of $\mathbf{v}$, an entirely different set of perturbations than those in $\mathcal{L}(\mathbf{u})$; whereas if $p = q$ then $(u_{\mathbf{u}}, \ldots, u_{\mathbf{u}}) \in \mathcal{L}(\mathbf{u}) \cap \mathcal{L}(\mathbf{v})$.

DEFINITION: $\mathcal{D}$ is (sufficiently) *rich* at $\mathbf{u}$ if

(R.1) for every individual i, $\{\mathbf{u}, \mathcal{U}, i\} \subset \mathcal{D}$

(R.2) for every i and every $\mathbf{v} \in \{\mathbf{u}, \mathcal{U}, i\}$, $\mathcal{L}(\mathbf{v}) \subset \mathcal{D}$.

That is, i can be *any* type in $\mathcal{U}$, provided others are at $\mathbf{u}_{-i} := (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n)$. Further, for any population $\mathbf{v}$ resulting from any one individual i switching, $\mathcal{L}(\mathbf{v})$ is also in $\mathcal{D}$.

We impose the following condition on $\mathbf{u}$ itself. The economy excluding individual i, $\mathbf{u}_{-i}$, will be called *regular* if it has only a finite number of equal-wealth Walrasian equilibria when its aggregate endowment equals $\omega - \bar{\omega}$. It is known that such economies are typical (e.g., see Debreu (1970) or Mas-Colell (1985, Chapter 5)). We call the economy $\mathbf{u}$ *regular* if, for each i, $\mathbf{u}_{-i}$ is regular.

Let $(\mathbf{u}_{-i}, v)$ denote the profile in which i is of type v and others are of types $\mathbf{u}_{-i}$. For any given $\mathbf{u}_{-i}$, the range of the mechanism f for individual i is

$$R_i(\mathbf{u}_{-i}) = \{f_i(\mathbf{u}, v, i) : (\mathbf{u}, v, i) \in \mathcal{D}\}.$$

An immediate but important observation is that f will be incentive compatible for i at $(\mathbf{u}_{-i}, u)$ if and only if it assigns i his u-best point in the set $R_i(\mathbf{u}_{-i})$; that is, it will be incentive compatible if and only if

$$f_i(\mathbf{u}_{-i}, u) \in \arg\max_x u(x) \text{ s.t. } x \in R_i(\mathbf{u}_{-i}).$$

The idea is simple: If $u(f_i(\mathbf{u}, u, i)) < u(f_i(\mathbf{u}, v, i))$ then it would not be incentive compatible for i to truthfully announce u. Thus $R_i(\mathbf{u}_{-i})$ may be regarded as i's *opportunity set* at $\mathbf{u}_{-i}$.

Let H_p denote the hyperplane through the origin with normal p, i.e., $H_p := \{y : py = 0\}$. Our main result shows that the range of the mechanism for all individuals is contained in a common hyperplane, namely, a hyperplane through the economy's average endowment $\bar{\omega}$.

Theorem 1 *Let* $\mathbf{u}$ *be any regular economy such that* $\mathcal{D}$ *is rich at* $\mathbf{u}$. *If* f *is IC on* $\mathcal{D}$ *then for every individual* i,

$$R_i(\mathbf{u}_{-i}) \subset H_p + \{\bar{\omega}\},$$

where $p = p(\mathbf{u})$.

The proof of the linearity of $R_i(\mathbf{u}_{-i})$, in Section 7, consists of three steps: (1) The range $R_i(\mathbf{u}_{-i})$ is not locally stictly convex (Lemmas 1 and 2); (2) The mechanism must always assign an equal-wealth Walrasian allocation (Lemma 3); (3) The range is not locally stictly concave (Lemmas 4 and 5). Each of these lemmas require only one person changes from $\mathbf{u}$, with the exception of Lemma 3. This lemma is proved via a sequence of $n-1$ manipulations. At each step, one yet unmanipulated utility function is replaced by an essentially flat utility function common to all the previous manipulations. In this part of the proof more that one person changes from $\mathbf{u}$, but only in a limited way: Even when more than one person moves away from $\mathbf{u}$, everyone still shares a common MRS at $f(\mathbf{u})$. Requirement (R.2) allows for the needed flexibility.

3.2 The full appropriation connection

The conclusion of Theorem 1 can be restated in a way that highlights its links with the quasi-linear model. Given any profile for others, $\mathbf{u}_{-i}$, in an IC mechanism individual i must "fully appropriate" the consequences of his actions, in the sense that others are neither benefited nor hurt by his announced type. More precisely, notice that $A_{-i}(\mathbf{u}) := \sum_{j \neq i} A(f_j(\mathbf{u}), u_j)$ represents the aggregate at-least-as-good-as-$f(\mathbf{u})$ set for individuals other than i. If the mechanism is incentive compatible, the boundary of this set becomes i's opportunity set (see Figure 1). That is, the mechanism acts as if i completely controls the allocation of the economy's entire resources, ω, subject only to the constraint that each individual $j \neq i$ must achieve at least utility $u_j(f_j(\mathbf{u}))$.

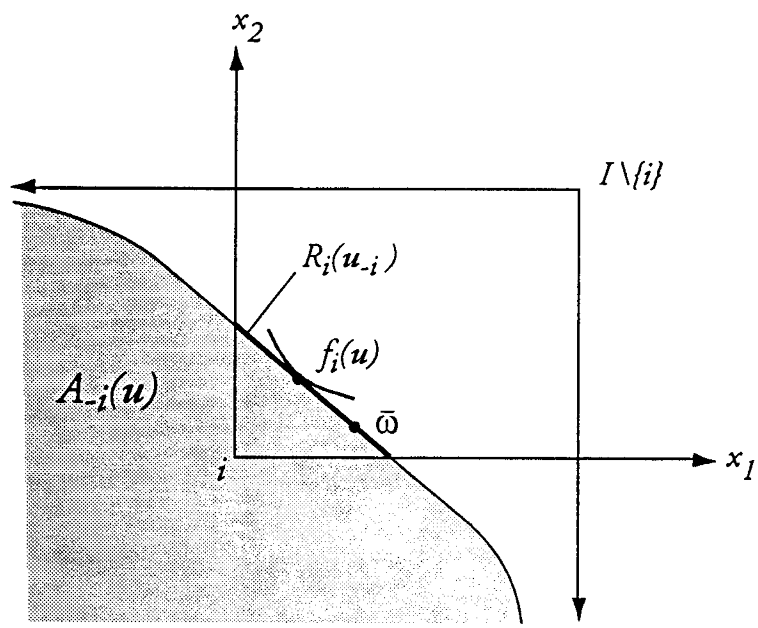

Figure 1: INDIVIDUAL i ACTS AS A FULL APPROPRIATOR. The size of the Edgeworth box is ω. Hence, $f_i(\mathbf{u})$ from i's perspective equals $\omega - f_i(\mathbf{u})$ from $I\setminus\{i\}$'s perspective.

Corollary 1 (full appropriation) *Suppose f is IC on the domain $\mathcal{D}$. Let $\mathbf{u}$ be any regular population such that $\mathcal{D}$ is rich at $\mathbf{u}$. Then, for each i and $v \in \mathcal{U}$*

$$f_i(\mathbf{u}, v, i) \in \arg\max_{x \in \Omega} v(x) \ \ s.t. \ x \in \{\omega\} - A_{-i}(\mathbf{u}).$$

3.3 Impossibility on the universal domain

In a finite economy, if there is a one person change from $\mathbf{u}$ to $\mathbf{v} \in \{\mathbf{u}, \mathcal{U}, i\}$ then efficiency demands that the change be accommodated by a change in the allocation, i.e., $f(\mathbf{u}) \neq f(\mathbf{v})$. But the change in the allocation will typically conflict with the demands of IC that prices do not change—that i should have no monopoly power. The two demands are compatible only if the change in the allocation does not require a change in the MRS's of $\mathbf{u}_{-i}$, i.e., only if the preferences in $\mathbf{u}_{-i}$ exhibit linear segments. This will be illustrated in Section 5, where an example of a rich, perfectly competitive domain will be presented.

The impossibility of IC on the *universal domain* $\mathcal{U}^n$ follows readily from Theorem 1 since, in most populations $\mathbf{u} \in \mathcal{U}^n$, some individuals will have monopoly power. Indeed, impossibility can be shown even on relatively small domains. Let $\mathcal{U}^*$ denote the set of all $u \in \mathcal{U}$ that are *strictly* quasi-concave in the interior of Ω. Let w^p be any element of $\mathcal{U}^*$ satisfying $\nabla w^p(\bar{\omega}) = p$. The following is a sufficient condition for IC to be unachievable.

Corollary 2 *Suppose* $\mathbf{u}$ *is regular. Let* $\mathbf{v} = (\mathbf{u}, v, i)$ *and suppose* $p = p(\mathbf{u}) \neq q = q(\mathbf{v})$. *Then, any mechanism* f *on the domain* $\mathcal{L}(\mathbf{u}) \cup \mathcal{L}(\mathbf{v}) \cup_{r \in [p,q]} \mathcal{L}((\mathbf{v}, w^r, i))$ *will fail to be IC.*

The next result shows that the conditions described in Corollary 2 are ubiquitous. (See Hurwicz and Walker (1990) for an analogous result in the quasi-linear model.)

Corollary 3 *Suppose* $\mathbf{u}$ *is regular, with* $u_j \in \mathcal{U}^*$ *for each individual* j. *Let* v *be any preferences such that* $\nabla v(f_i(\mathbf{u})) \neq \nabla u_i(f_i(\mathbf{u}))$ *for some individual* i. *Then, letting* $\mathbf{v} = (\mathbf{u}, v, i)$, $p = p(\mathbf{u})$ *and* $q = p(\mathbf{v})$, *any mechanism* f *on the domain* $\mathcal{L}(\mathbf{u}) \cup \mathcal{L}(\mathbf{v}) \cup_{r \in [p,q]} \mathcal{L}((\mathbf{v}, w^r, i))$ *will fail to be IC.*

Intuition for Corollary 3 comes from Figure 1. As the figure illustrates, IC implies each individual i must face a linear opportunity set. But if all individuals $j \neq i$ have *strictly* quasi-concave preferences then the boundary of $A_{-i}(\mathbf{u})$ will be strictly convex. So, acting as a full appropriator, i will not face a linear budget line. The underlying idea is that if everyone else's preferences are strictly quasi-concave, then when i changes his quantities demanded, market-clearing prices *must* change since others will not be willing to accommodate him at the original prices.

4 Transferable Utility Economies

In this section, we restrict preferences to the quasi-linear form, that is, $u \in \mathcal{U}$ implies

$$u(x_1, \ldots, x_\ell) = \tilde{u}(x_1, \ldots, x_{\ell-1}) + x_\ell.$$

Further, assume individuals are not restricted in the quantity of the last commodity that they can supply, so the consumption set Ω now equals $\mathbf{R}_+^{\ell-1} \times \mathbf{R}$. Finally, take the aggregate endowment ω as strictly positive for all goods except the last, with $\omega_\ell = 0$.

As above, the domain of the mechanism f will be $\mathcal{D} \subseteq \mathcal{U}^n$; however, in this section $\mathcal{U}$ consists of all continuous, quasi-concave, strictly increasing utility functions on Ω with the quasi-linear form. This domain permits the following simplifications: we will not need to assume f is continuous, nor will we need to impose any smoothness or interiority assumptions on the preferences in $\mathcal{U}$.

Let u^p be the preferences defined by

$$u^p(x) = px \quad \text{for all } x \in \mathbf{R}_+^\ell.$$

Let

$$\begin{aligned} \mathcal{L}^1(\mathbf{u}, p) = \{\mathbf{u}^S &= (u_1^S, \ldots, u_i^S, \ldots, u_n^S) : S \subseteq I, \\ & u_i^S = u_i \text{ for all } i \in S,\ u_i^S = u^p \text{ for all } i \notin S\}. \end{aligned}$$

be the set of all possible replacements of utility functions in $\mathbf{u}$ by u^p. The role of $\mathcal{L}(\mathbf{u})$ in the ordinal model is played in the quasi-linear model by

$$\mathcal{L}(\mathbf{u}, p) = \{(\mathbf{u}^S, u, i) : \mathbf{u}^S \in \mathcal{L}^1(\mathbf{u}, p),\ i \in S,\ u \in [u_i, u^p]\}.$$

Richness with quasi-linear preferences is

DEFINITION: $\mathcal{D}$ is (sufficiently) *rich at* $\mathbf{u}$ if

(R.1*) for every individual i, $\{\mathbf{u}, \mathcal{U}, i\} \subset \mathcal{D}$, and

(R.2*) for some $p \in \mathcal{P}(\mathbf{u})$, for every i and every $\mathbf{v} \in \{\mathbf{u}, \mathcal{U}, i\}$, $\mathcal{L}(\mathbf{v}, p) \subset \mathcal{D}$.

As in the ordinal model, we have

Theorem 2 *Let* $\mathbf{u}$ *be any regular population such that* $\mathcal{D}$ *is rich at* $\mathbf{u}$. *If* f *is IC on the domain* $\mathcal{D}$, *then there exists a price vector* p *such that, for every individual* i,

$$R_i(\mathbf{u}_{-i}) \subset H_p + \{\bar{\omega}\}.$$

This theorem is proved in Appendix B. Although differentiability of preferences plays no explicit role, in the quasi-linear model it is sufficient for regularity (see Fact 2, Appendix B).

4.1 Perfect Competition as Full Appropriation

Our method of proof for the quasi-linear domain builds on the Groves characterization of IC mechanisms. For any population $\mathbf{u}$, the *gains from trade* in $\mathbf{u}$ is given by

$$G(\mathbf{u}) = \max \sum_i u_i(x_i) \text{ s.t. } \mathbf{x} \in X.$$

As is well known, for TU (transferable utility) economies efficient allocations are equivalent to ones that maximize the gains from trade.

For mechanisms achieving the maximum gains from trade, Holmström (1979) showed that if f is IC on the convex domain $\{\mathbf{u}, [u_i, u], i\}$ then, for every $v \in [u_i, u]$, individual i appropriates the whole gains from trade minus a lump sum:

$$v(f_i(\mathbf{u}, v, i)) = G(\mathbf{u}, v, i) - \sum_{j \neq i} u_j(f_j(\mathbf{u})). \tag{1}$$

So the share of the gains from trade going to others remains constant:

$$\sum_{j \neq i} u_j(f_j(\mathbf{u}, v, i)) = \sum_{j \neq i} u_j(f_j(\mathbf{u})). \tag{2}$$

For our purpose it will be more convenient to express (1) in the set theoretic terms of Corollary 1. This says that i receives her best consumption bundle, subject to others receiving at least utility $\sum_{j \neq i} u_j(f_j(\mathbf{u}))$:

$$f_i(\mathbf{u}, v, i) \in \arg\max_{x \in \Omega} v(x) \quad \text{s.t.} \quad (\omega - x) \in A_{-i}(\mathbf{u}).$$

Or equivalently,

$$R_i(\mathbf{u}_{-i}) \subset \{\omega\} - A_{-i}(\mathbf{u}).$$

See Figure 2 below.

Hence, the Groves characterization demonstrates that for a mechanism to be incentive compatible it must assign "utility rights" defined by $A_{-i}(\mathbf{u})$ and then allow each individual to fully appropriate all the added gains that u_i contributes to the social total. Theorem 2

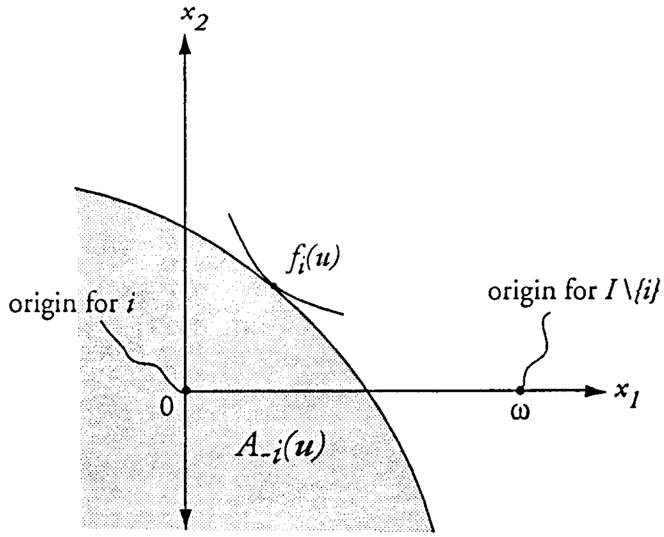

Figure 2: INDIVIDUAL i FULLY APPROPRIATES THE GAINS FROM TRADE MINUS A LUMP SUM, LEAVING ALL OTHERS AT UTILITY LEVEL $\sum_{j \neq i} u_j(f_j(\mathrm{u}))$

tells us that if the mechanism is budget balancing, the incentive compatible opportunity set for each i must be linear in the positive orthant. In other words, the only environments in which full appropriation by individuals can be feasible are the perfectly competitive ones.

5 Perfectly Competitive Domains for Finite Economies: An Example

In this section we construct an example of an incentive compatible mechanism with a rich domain. As Theorem 1 requires, the example will involve a family of perfectly competitive economies. Corollary 3 implies that if everyone's preferences are *strictly* quasi-concave, monopoly power is inevitable in finite economies and therefore IC mechanisms cannot be achieved. Nevertheless, we will show that constructing a rich IC example is easy: *any* finite economy can be perturbed such that it and a class of nearby economies form a domain that satisfies the requirements of Theorems 1 or 2. Further, the distance between the original economy and the perturbed economy goes to zero as the number of individuals increases.

Below attention is limited to ordinal economies, with $n \geq 3$. In the construction, $B(x, \epsilon)$ will denote an ϵ-ball centered at x, i.e., $B(x, \epsilon) := \{y : \| y - x \| < \epsilon$.

To begin, pick any population u' in $\mathcal{U}^n$ and let (x^*, p) be any equal-wealth Walrasian equilibrium for $\mathbf{u}'$. In an epsilon neighborhood of x_i^*, flatten i's indifference curve through x_i^*, keeping its slope equal to p. Let $\mathbf{u}^* = (u_1^*, \ldots, u_n^*)$ denote such a perturbed population. So, by construction, there exists an $\epsilon > 0$ such that for each i

$$y \in B(x_i^*, \epsilon) \cap (H_p + \{\bar{\omega}\}) \Rightarrow u_i^*(y) = u_i^*(x_i^*).$$

Notice that $(\mathbf{x}^*, p)$ remains an equilibrium for the perturbed economy.

Let δ be the diameter of a ball centered at $\bar{\omega}$ that includes $\mathbf{R}_+^\ell \cap (H_p + \{\bar{\omega}\})$, that is,

$$\mathbf{R}_+^\ell \cap (H_p + \{\bar{\omega}\}) \subset B(\bar{\omega}, \frac{\delta}{2}).$$

See Figure 3. To ensure that no one will be able to influence prices, assume n is sufficiently large that

$$\epsilon > \frac{2\delta}{n-2}. \qquad (\star)$$

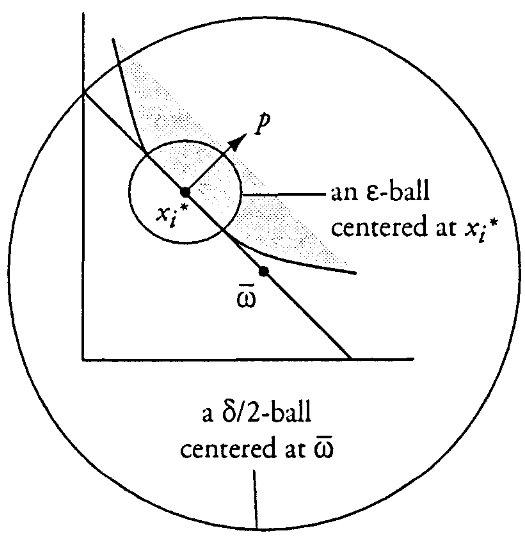

Figure 3: EACH INDIVIDUAL'S INDIFFERENCE CURVE THROUGH x_i^* HAS BEEN FLATTENED TO ENSURE NO ONE WILL BE ABLE TO INFLUENCE PRICES

Let $\mathcal{D}$ be the set of populations satisfying (R.1-2) when $\mathbf{u} = \mathbf{u}^*$, i.e., the smallest domain that is rich at $\mathbf{u}^*$. We will show that there exists an IC mechanism on $\mathcal{D}$. In particular, there is a mechanism f such that for every population $\mathbf{u} \in \mathcal{D}$, $p(\mathbf{u}) = p$ and $(f(\mathbf{u}), p) \in \mathrm{WE}(\mathbf{u})$. That is, f is a perfectly competitive mechanism on $\mathcal{D}$.

The construction of f follows.

1. Let $f(\mathbf{u}^*) = \mathbf{x}^*$.

2. For any i and $\mathbf{u} \in \{\mathbf{u}^*, \mathcal{U}, i\}$, let

$$f_i(\mathbf{u}) \equiv x_i \in \arg\max_{x \in \Omega} u_i(x) \text{ s.t. } px \leq p\bar{\omega}.$$

   And for each $j \neq i$, let

$$f_j(\mathbf{u}) \equiv x_j = x_j^* - \frac{z}{n-1},$$

   where $z = x_i - x_i^*$. Since $z \in B(0, \frac{\delta}{2})$, $\| z \| < \delta$. So the inequality ($\star$) implies

$$\| \frac{z}{n-1} \| < \frac{\delta}{n-1} < \frac{\epsilon}{2}.$$

   That is, for all individuals $\nabla u_j(x_j) = \nabla u_j(x_j^*) = p$, so $(\mathbf{x}, p) \in \mathrm{WE}(\mathbf{u})$.

3. For any i, $\mathbf{u} \in \{\mathbf{u}^*, \mathcal{U}, i\}$, and $\mathbf{u}' \in \mathcal{L}(\mathbf{u})$, let $f(\mathbf{u}') = f(\mathbf{u}) \equiv \mathbf{x}$. It is easy to verify that $(\mathbf{x}, p) \in \mathrm{WE}(\mathbf{u}')$.

The example shows that small flats for all individuals, as illustrated in Figure 3, results in each individual facing a large flat segment as his opportunity set, as illustrated in Figure 1. Also notice from the inequality ($\star$) that as $n \to \infty$, ϵ can be chosen to go to zero; hence, the population $\mathbf{u}^*$ approaches the population $\mathbf{u}'$ with smooth and perhaps *strictly convex* preferences. Alternatively expressed, an arbitrary population selected from $\mathcal{U}^n$ approaches a perfectly competitive population as $n \to \infty$. The example therefore helps us understand why a Walrasian mechanism is IC in economies with a continuum of agents and smooth preferences—without the need for any further "flattening" of preferences: generically, in the continuum each infinitesimal individual does face a linear opportunity set, hence each infinitesimal individual can fully appropriate his social contribution as illustrated in Figure 1 (see Section 6).

6 On the Asymptotic Elimination of the Incentive/Efficiency Trade-off

The characterization of efficiency and incentive compatibility as perfect competition suggests that the elimination of a trade-off between these two objectives will be a variation on the familiar theme of finite economies becoming more nearly perfectly competitive as the size of the population increases. In comparison to the standard approach which identifies perfect competition with the continuum, we shall proceed by showing that continuum economies are perfectly competitive because they resemble perfectly competitive finite economies. The crucial difference is that whereas, among finite populations of a given size, perfectly competitive economies are sparse, perfectly competitive continuum economies are, in a standard mathematical sense, the complement of a sparse set. The intuition for the turnabout comes from the geometry of the Example in Section 5: generically in continuum economies, infinitesimally-sized flats in indifference curves—that is, smooth indifference curves—suffice for perfect competition.[3]

6.1 Preliminaries

$\mathcal{U}$ remains as above the set of continuously differentiable, quasi-concave utility functions that are strictly increasing and satisfy the interiority condition. $\mathcal{U}$ is separable and metrizable. Let M be the set of probability measures μ on $\mathcal{U}$ endowed with the topology of weak convergence. It is important to observe that in this topology $\mu_k \to \mu$ need not imply that $\operatorname{supp}\mu_k$ converges in the Hausdorff metric to $\operatorname{supp}\mu$, although there will exist sets $R_k \subset \operatorname{supp}\mu_k$ such that $R_k \to \operatorname{supp}\mu$ and $\mu_k(R_k) \to \mu(\mathcal{U})$. The elements of $\operatorname{supp}\mu_k$ not in R_k may be regarded as the non-representative members of μ_k in the sense that they are not close to the types of individuals appearing in μ. To describe misrepresentation of preferences, measures with such non-representative types must be allowed.

Recall that $\omega \in \mathbf{R}^{\ell}_{++}$. Let $S = \{p : p \in \mathbf{R}^{\ell}_{++} \;\&\; p\cdot\omega = 1\}$ be the set of normalized prices. At prices p, the set of utility-maximizing choices for an individual with characteristics (u, ω) is

$$\psi(p, u) = \arg\max\{u(x) : px = p\omega = 1\}.$$

The aggregate demand correspondence for the economy μ when individuals are each endowed with ω is

$$\Psi(p, \mu) = \int \psi(p, u)\, d\mu(u).$$

Since M is a set of probability measures, $\int \omega\, d\mu = \omega$ for all $\mu \in M$. Therefore,

$$\Pi(\mu) = \{p : \omega \in \Psi(p, \mu)\}$$

[3]Differentiability of preferences is only generically sufficient (not always sufficient) because here we are concerned with asymptotic results; so we view continuum economies as the limit of large finite economies.

is the set of Walrasian equilibrium prices for the population μ when each individual is endowed with ω.

It is well-known that

FACT 1 $\Psi : S \times M \to 2^{\mathbf{R}^\ell}$ *is a non-empty, compact and convex-valued correspondence that is jointy upper hemi-continuous; and,* $\Pi : M \to 2^S$ *is a non-empty compact-valued upper hemi-continuous correspondence.*

Throughout the following π denotes a selection from Π.

6.2 Analysis

Define M_n as the subset of probability measures $\mu_n = \sum \alpha_m \delta_{u_m}$ with finite support, where each α_m is an integer multiple of n^{-1} and δ_u is the measure with unit mass centered at u. M_n is interpreted as the set of possible distributions of preferences for populations consisting of n individuals, where each individual has weight n^{-1}. Regarded as a finite economy with n individuals, we should denote μ_n as, for example, (μ_n, n) to distinguish it from a continuum economy having finite support. However, unless the contrary is explicitly stated, we shall regard μ_n as a finite economy with n individuals.

Let

$$J(\mu_n) = \{\Delta u := (u, v) \in \operatorname{supp} \mu_n \times \mathcal{U}\};$$

$\Delta u \in J(\mu_n)$ represents a perturbation of μ_n to the "adjacent" population

$$\mu_n + \Delta u := \mu_n - n^{-1}(\delta_u - \delta_v),$$

accessible by a one-person perturbation from μ_n. In relation to the notation used above, $\mathbf{u} \sim \mu_n$, $(\mathbf{u}, v, i) \sim \mu_n + \Delta u$ and $(\mathbf{u}, \mathcal{U}, i) \sim \mu_n + J(\mu_n)$. Since μ_n is a distribution, individual characteristics are already in anonymous form; e.g., $(\mathbf{u}, v, i)$ and $(\mathbf{u}, v, j)$ both have the same distribution $\mu_n + \Delta u$ if $u = u_i = u_j$.

Given any price selection π, let

$$d_\pi(\mu_n) = \text{diameter}\{\pi(\mu_n + \Delta u) : \Delta u \in J(\mu_n)\},$$

i.e., the diameter of the smallest ball containing the set of market-clearing prices for μ_n and all of its adjacent populations. Call $\mu_n^* \in M_n$ a *perfectly competitive population* relative to the price selection π if $d_\pi(\mu_n^*) = 0$; and let $M_n^*(\pi) \subset M_n$ denote all such perfectly competitive populations. From Corollary 2 and the Example in Section 5, if μ_n^* is perfectly competitive then some individuals $u \in \text{supp}\,\mu_n^*$ must have indifference curves with flat segments through $x \in \psi(p, u)$, where $p = \pi(\mu_n^*)$. Restating our main result for finite economies, for a mechanism to be efficient and incentive compatible, it must be an equal-wealth Walrasian mechanism that is perfectly competitive, which means its domain must be in $M_n^*(\pi)$ for some price selection π.

Notice that the competitivity of μ_n^* depends on the choice of π; e.g., some economies may be perfectly competitive for one choice of π but not for another. To establish an asymptotic result, we will show that there exists *some* price selection under which the Walrasian mechanism will be (1) nearly perfectly competitive and, hence, (2) nearly incentive compatible in most large finite economies. The key to establishing both claims is the following result on the existence of a price selection that is generically continuous.

Theorem 3 *There exists a selection $\pi \in \Pi$ that is continuous on a dense G_δ subset of M.*

The proofs of all results in this section are given in Appendix C. To establish genericity, elements $u \in \mathcal{U}$ are replaced by their money-metric representations u_p (see below). Hence $\mathcal{U}$ is replaced by $\mathcal{U}_p$ (for some arbitrarily chosen $p \gg 0$), and we set $M = M(\mathcal{U}_p)$. The proofs do not involve the concept of regular economies.

First we show that if π is continuous at μ then all large finite economies near μ will be nearly perfectly competitive. For this purpose, it will be convenient to work with an explicit metric on M: we will use the Prohorov metric ρ (see Hildenbrand (1974), p. 49). Under this metric, given any n-person economy μ_n and any one-person perturbation $\Delta u \neq (u, u)$, $\rho(\mu_n, \mu_n + \Delta u) = 1/n$.[4]

DEFINITION: Given any price selection π and any sequence of finite economies $\mu_n \to \mu$, we will say that the economies μ_n are becoming *increasingly competitive* as the population increases if

$$d_\pi(\mu_n) \to 0 \text{ as } n \to \infty.$$

The continuum economy μ is a *perfectly competitive limiting economy* relative to π if, for every sequence $\mu_n \to \mu$, the economies μ_n are becoming increasingly competitive.

Proposition 1 *If π is continuous at μ then μ is a perfectly competitive limiting economy.*

[4]Recall if $\Delta u = (u, v)$ then $\mu_n + \Delta u = \mu_n - n^{-1}(\delta_u - \delta_v)$.

The Proposition implies that under the price selection π in Theorem 3, nearly perfectly competitive economies are dense in M. More formally, using this selection, for any $\epsilon > 0$ define $M^{\epsilon}_{\mathrm{PC}} = \cup_n \{\mu_n : d_\pi(\mu_n) < \epsilon\}$. Observe that if ϵ is arbitrarily small then any economy $\mu_n \in M^{\epsilon}_{\mathrm{PC}}$ is virtually a perfectly competitive economy, i.e., virtually a $\mu_n \in M^*_n(\pi)$. Proposition 1 and Theorem 3 together immediately imply that for *any* $\epsilon > 0$, $M^{\epsilon}_{\mathrm{PC}}$ is dense in M.

It only remains to show that, for any nearly perfectly competitive finite economy, the (continuous) Walrasian mechanism is nearly incentive compatible. For this purpose, it will be useful to choose a canonical function to represent ordinal preferences. Re-represent the preferences underlying u according to the *money-metric* utility function

$$u_p(x) = \inf\{py : u(y) \geq u(x)\},$$

where $p \in S$. (Note: The definition of the linear preferences $u^p(x) = px$ in Section 4 is not to be confused with $u_p(x)$.)

It is readily verified that for $x \gg 0$,

- $u_p(x) \leq px$
- $u_p(x) = px$ if and only if $\nabla u_p(x) = p$.

Hence the money metric utility representation implies that the set of utility-maximizing choices for any individual with characteristics (u, ω) facing prices p satisfies

$$\psi(p, u) = \{x : u_p(x) = px = p\omega = 1\}.$$

Proposition 2 *Let $d = d_\pi(\mu_n)$, let $p = \pi(\mu_n)$, and let $\alpha = \min\{p_1, \ldots, p_\ell\}$. Suppose $d < \alpha$. Then for any $\Delta u = (u, v) \in J(\mu_n)$ and any $x \in \psi(q, v)$, where $q = \pi(\mu_n + \Delta u)$:*

$$u_p(x) - u_p(\psi(p, u)) < \frac{d}{\alpha - d}.$$

The Proposition says that the gain from any misrepresentation by any individual in μ_n is bounded by an amount that goes to zero with $d_\pi(\mu_n)$.

To apply the Proposition, consider the price selection of Theorem 3 and any sequence $\mu_n \to \mu$. If π is continuous at μ (a generic property) then Proposition 1 implies the economies μ_n are becoming increasingly competitive, i.e., $d_\pi(\mu_n) \to 0$. Hence, Proposition 2 implies the gains from misrepresentation are going to zero as $n \to \infty$. In terms of Theorem 1, each individual's opportunity set $R_i(\mathrm{u}_{-i})$ under a continuous Walrasian mechanism is becoming increasingly linear; hence his incentive to misrepresent is going to zero.

REMARK (*The definitions of perfect competition and incentive compatibility in the continuum*): The Walrasian definition of perfectly competitive equilibrium focuses entirely on the coordination of demands and supplies in a single economy. By contrast, the definition of incentive compatibility involves restrictions with respect to nearby economies — resulting from misrepresentation. Similarly, perfect competition as we use it here involves restrictions related to nearby economies — reached by some individual experimenting by altering his demands or supplies. Thus, to establish our asymptotic results, we view a continuum economy μ as perfectly competitive *not* because individuals are of negligible size, but because for any $\mu_n \to \mu$, $d_\pi(\mu_n) \to 0$. Similarly, the Walrasian mechanism is incentive compatible at μ only if for all nearby finite economies μ_n, the gains from misrepresentation are small.[5]

The literature on envy-free allocations in the continuum represents an interesting way around the problem of explicitly dealing with perturbations by individuals. "Envy" is a virtual condition: would A rather have what he is given or what B gets. There is no need to ask whether it is possible to give A what B is getting without disturbing B or anyone else. Nevertheless, the issue of perturbations is treated implicitly. A *caveat* is made that the analysis should be undertaken using a continuous mechanism. See Champsaur and Laroque (1982). Our characterization of efficiency and incentive compatibility in the finite model along with this study into the asymptotic behavior of the Walrasian mechanism builds a bridge to the envy-free characterization of Walrasian equilibrium in the continuum, one that calls explicit attention to the importance of perfect competition.

7 Proof of Theorem 1 and Its Corollaries

As mentioned in Section 3, Theorem 1 will be proved in three steps.

STEP 1: In Figure 4, point x lies on the concave part of $R_i(\mathrm{u}_{-i})$, but $R_i(\mathrm{u}_{-i})$ also has a convex-shaped segment. Our first step (Lemmas 1–2) is to rule out the convex possibility.

[5] Incentive compatibility can be directly defined for continuum economies by considering "perturbations in the continuum," as in Makowski and Ostroy (1992) and Gretsky, Ostroy and Zame (1995).

To illustrate the idea, observe that in the figure the indifference curve of u through x is also

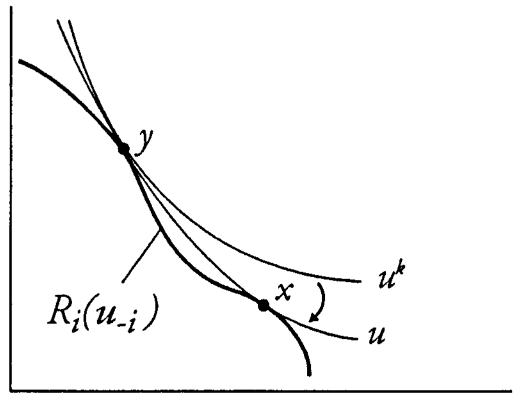

Figure 4: $R_i(\mathrm{u}_{-i})$ CANNOT HAVE A CONVEX-SHAPED SEGMENT AS ILLUSTRATED

tangent to $R_i(\mathrm{u}_{-i})$ at y. The mechanism must choose one of these two points. Suppose it chooses x. Let u^k be a sequence of preferences converging to u, as illustrated. Incentive compatibility requires the mechanism to choose y for all u^k, but it chooses x for u. This is compatible with our assumption that the mechanism is continuous in terms of utilities because, even though there is a discontinuous change in the allocation, the individual's utility varies continuously. However, the jump from y to x increases the individual's wealth, which must discontinuously change some other individuals' utilities. (See the proof of Lemma 2 for details.)

We first show that for individuals with preferences in $\mathcal{U}^*$ (strictly quasi-concave), our assumption that the mechanism f is continuous in utilities implies f also will be continuous in allocations.

Lemma 1 (continuity) *Suppose* $\mathrm{u}, \mathrm{u}^k \in \mathcal{D}$, *where* $\mathrm{u}^k = (\mathrm{u}, u^k, i)$. *Also assume* $u_i \in \mathcal{U}^*$ *and* $f_i(\mathrm{u}) \gg 0$. *Then* $u^k \to u_i$ *implies* $f_i(\mathrm{u}^k) \to f_i(\mathrm{u})$.

Proof: Let $x = f(\mathrm{u})$, $x^k = f(\mathrm{u}^k)$, $p = p(\mathrm{u})$, $p^k = p(\mathrm{u}^k)$. Suppose $x_i^k \not\to x_i$. Then there would be a convergent subsequence, say $s(k)$, on which $p^{s(k)} \to p^0$, $x^{s(k)} \to x^0$, and $x_i^0 \neq x_i$. Since each p^k supports x^k at u^k, p^0 supports x^0 at u, i.e.,

$$p^0 \omega \leq p^0 \sum_{j=1}^{n} A_j(x_j^0, u_j).$$

Hence, since $\sum_j x_j^0 = \omega$,

$$p^0 x_j^0 \leq p^0 A_j(x_j^0, u_j) \text{ for each individual } j.$$

Further, since continuity implies $u_j(x_j^k) \to u_j(x_j)$ for each j,

$$u_j(x_j^0) = u_j(x_j) \text{ for each } j.$$

Hence, $p^0 x_j^0 \leq p^0 x_j$ for each j. And since u_i is *strictly* quasi concave in the interior of Ω, where both x_i and x_i^0 reside, $p^0 x_i^0 < p^0 x_i$. Summing shows $p^0 \sum_j x_j^0 < p^0 \sum_j x_j$. But feasibility implies $\sum_j x_j^0 = \sum_j x_j$, a contradiction. Q.E.D.

We now show that if i may have any first-order similar preferences at $\mathbf{u}$ then the convex hull of $R_i(\mathbf{u}_{-i})$, denoted con $R_i(\mathbf{u}_{-i})$, is supported by $p(\mathbf{u})$ at $f_i(\mathbf{u})$. Thus $R_i(\mathbf{u}_{-i})$ cannot have a convex segment as illustrated in Figure 4.

Lemma 2 *If f is IC on the domain $\{\mathbf{u}, N_i(\mathbf{u}), i\}$ and $f_i(\mathbf{u}) \gg 0$ then*

$$p(\mathbf{u}) \cdot R_i(\mathbf{u}_{-i}) \leq p(\mathbf{u}) \cdot f_i(\mathbf{u}).$$

Proof: Let $x = f_i(\mathbf{u})$, $p = p(\mathbf{u})$. Let u be any utility function in $\mathcal{U}^*$ whose indifference curve through x is nested inside u_i's indifference curve through x; so, $u_i(y) \leq u_i(x)$ impliies $u(y) < u(x)$ unless $y = x$. Notice that incentive compatibility implies $f_i(\mathbf{u}_{-i}, u) = x$.

Suppose there were a $y \in R_i(\mathbf{u}_{-i})$ such that $py > px$. Then there would be a utility funtion $v \in \mathcal{U}$ such that $v(x) < v(y)$ for some $y \in R_i(\mathbf{u}_{-i})$, but the indifficerence curve of v through x coincides with the indifference curve of u through x in a ball around x with radius $\epsilon > 0$, i.e., in $B(x, \epsilon) := \{x' : \| \ x' - x \ \| < \epsilon\}$. Notice that incentive compatibility implies $f_i(\mathbf{u}_{-i}, v) \notin B(x, \epsilon)$.

Now consider the preferences $u^\alpha \equiv \alpha v + (1 - \alpha) u$, where $\alpha \in [0, 1]$. By construction u^α is strictly quasi-concave in $\mathbf{R}^\ell_{++}$. Further, for small values of α, x is strictly preferred by u^α over all other points in $R_i(\mathbf{u}_{-i})$. But at some ciritical value for α, say α^*, the individual is just indifferent between x and some other point(s) in $R_i(\mathbf{u}_{-i})$. And beyond this value, x is dominated by some other point $y \in R_i(\mathbf{u}_{-i})$, where $y \notin B(x, \epsilon)$. So, incentive compatibility implies $f(\mathbf{u}_{-i}, u^\alpha) = x$ for all α in $[0, \alpha*)$, while $f(\mathbf{u}_{-i}, u^\alpha) \notin B(x, \epsilon)$ for all $\alpha \in (\alpha*, 1]$. Thus, whether or not the mechanism assigns x to u^{α^*}, $f_i(\mathbf{u}_{-i}, u^\alpha)$ is not continuous at α^*, contradicting Lemma 1. Q.E.D.

Step 2: Our next step is to show that any incentive compatible mechanism must satisfy an equal-wealth condition (Lemma 3 below).

$\mathcal{L}(\mathbf{u})$ is a relatively small domain; so it is easy to find an IC mechanism on $\mathcal{L}(\mathbf{u})$. For example, let $\mathbf{x}$ by an equal-wealth Walrasian equilibrium allocation for $\mathbf{u}$. Define $f(\mathbf{u}') = \mathbf{x}$ for all $\mathbf{u}' \in \mathcal{L}(\mathbf{u})$. It is easy to check that this f is efficient and (trivially) IC on $\mathcal{L}(\mathbf{u})$. Nevertheless, $\mathcal{L}(\mathbf{u})$ is sufficiently rich to assure that the equal-wealth Walrasian allocations are the *only* incentive compatible allocations for $\mathbf{u}$.

Lemma 3 *f is IC on $\mathcal{L}(\mathbf{u})$ implies $(f(\mathbf{u}), p(\mathbf{u})) \in \mathrm{WE}(\mathbf{u})$.*

The basic idea behind this key lemma is simple. Technical complications arise only because, for the sake of generality, we permit f to assign "non-subsistance" boundary allocations to some individuals. To give the reader the main idea behind Lemma 3, here we present a proof of the lemma under the extra assumption that $f_i(\mathbf{u}') \gg 0$ for all individuals i and all profiles $\mathbf{u}'$ in $\mathcal{L}(\mathbf{u})$. The appendix contains a proof of Lemma 3 without this simplifying assumption.

Lemma 3′ *f is IC on $\mathcal{L}(\mathbf{u})$ implies*

$$(f(\mathbf{u}), p(\mathbf{u})) \in \mathrm{WE}(\mathbf{u})$$

assuming $f_i(\mathbf{u}') \gg 0$ for all individuals i and all profiles $\mathbf{u}'$ in $\mathcal{L}(\mathbf{u})$.

Proof: Let $p = p(\mathbf{u})$ and $u^* = u_{\mathbf{u}}$. Since p supports $f(\mathbf{u})$ at $\mathbf{u}$, we need only verify that $pf_i(\mathbf{u}) = p\bar{\omega}$ for all i. Assume the contrary. Then, since feasibility implies $\sum_{i=1}^{n} f_i(\mathbf{u}) = n\bar{\omega}$, there must be an individual, say $i = 1$, with $pf_1(\mathbf{u}) > p\bar{\omega}$. Let $\mathbf{u}^1 = (u^*, u_2, \ldots, u_n)$. Lemma 2 implies $pf_1(\mathbf{u}^1) = pf_1(\mathbf{u}) > p\bar{\omega}$ and $\nabla u^*(f_1(\mathbf{u}^1)) \propto p$; so $p = p(\mathbf{u}^1)$.

Since $f(\mathbf{u}^1)$ is feasible and $pf_1(\mathbf{u}^1) > p\bar{\omega}$, there must be some other individual, say $i = 2$, with $pf_2(\mathbf{u}^1) < p\bar{\omega}$. Let $\mathbf{u}^2 = (u^*, u^*, u_3, \ldots, u_n)$. As above, Lemma 2 implies $pf_2(\mathbf{u}^2) = pf_2(\mathbf{u}^1) < p\bar{\omega}$ and $\nabla u^*(f_2(\mathbf{u}^2)) \propto p$; so $p = p(\mathbf{u}^2)$. Further, since individuals 1 and 2 have the same preferences in $\mathbf{u}^2$, anonymity implies $pf_1(\mathbf{u}^2) = pf_2(\mathbf{u}^2) < p\bar{\omega}$.

Thus, since $f(\mathbf{u}^2)$ is feasible, there must be a third consumer, say $i = 3$, for whom $pf_3(\mathbf{u}^2) > p\bar{\omega}$. Let $\mathbf{u}^3 = (u^*, u^*, u^*, u_4, \ldots, u_n)$. As above, Lemma 2 implies $pf_3(\mathbf{u}^3) = pf_3(\mathbf{u}^2) > p\bar{\omega}$ and $\nabla u^*(f_3(\mathbf{u}^3)) \propto p$; so $p = p(\mathbf{u}^3)$. Further, since individuals 1 through 3 have the same preferences in $\mathbf{u}^3$, anonymity implies $pf_1(\mathbf{u}^3) = pf_2(\mathbf{u}^3) = pf_3(\mathbf{u}^3) > p\bar{\omega}$.

Repeat the above procedure $n - 3$ more times to form $\mathbf{u}^4, \ldots, \mathbf{u}^k, \ldots \mathbf{u}^n$, where $\mathbf{u}^k = (u^*, \ldots, u^*, u_{k+1}, \ldots, u_n)$. One thus finds, depending on whether n is odd or even, that $pf_i(\mathbf{u}^n) > p\bar{\omega}$ for all i or $pf_i(\mathbf{u}^n) < p\bar{\omega}$ for all i, contradicting the feasibility of $f(\mathbf{u}^n)$, where

$\mathbf{u}^n = (u^*, u^*, \ldots, u^*)$. Q.E.D.

STEP 3: The final step is to show that $R_i(\mathbf{u}_{-i})$ must be contained in a hyperplane through $\bar{\omega}$. For this, we will show that

1. $\bar{\omega} \in R_i(\mathbf{u}_{-i})$ (Lemma 4).
2. For all $x_i \in R_i(\mathbf{u}_{-i})$ there exists a supporting hyperplane H to $R_i(\mathbf{u}_{-i})$ at x_i such that $\bar{\omega} \in H$ (hence H supports $R_i(\mathbf{u}_{-i})$ at $\bar{\omega}$).
3. If H and H' support $R_i(\mathbf{u}_{-i})$ at $\bar{\omega}$, then, under a regularity assumption, $H = H'$ (Lemma 5).

Recall w^p denotes any function in $\mathcal{U}^*$ satisfying $\nabla w^p(\bar{\omega}) = p$.

Lemma 4 *Select any population* $\mathbf{u}$ *and individual* i. *Let* $p = p(\mathbf{u})$ *and* $\mathbf{v} = (\mathbf{u}, w^p, i)$. *Then* f *is IC on* $\mathcal{L}(\mathbf{u}) \cup \mathcal{L}(\mathbf{v})$ *implies*

$$\bar{\omega} \in R_i(\mathbf{u}_{-i}).$$

Indeed, $\bar{\omega} = f_i(\mathbf{v})$.

Proof: Let $q = p(\mathbf{v})$, $\mathbf{y} = f(\mathbf{v})$, and $v = w^p$. Lemma 3 implies $(\mathbf{y}, q) \in \mathrm{WE}(\mathbf{v})$. Hence, since the indifference curve of v through $\bar{\omega}$ is strictly convex, $v(y_i) > v(\bar{\omega})$ unless $y_i = \bar{\omega}$. But since $p = \nabla v(\bar{\omega})$ by construction, $v(y_i) > v(\bar{\omega}) \Rightarrow py_i > p\bar{\omega} \Rightarrow y_i \notin R_i(\mathbf{u}_{-i})$ since $pR_i(\mathbf{u}_{-i}) \leq pf_i(\mathbf{u}) = p\bar{\omega}$ (using Lemmas 2–3), a contradiction. So, $y_i = \bar{\omega}$. Q.E.D.

Now consider any $x_i = f_i(\mathbf{u})$, and let $p = p(\mathbf{u})$. Lemma 2 implies $H_p + \{\bar{\omega}\}$ supports $R_i(\mathbf{u}_{-i})$ at x_i, and Lemma 3 implies $\bar{\omega} \in H_p + \{\bar{\omega}\}$. Hence, Lemma 4 implies $H_p + \{\bar{\omega}\}$ supports $R_i(\mathbf{u}_{-i})$ at $\bar{\omega}$. But we have not yet excluded the possibility of $R_i(\mathbf{u}_{-i})$ having a kink at $\bar{\omega}$. We now will show that if $\mathcal{D}$ is sufficiently rich then such a kink may be ruled out, at least for most economies.

Figure 5 illustrates the consequences of a kink at $\bar{\omega}$: the individual would receive the same allocation, $\bar{\omega}$, whether he announces u, v, or any preferences "between" them. Let $u^\alpha = \alpha u + (1-\alpha)v$, where $\alpha \in [0,1]$; and let $p(\alpha) = \alpha p + (1-\alpha)q$, so $p(\alpha) \propto \nabla u^\alpha(\bar{\omega})$. Let $(x_1(\alpha), \ldots, x_{i-1}(\alpha), x_{i+1}(\alpha), \ldots, x_n(\alpha))$ be the allocation to all other individuals as i varies her preferences while others' remain fixed. Feasibility implies

(a) $\sum_{j \neq i} x_j(\alpha) = (n-1)\bar{\omega}$;

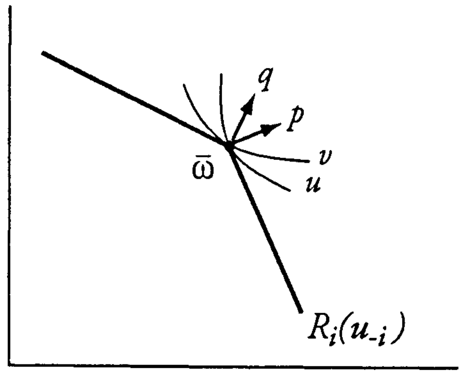

Figure 5: IF $R_i(\mathrm{u}_{-i})$ HAS A KINK THEN THE ECONOMY WITHOUT i WOULD HAVE A CONTINUUM OF WALRASIAN EQUILIBRIA

while Step 2 implies

(b) $p(\alpha) \cdot x_j(\alpha) = p(\alpha) \cdot \bar{\omega}$ for all $j \neq i$.

Conditions (a) and (b) say that the economy consisting of all individuals except i has a continuum of equal-wealth Walrasian equilibria, one for each price vector $p(\alpha)$. So, for regular economies, $R_i(\mathrm{u}_{-i})$ will not have a kink at $\bar{\omega}$.

Lemma 5 *Fix* u *and* i. *Suppose* u_{-i} *is regular. Let* $v \in \mathcal{U}$, $\mathrm{v} = (\mathrm{u}, v, i)$, $p = p(\mathrm{u})$, *and* $q = p(\mathrm{v})$. *Then* f *is IC on* $\mathcal{L}(\mathrm{u}) \cup \mathcal{L}(\mathrm{v}) \cup_{r \in [p,q]} \mathcal{L}((\mathrm{v}, w^r, i))$ *implies* $p = q$.

Proof: Let $R = R_i(\mathrm{u}_{-i})$. By Lemma 3, $pf_i(\mathrm{u}) = p\bar{\omega}$ and $qf_i(\mathrm{v}) = q\bar{\omega}$; and by Lemma 4, $\bar{\omega} \in R$. Hence, Lemma 2 implies $p \cdot \mathrm{con}\, R < p\bar{\omega}$ and $q \cdot \mathrm{con}\, R < q\bar{\omega}$, i.e., both p and q support the convex set $\mathrm{con}\, R$ at $\bar{\omega}$. But the set of supports is convex; so any $r \in [p,q]$ also supports $\mathrm{con}\, R$ at $\bar{\omega}$. Suppose $p \neq q$. Consider the populations $\mathbf{w}^r = (\mathbf{u}, w^r, i)$, for $r \in [p,q]$. Clearly, for any such population, $f_i(\mathrm{w}^r) = \bar{\omega}$, hence $p(\mathrm{w}^r) = r$. Now Lemma 3 implies $(f(\mathrm{w}^r), r) \in \mathrm{WE}(\mathrm{w}^r)$ with $f_i(\mathrm{w}^r) = \bar{\omega}$. Hence, for any $r \in [p,q]$, the pair $((f_1(\mathrm{w}^r), \ldots, f_{i-1}(\mathrm{w}^r), f_{i+1}(\mathrm{w}^r), \ldots, f_n(\mathrm{w}^r)), r)$ is an equal-wealth Walrasian equilibrium for the economy $\mathbf{u}_{-i}$ when its aggregate endowment equals $\omega - \bar{\omega}$, contradicting the regularity of $\mathbf{u}_{-i}$. Q.E.D.

Proof of Theorem 1 Lemma 3 implies that for any i and $\mathbf{v} = (\mathbf{u}, v, i)$,

$$(f(\mathrm{v}), p(\mathrm{v})) \in \mathrm{WE}(\mathrm{v}).$$

Further, Lemma 5 implies that $p(\mathrm{v}) = p$. Hence $f_i(\mathrm{v}) \in H_p + \{\bar{\omega}\}$. Q.E.D.

Proof of Corollary 1 Lemma 3 and Theorem 1 imply that for any i and $\mathbf{v} = (\mathbf{u}, v, i)$,

$$(f(\mathbf{v}), p) \in \mathrm{WE}(\mathbf{v}),$$

where $p = p(\mathbf{u})$. Since p remains constant when i switches to v, for each $j \neq i$

$$u_j(f_j(\mathbf{v})) = u_j(f_j(\mathbf{u})).$$

Hence, efficiency implies

$$v(f_i(\mathbf{v})) \geq v(x)$$

for all non-negative x in $\{\omega\} - A_{-i}(\mathbf{u})$. Q.E.D.

Proof of Corollary 2 Follows immediately from Lemma 5. Q.E.D.

Proof of Corollary 3 In view of Lemmas 3 and 5, it will suffice to show that $(f(\mathbf{v}), p) \notin \mathrm{WE}(\mathbf{v})$. Since $\nabla u_i(f_i(\mathbf{u})) \neq \nabla v(f_i(\mathbf{u})$, $f_i(\mathbf{u})$ is not i's Walrasian demand when he is of type v and faces prices p. But the demand of each individual $j \neq i$ is unique at prices p; so markets will not clear at p when i is of type v. Q.E.D.

8 Remarks on the Indispensability of Perfect Competition

In Section 1 we pointed to three branches of the mechanism design literature that are linked to our results on the indispensability of perfect competition — the impossibility theorems for finite ordinal models, the Groves/full appropriation characterization of weakly efficient dominant strategy mechanisms for finite quasi-linear models, and the envy-free characterization of Walrasian equilibria for continuum models. In this Section, we compare our results to two contributions containing apparently different conclusions.

One source of the difference between our results and those below can be explained by a difference in approach. Whereas most of the mechanism design literature is naturally enough concerned with the mechanism itself, we are interested in the kind of economic environment that would support a particular kind of mechanism. For us, the question is not whether a certain kind of mechanism exists, but why certain kinds of environments do or do not admit certain kinds of mechanisms. Given (i) the evident conclusion that a perfectly competitive environment yields a mechanism that is both efficient and incentive compatible and (ii) the demonstrations that in finite economies mechanisms with both properties typically do not exist, we set out to demonstrate the equivalence between perfectly competitive mechanisms and ones satisfying both desiderata.

Hurwicz and Walker (1990) establish generic impossibility theorems for efficient incentive compatible mechanisms in finite quasi-linear models having public or private goods. In the course of their demonstrations, they make the following observation. The failure of incentive compatible mechanisms arises from the existence of conflicts among consumers. Indeed, once these conflicts disappear, such mechanisms are possible. They prove this intuition in the following way. The set of commodities $\{1, \ldots, \ell\}$ is partitioned into non-empty subsets $I_1, \ldots, I_n$, and each consumer i's utility function is assumed to be sensitive to changes only in the quantities of commodities in I_i. So the set $\mathcal{U}_i$ of utility functions available to consumer i is $\{u : \mathbf{R}^\ell \to \mathbf{R} : c \notin I_i \Rightarrow \partial u/\partial x_c \equiv 0$, and u is not constant$\}$. For a vector of utility functions $(u_1, \ldots, u_n) \in \prod_{i=1}^n \mathcal{U}_i$, the mechanism assigns each consumer all the available endowments of all the goods to which his utility may be sensitive (and throws away all goods no one wants). This is obviously an efficient incentive compatible mechanism, but the authors also prove that this is the only case where such a mechanism exists.

Despite the disagreement between this claim and our demonstration in Section 4 that economies admitting efficiency and incentive compatibility exist without the restriction to non-overlapping preferences among commodities, there is no real contradiction here. H & W require that a mechanism must be efficient and incentive compatible on a Cartesian product $\prod_{i=1}^n \mathcal{U}_i$ of sets of utility functions, whereas we only require that it achieve these properties on a restricted neighborhood $\mathcal{D}$ (which includes the possibility that each individual i can announce any element of $\mathcal{U}_i$). The H & W characterization shows that eliminating conflict is one way to avoid its possible inefficiency consequences, but full appropriation and perfect competition is another much less restrictive condition in which conflict can be efficiently resolved.

H & W allude to a second case under which efficient incentive compatible mechanisms exist, namely, when the economy has a continuum of agents (1990, Section 8). The authors suggest that there is no conflict of interest among consumers in the continuum, as no one can affect another's well-being. This case, however, supports our approach. In continuum economies, non-overlapping preferences among commodities is not necessary for the generic coexistence of efficiency and incentive compatibility. In contrast to our finite characterization, the finite economies offered by H & W — with non-overlapping preferences — do not converge to typical continuum economies.

Another characterization of incentive compatible mechanisms for finite ordinal models is offered by Barberà and Jackson (1995). B & J mechanisms are also defined on a Cartesian product, but their mechanisms need not be efficient. They characterize all incentive compatible mechanisms as fixed-proportion trading schemes. On its face, their mechanisms appear to mimic trade according to prices and therefore to exhibit an underlying similarity to the results of this paper, but the resemblence is only superficial. Fixed-proportions schemes permit consumers to move only in a finite number of directions rather than the infinite directions possible with hyperplanes, as is the case when they are constrained by prices. A second difference is that these proportions are determined independently of individuals' declared utility functions, in contrast to perfect competition where prices depend on consumers' utilities. These two factors imply that B & J mechanisms do not converge to the general incentive compatible mechanism for continuum economies; similarly, the lack of efficiency is not reduced when the number of individuals increases.

The disparity in asymptotic conclusions follows from their requirement that the incentive compatibility condition apply exactly throughout the entire domain. A comparison of the conclusions of B & J with the conclusions in this paper and elsewhere illustrates that the consequences of imposing exact incentive compatibility everywhere can be quite different from the standard adopted here of asymptotically achieving the condition generically.

A Proof of Lemma 3

Let $p = p(\mathbf{u})$ and $u^* = u(\mathbf{u})$. Since p supports $f(\mathbf{u})$ at $\mathbf{u}$, we need only verify that $pf_i(\mathbf{u}) = p\bar{\omega}$ for all i.

Assume the contrary. We will construct a sequence of populations in $\mathcal{L}^0(\mathbf{u})$, $\{\mathbf{u}^k\}_{k=1,2,\ldots,n}$, satisfying for each k:

- $p = p(\mathbf{u}^k)$
- $u_i^k = u^*$ for $i = 1, 2, \ldots, k$
- $0 < pf_1(\mathbf{u}^k) = pf_2(\mathbf{u}^k) \cdots = pf_k(\mathbf{u}^k) \neq p\bar{\omega}$,

where $\mathbf{u}^k = (u_1^k, u_2^k, \ldots, u_n^k)$. Notice that by construction $p\sum_{i=1}^n f_i(\mathbf{u}^n) \neq np\bar{\omega}$, which contradicts the feasibility of $f(\mathbf{u}^n)$, i.e., the fact that $\sum_{i=1}^n f_i(\mathbf{u}^n) = \omega = n\bar{\omega}$. Thus the construction suffices to prove the lemma.

Preliminary to the construction, observe that since each $\mathbf{u}^k$ is in $\mathcal{L}^0(\mathbf{u})$, for each population $\mathbf{u}^k$ and individual i

$$\{x : \nabla u_i^k(x) \propto p\} \subset C(\mathbf{u}).$$

So, for each k and each i such that $f_i(\mathbf{u}^k) \gg 0$,

$$\nabla u^*(f_i(\mathbf{u}^k)) \propto p.$$

Since feasibility implies $\sum_{i=1}^n f_i(\mathbf{u}) = n\bar{\omega}$, there must be an individual, say $i = 1$, with $pf_1(\mathbf{u}) > p\bar{\omega}$. Let $\mathbf{u}^1 = (u^*, u_2, \ldots, u_n)$. Lemma 2 implies $pf_1(\mathbf{u}^1) = pf_1(\mathbf{u}) > p\bar{\omega}$ and $\nabla u^*(f_1(\mathbf{u}^1)) \propto p$; so $p = p(\mathbf{u}^1)$.

We proceed inductively. Suppose we have constructed $\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^k$. We show how to construct $\mathbf{u}^{k+1}$. Since $\sum_{i=1}^n f_i(\mathbf{u}^k) = n\bar{\omega}$ (by feasibility) and $p\sum_{i=1}^k f_i(\mathbf{u}^k) \neq kp\bar{\omega}$ (by construction), there must exist some individual $i > k$ such that $pf_i(\mathbf{u}^k) \neq p\bar{\omega}$. There are three possible cases:

(a) $0 < pf_i(\mathbf{u}^k) \neq p\bar{\omega}$ for some $i > k$

(b) $pf_i(\mathbf{u}^k) = p\bar{\omega}$ for some $i > k$, $pf_i(\mathbf{u}^k) = 0$ for some $i > k$, and $pf_i(\mathbf{u}^k) \in \{0, p\bar{\omega}\}$ for all $i > k$

(c) $pf_i(\mathbf{u}^k) = 0$ for all $i > k$.

If case (a) applies, say for individual $i = k+1$, let $\mathbf{u}^{k+1} = (u^*, \ldots, u^*, u_{k+2}^k, \ldots, u_n^k)$. Lemma 2 implies $pf_{k+1}(\mathbf{u}^{k+1}) = pf_{k+1}(\mathbf{u}^k) \neq p\bar{\omega}$ and $\nabla u^*(f_{k+1}(\mathbf{u}^{k+1})) \propto p$; so $p = p(\mathbf{u}^{k+1})$. Further, since individuals 1 through $k+1$ have the same preferences in $\mathbf{u}^{k+1}$, anonymity implies $0 < pf_1(\mathbf{u}^{k+1}) = \cdots = pf_{k+1}(\mathbf{u}^{k+1}) \neq p\bar{\omega}$.

Suppose instead that case (b) applies. In particular, say that for individual n, $pf_n(\mathbf{u}^k) = p\bar{\omega}$. Let $v^\alpha = (1-\alpha)u_n + \alpha u^*$ and let $\mathbf{v}^\alpha = (\mathbf{u}^k, v^\alpha, n)$, where $\alpha \in [0,1]$. Observe that Lemma 2 implies p supports $f(\mathbf{v}^\alpha)$ for any $\alpha \in [0,1]$. Further, by anonymity,

$$pf_1(\mathbf{v}^1) = \cdots = pf_k(\mathbf{v}^1) = pf_n(\mathbf{v}^1) = p\bar{\omega}.$$

Hence, either

- there is an $i > k$, say $i = k+1$, for whom $pf_{k+1}(\mathbf{v}^1) > p\bar{\omega}$, or
- $pf_i(\mathbf{v}^1) = p\bar{\omega}$ for all individuals i.

In the first event, let $\mathbf{u}^{k+1} = (u^*, \ldots, u^*, u^k_{k+2}, \ldots, u^k_{n-1}, v^1)$. Lemma 2 implies $pf_{k+1}(\mathbf{u}^{k+1}) = pf_{k+1}(\mathbf{v}^1) > p\bar{\omega}$ and $\nabla u^*(f_{k+1}(\mathbf{u}^{k+1})) \propto p$; so $p = p(\mathbf{u}^{k+1})$. Further, since individuals 1 through $k+1$ have the same preferences in $\mathbf{u}^{k+1}$, anonymity implies $pf_1(\mathbf{u}^{k+1}) = \cdots = pf_{k+1}(\mathbf{u}^{k+1}) > p\bar{\omega}$.

In the second event, the continuity of the mechanism implies that for some $\beta \in [0,1]$ there is an $i > k$, say $i = k+1$, such that $0 < pf_{k+1}(\mathbf{v}^\beta) < p\bar{\omega}$. In this event, let $\mathbf{u}^{k+1} = (u^*, \ldots, u^*, u^k_{k+2}, \ldots, u^k_{n-1}, v^\beta)$. As above, Lemma 2 implies $pf_{k+1}(\mathbf{u}^{k+1}) = pf_{k+1}(\mathbf{v}^\beta)$ and $\nabla u^*(f_{k+1}(\mathbf{u}^{k+1})) \propto p$; so $p = p(\mathbf{u}^{k+1})$. Further, since individuals 1 through $k+1$ have the same preferences in $\mathbf{u}^{k+1}$, anonymity implies $0 < pf_1(\mathbf{u}^{k+1}) = \cdots = pf_{k+1}(\mathbf{u}^{k+1}) < p\bar{\omega}$.

Finally, suppose case (c) applies. Consider again the populations $\mathbf{v}^\alpha = (\mathbf{u}^k, v^\alpha, n)$. We first show that for any $\epsilon > 0$ there is an $\alpha \in [0,1]$ such that

$$0 < \| \sum_{i=k+1}^{n} f_i(\mathbf{v}^\alpha) \| < \epsilon.$$

By assumption, $u_i(f_i(\mathbf{v}^0)) = 0$ for all $i > k$. But $u_i(f_i(\mathbf{v}^1)) \gg 0$ for some $i > k$ (otherwise, annonymity implies $f_i(\mathbf{v}^1) = 0$ for all i, contradicting the efficiency of $f(\mathbf{v}^1)$). Let $\alpha^* = \max \alpha$ s.t. $u_i(f_i(\mathbf{u}^\alpha) = 0$ for all $i > k$. The continuity of the mechanism implies there is a sequence $\alpha^t \to \alpha^*$ such that $u_i^t \equiv u_i(f_i(\mathbf{v}^{\alpha^t})) \to 0$ for all $i > k$ but, for each t, $u_i^t > 0$ for some $i > k$. Let

$$\begin{aligned} f^t &\equiv f(\mathbf{v}^{\alpha^t}) \to f^* \text{ and} \\ \frac{p^t}{\| p^t \|} &\equiv \frac{p(\mathbf{v}^{\alpha^t})}{\| p(\mathbf{v}^{\alpha^t}) \|} \to p^* \end{aligned}$$

along a subsequence. Since $u_i^t \to 0$ for each $i > k$, $p^* {f_i}^* = 0$ for each $i > k$. Further, strict monotonicity implies $p^* \gg 0$. Hence ${f_i}^* = 0$ for all $i > k$. So for some α sufficiently close to α^* along this subsequence, the desired inequality will be satisfied.

Now fix ϵ sufficiently small so that $B(\omega, \epsilon) \subset C(\mathbf{u})$ and $\bar{\omega} \notin B(0, \epsilon)$; and let $\beta \in [0,1]$ be sufficiently close to α^* so that the inequality of above is satisfied. Let $\mathbf{v} = \mathbf{v}^\beta$. We show that $p(\mathbf{v}) = p$. Assume the contrary. Then $f_1(\mathbf{v}) \notin C(\mathbf{u})$. Hence, since all individuals $i \leq k$ are identical with strictly quasi-concave preferences outside this cone, $\sum_{i=1}^{k} f_i(\mathbf{v}) = ky$ for some $y \in \Omega$. Feasibility implies $ky + \sum_{i=k+1}^{n} f_i(\mathbf{v}) = \omega$. Hence, $\| \sum_{i=k+1}^{n} f_i(\mathbf{v}) \| = \| \omega - ky \|$. By construction, the LHS is less than ϵ. But since $ky \notin C(\mathbf{u})$, $\| \omega - ky \| > \epsilon$, a contradiction.

We conclude that there is an individual $i > k$, say $i = k+1$, for whom $0 < pf_{k+1}(\mathbf{v}) < p\bar{\omega}$. Let $\mathbf{u}^{k+1} = (u^*, \ldots, u^*, u^k_{k+1}, \ldots, u^k_{n-1}, v^\beta)$. Since $p(\mathbf{v}) = p$, Lemma 2 implies $pf_{k+1}(\mathbf{u}^{k+1}) =$

$pf_{k+1}(\mathbf{v})$ and $\nabla u^*(f_{k+1}(\mathbf{u}^{k+1})) \propto p$; so $p = p(\mathbf{u}^{k+1})$. Further, since individuals 1 through $k+1$ have the same preferences in $\mathbf{u}^{k+1}$, anonymity implies $0 < pf_1(\mathbf{u}^{k+1}) = \cdots = pf_{k+1}(\mathbf{u}^{k+1}) < p\bar{\omega}$. Q.E.D.

B Proof of Theorem 2

B.1 Background

As noted in Section 4, for quasi-linear economies we can build on the Groves characterization. Thus we know from the outset that any IC mechanism must involve full appropriation. We wish to show that if the mechanism is budget balancing, it must also be perfectly competitive. Since we want our characterization to be non-empty, it will have to apply on a restricted domain. Fortunately, the Groves characterization also applies on a restricted—in particular, any convex—domain, as shown by Holmström (1979). The following summarizes the facts we shall use from Holmström, translated into set theoretic terms (see Section 4.1).

FACT 2: *f is IC on $\{\mathbf{u}, [u_i, u], i\}$ implies for all $v \in [u_i, u]$:*

(F.1) $\sum_{j \neq i} u_j(f_j(\mathbf{u}, v, i)) = \sum_{j \neq i} u_j(f_j(\mathbf{u}))$

(F.2) $f_i(\mathbf{u}, v, i) \in \arg\max_{x \in \Omega}\ v(x)$ s.t. $x \in \{\omega\} - A_{-i}(\mathbf{u})$.

This fact will allow us to verify the conclusions of Lemmas 2 and 3 (i.e., Steps 1 and 2 for ordinal economies) without assuming the continuity of f or the differentiability of preferences. Hence the dispensability of these assumptions in the TU setting.

B.2 The proof

Analogous to the proof of Theorem 1 for ordinal economies, the proof of Theorem 2 involves three steps. We first give the TU analogs for Lemmas 2 and 3 (Lemmas 6 and 7 below).

Lemma 6 *f is IC on $\{\mathbf{u}, [u_i, u^p], i\}$, where $p \in \mathcal{P}(\mathbf{u})$, implies*

$$pR_i(\mathbf{u}_{-i}) \leq pf_i(\mathbf{u}).$$

Proof: Let $x = f(\mathbf{u})$ and let $B = \{\omega\} - \sum_{j \neq i} A(x_j, u_j)$. By definition,

$$p\omega \leq p \sum_j A(x_j, u_j).$$

Hence $pB \leq pA(x_i, u_i)$ and, in particular, $pB \leq px_i$. Thus, (F.2) implies $pf_i(\mathbf{u}, u^p, i) = px_i$. So if $py > px_i$ for some $y \in R_i(\mathbf{u}_{-i})$, then $pf_i(\mathbf{u}, u^p, i) > px_i$, a contradiction. Q.E.D.

We now display the analog of Lemma 3. The proof is similar to that of Lemma 3′, but without the differentiability or interiority assumptions.

Lemma 7 *f is IC on $\mathcal{L}(\mathbf{u}, p)$, where $p \in \mathcal{P}(\mathbf{u})$, implies*

$$(f(\mathbf{u}), p) \in \mathrm{WE}(\mathbf{u}).$$

Proof: Let $x = f(\mathbf{u})$ and let $B = \{\omega\} - \sum_{j \neq i} A(x_j, u_j)$. Since p supports x at $\mathbf{u}$, $px_i \leq pA(x_i, u_i)$ for all i. Hence, we need only verify that $px_i = p\bar{\omega}$ for all i.

Assume the contrary. Then there is an individual, say $i = 1$, with $px_1 > p\bar{\omega}$. Let $\mathbf{u}^1 = (u^p, u_2, \ldots, u_n)$. Lemma 6 implies $pf_1(\mathbf{u}^1) = pf_1(\mathbf{u}) > p\bar{\omega}$.. Further, p supports $x^1 \equiv f(\mathbf{u}^1)$ at $\mathbf{u}^1$. To verify observe that $u^p(x_1^1) = u^p(x_1) = px_1$; and, by (F.1), $\sum_{j \neq i} A(x_j^1, u_j) = \sum_{j \neq i} A(x_j, u_j)$. Thus, since $\sum_j x_j^1 = \omega$, $p\omega \leq p \sum_j A(x_j, u_j)$ implies

$$p \sum_{j \neq i} x_j^1 \leq p \sum_{j \neq i} A(x_j^1, u_j).$$

And trivially,

$$px_1^1 \leq pA(x_1^1, u^p).$$

Summing shows

$$p \sum_j x_j^1 = p\omega \leq p \sum_j A(x_j^1, u_j^1).$$

Hence, there must be an individual, say $i = 2$, with $px_2^1 < p\bar{\omega}$. Let $\mathbf{u}^2 = (u^p, u^p, u_3, \ldots, u_n)$. Lemma 6 implies $pf_2(\mathbf{u}^2) = pf_2(\mathbf{u}^1) < p\bar{\omega}$.. Further, p supports $x^2 \equiv f(\mathbf{u}^2)$ at $\mathbf{u}^2$ (verified as above).

Since individuals 1 and 2 have the same preferences in $\mathbf{u}^2$, anonymity implies $px_1^2 = px_2^2 < p\bar{\omega}$. Thus there must be a third consumer, say $i = 3$, for whom $px_3^2 > p\bar{\omega}$. Repeat the above procedure $n - 2$ more times, to form $\mathbf{u}^3, \ldots, \mathbf{u}^n$. One thus finds, depending on whether n is odd or even, that $px_i^n > p\bar{\omega}$ for all i or $px_i^n < p\bar{\omega}$ for all i, contradicting the feasibility of x^n, where $x^n = f(\mathbf{u}^n) = f(u^p, \ldots, u^p)$. Q.E.D.

Let $v^p(x) \equiv \tilde{v}^p(x_1, \ldots, x_{\ell-1}) + x_\ell$ be an continuously differentiable function in $\mathcal{U}$ satisfying $\tilde{v}^p$ is strictly quasi-concase and $\nabla v^p(\bar{\omega}) = p$.

Lemma 8 *Select any population* $\mathbf{u}$ *and individual* i. *Let* $p \in \mathcal{P}(\mathbf{u})$ *and* $\mathbf{v} = (\mathbf{u}, v^p, i)$. *Then* f *is IC on* $\mathcal{L}(\mathbf{u}, p) \cup \mathcal{L}(\mathbf{v}, p)$ *implies*

$$\bar{\omega} \in R_i(\mathbf{u}_{-i}).$$

Indeed, $\bar{\omega} = f_i(\mathbf{v})$.

Proof: See the proof of Lemma 4 for ordinal economies. Q.E.D.

Lemma 9 *Fix* $\mathbf{u}$ *and* i. *Suppose* $\mathbf{u}_{-i}$ *is regular. Let* $v \in \mathcal{U}$, $\mathbf{v} = (\mathbf{u}, v, i)$, $p \in \mathcal{P}(\mathbf{u})$, *and* $q \in \mathcal{P}(\mathbf{v})$. *Then* f *is IC on* $\mathcal{L}(\mathbf{u}, p) \cup \mathcal{L}(\mathbf{v}, q) \cup_{r \in [p,q]} \mathcal{L}((\mathbf{u}, w^r, i), r)$ *implies* $p = q$.

Proof: See the proof of Lemma 5 for ordinal economies. Q.E.D.

As noted in Section 4, while differentiability plays no explicit role in the proof of Lemma 9, it is sufficient (although not necessary) for the regularity of $\mathbf{u}$.

FACT 3 *If* u_i *is differentiable for each* i, *then* $\mathbf{u}$ *will be regular.*

Proof: Suppose both $(\mathbf{x}, p)$ and $(\mathbf{y}, q)$ are equal-wealth Walrasian equilibria for $\mathbf{u}_{-i}$ when its aggregate endowment equals $\omega - \bar{\omega}$. Let $\omega^* = \omega - \bar{\omega}$. Then by assumption

$$\begin{aligned} p\omega^* &\leq p \sum_{j \neq i} A(x_j, u_j) \\ q\omega^* &\leq q \sum_{j \neq i} A(y_j, u_j) = q \sum_{j \neq i} A(x_j, u_j), \end{aligned}$$

where the last equality follows for the quasi-linearity of preferences (i.e., the vertical parallelism of indifference curves). Thus, since $\sum_{j \neq i} x_j = \omega^*$, by a familiar argument:

$$px_j \leq pA(x_j, u_j) \text{ for each } j \neq i \tag{1}$$

$$qx_j \leq qA(x_j, u_j) \text{ for each } j \neq i. \tag{2}$$

Suppose $p \neq q$. Hence for some commodity h ($h \neq \ell$), $p_h \neq q_h$. But since $\omega^*_h > 0$, there must be an individual j such that $x_{jh} > 0$. Eq. (1) implies that for this individual $\partial u_j(x_j)/\partial x_{jh} = p_h$, while Eq. (2) implies $\partial u_j(x_j)/\partial x_{jh} = q_h$, a contradiction. Q.E.D.

Proof of Theorem 2: Let p be as described in (R.2*). Now proceed as in the proof of Theorem 1 for ordinal economies. Q.E.D.

C Proof of Theorem 3 and Propositions 1 and 2

Proof of Theorem 3: First, we show that there exists a selection π that is continuous on a maximal G_δ set. Let π_α be a selection and M_α the set on which it is continuous. It is readily seen that M_α is G_δ. (Let $\mathcal{O}_n$ be the union of all open sets $\mathcal{O}$ such that $d_{\pi_\alpha}(\mathcal{O}) < n^{-1}$. Then $\mathcal{O}_n$ is open and $M_\alpha = \cap_n \mathcal{O}_n$.)

Define an ordering on selections; denote $\pi_{\alpha'} \succeq \pi_\alpha$ if $M_\alpha \subset M_{\alpha'}$ and $\pi_{\alpha'}$ agrees with π_α on M_α. By Zorn's Lemma there exists a π continuous on M^* that is maximal in the ordering. Because M^* is the set of continuity points of π, it too is a G_δ set. (Appeal to Zorn's Lemma is used by Mas-Colell (1985, 5.8.18, p. 234) in establishing the existence of a continuous selection on the space of regular economies.)

It remains to show that M^* is dense. The following results will establish this claim.

(1) $\mathcal{U}_p$ is a complete metric space. To prove this, it suffices to show that $\mathcal{U}_p$ is a G_δ set in the space of C^1 functions on $\mathbb{R}^\ell_{++}$ since a G_δ set in a complete metric space is complete (Mas-Colell, A.3.4, p.10)). Proposition 2.4.5 in Mas-Colell (1985, p. 72) demonstrates that the set $\mathcal{U}_d$ of C^1 strictly monotone utility functions on $\mathbb{R}^\ell_{++}$ satisfying the interiority condition and the normalization in which $u(x) = \lambda$ iff $u(x) = u(\lambda e)$, where $e = (1, 1, \ldots, 1)$, is G_δ. Further, the subset of quasi-concave functions of $\mathcal{U}_d$, call it $\mathcal{U}_{dc}$, is readily seen to be complete. Also, the map $\phi : \mathcal{U}_{dc} \to \mathcal{U}_p$, where $\phi(u_d) = u_p$ means $u_{dc}(x) = u_{dc}(y)$ iff $u_p(x) = u_p(y)$, is homeomorphic. Hence, $\mathcal{U}_p$ is G_δ.

(2) $M = M(\mathcal{U}_p)$ is a separable and complete metric space because M is defined on a separable complete metric space $\mathcal{U}_p$ (see Mas-Colell (1985, E.3.1, p. 24)).

(3) By a straightforward extension of a Theorem of Fort (see Hildenbrand (1974, p. 31)), since $\Pi : M \to 2^S$ is an upper hemi-continuous correspondence from a separable complete metric space into a totally bounded set, Π is continuous on a dense G_δ set.

Now, suppose the contrary that M^* is not dense and therefore does not intersect an open set $\mathcal{O}$ in M. By (3), there a $\bar{\mu} \in \mathcal{O}$ that is a continuity point of Π. Let $\pi(\bar{\mu}) \in \Pi(\bar{\mu})$. For each other $\mu \in \mathcal{O}$, select $\pi(\mu) \in \Pi(\mu)$ such that $\|\pi(\mu) - \pi(\bar{\mu})\| = \min \|\Pi(\mu) - \pi(\bar{\mu})\|$. In case there is more than one such p, choose the one with the smallest value of p_1, and then if necessary the smallest value of p_2, etc., to select a unique element from $\Pi(\mu)$. Since Π is continuous at $\bar{\mu}$ and $\pi(\mu)$ is minimum distance from $\pi(\bar{\mu})$, π is evidently continuous at $\bar{\mu}$. But this contradicts the hypothesis that the set of continuity points M^* is maximal. Q.E.D.

Proof of Proposition 1: Let $\mu_n \to \mu$. We need to show that for any $\epsilon > 0$ there is an N such that $n > N$ implies $d_\pi(\mu_n) < \epsilon$.

The continuity of π implies there is a $\delta > 0$ such that $\mu' \in B_\delta(\mu)$ implies $\| \pi(\mu') - \pi(\mu) \| < \epsilon$.

Let N be sufficiently large that $N > 2/\delta$ and $\rho(\mu_n, \mu) < \delta/2$ for all $n > N$. Then for any $n > N$ and any perturbation $\Delta u \in H(\mu_n)$, $\rho(\mu_n, \mu_n + \Delta u) = 1/n < \delta/2$. Hence, for any $n > N$, $\{\mu_n + \Delta u : \Delta u \in H(\mu_n)\} \subset B_{\delta/2}(\mu_n) \subset B_\delta(\mu)$. Q.E.D.

Proof of Proposition 2: An element $x \in \psi(q, v)$ is chosen from the set

$$B_q = \{x' \in \mathrm{R}^\ell_+ : qx' = 1\} = \text{convex hull } \{q_1^{-1}, q_2^{-1}, \ldots, q_\ell^{-1}\}.$$

Therefore,

$$\begin{aligned} u_p(x) := \inf\{py : u(y) \geq u(x)\} &\leq \max\{px' : x' \in B_q\} \\ &= \max_c \frac{p_c}{q_c} \\ &< \max_c \frac{p_c}{p_c - d} \\ &\leq \frac{\alpha}{\alpha - d}. \end{aligned}$$

The next to last inequality follows from fact that $\|q - p\| < d < \alpha$ and the last from the fact that $p_c \geq \alpha$. So, $u_p(x) - 1 < \alpha/(\alpha - d) - 1 = d/(\alpha - d)$. Observing that $u_p(\psi(p, u)) = p\omega = 1$ completes the proof. Q.E.D.

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