Money In A Real Business Cycle Model

by

Roger E.A. Farmer

University of California, Los Angeles

Abstract

This paper constructs a real business cycle model in which real money balances yield utility. I calibrate the model to fit the first moments of US data and I simulate a set of impulse response functions that are generated by the model for GDP, the rate of interest, money growth and real balances. These theoretical impulse responses are compared with actual impulse responses from US data. The model does a reasonably good job of capturing the dynamic interactions of money and real variables in US data. It differs from most existing approaches by choosing a parameterization of utility for which the model admits the existence of indeterminate equilibria. I argue that this fact is critical in explaining the monetary propagation mechanism.

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(1) Introduction

This paper is about the role of money in initiating and sustaining business cycles. I am going to construct a model that is close to a real business cycle economy and I am going to use it to analyze some features of time series data, particularly the interaction between money, prices, output and the rate of interest. The model is similar to work by other authors who have introduced money into an RBC economy. I have chosen to include a motive for holding money by including real balance as an argument of the utility function although this is not the main feature that distinguishes my work from related approaches.[1] My main innovation is to introduce a non separable specification of the utility function that allows me to make the case that money may be relatively important in financing transactions.

Previous authors have worked with models in which the motive for holding money is parameterized in a way that allows them to infer the properties of the demand function for money and the supply function of labor from first moments of the data. Since the share of social resources that is lost by using money is relatively small – these models imply that the marginal utility of money must be low. If the utility function for money is relatively simple, the properties of the second derivatives of utility, the parameters that determine the curvature of the function, can be directly inferred from the properties of the first moments of the data. It is these curvature parameters that determine the elasticities of the money demand function and the labor supply function. In most models that have been analyzed in the literature the choice of a simple functional form, for the function that describes the motive for holding money, places restrictions on the behavior of demand and supply functions that prevents these models from explaining the rich pattern of monetary and real interactions that we observe in data.

The most important difference of my approach, from standard real business cycle economies, is that the underlying general equilibrium model that I will construct contains a continuum of equilibria; in the language of general equilibrium theory the steady state equilibrium of the model is indeterminate.[2] The fact

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[1] Related work includes the calibrated cash in advance models of Cooley and Hansen (1989), or the liquidity effects model of Christiano and Eichenbaum (1992), Fuerst (1992) and Lucas (1990).

[2] The model that I will describe in this paper combines the real model of Benhabib and Farmer (1994), with the work on money in a production economy without capital, Benhabib and Farmer (1995). The current paper, and the work reported in Benhabib and Farmer (1995) differs from the related work by Beaudry and Devereux (1993) who exploit increasing returns to scale to generate indeterminacy. Unlike Beaudry and Devereux and Benhabib and Farmer (1994), the model in this paper maintains a technology with constant returns to scale.

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that the steady state equilibrium is indeterminate has two important consequences for the properties of dynamic stochastic equilibria. First, it makes it particularly easy to construct examples of business cycles that are driven by sunspot shocks. Second, and more importantly, the dynamic stochastic equilibria of the model display a rich internal propagation mechanism of a kind that mimics the dynamic characteristics of US data.

(2) The Model

I assume that the economy consists of a large number of representative families each of which maximizes the present discounted value of a lifetime utility function.

$$(1) \qquad \text{Max} \quad \tilde{U} \; = \; E_1 \left\{ \sum_{t=1}^{\infty} \beta^{t-1} U \left(C_t, \frac{M_t}{P_t}, L_t \right) \right\}$$

where C_t is consumption, L_t is labor supply and M_t/P_t is the real value of money balances. The period budget constraint of each family is given by:

$$(2) \qquad M_t \, + \, B_t \, + \, P_t K_{t+1} \; = \; M_{t-1} \, + \, B_{t-1}\left(1 + i_{t-1}\right) \, + \, P_t K_t\left(1 - \delta\right) \, + \, \left(Y_t \, - \, C_t \, + \, T_t\right)P_t, \quad t = 1, \dots \infty,$$

where M_t is the nominal stock of money and B_t is the nominal stock of government one period bonds held from period t to t+1. P_t is the price of commodities in terms of money and i_t is the rate of interest on bonds. K_t is the stock of capital, δ is the rate of depreciation, Y_t is GDP and T_t is a real transfer received by the representative family from the government at the beginning of period t. The household also faces a resource constraint described by equation (3),

$$(3) \qquad K_{t+1} \; = \; K_t\left(1 - \delta\right) \, + \, Y_t \, - \, C_t,$$

and a debt limit, inequality (4), that prevents the household from borrowing forever and never paying back its debt.

$$(4) \qquad \lim_{t \to \infty} Q_s^t \left(\frac{M_t \, + \, B_t}{P_t} \, + \, K_{t+1} \right) \; \geq \; 0, \qquad Q_s^t \; = \; \frac{P_t}{P_s} \frac{1}{\prod_{j=s}^{t-1}\left(1 + i_s\right)}.$$

The variable Q_s^t in this inequality is the date s value of a period t dollar.

To describe production, I assume that output is produced from labor and capital using the technology:

$$(5) \qquad Y_t \; = \; \left(A_t L_t\right)^{\alpha} K_t^{1-\alpha} V_t,$$

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where A_t is a non-stochastic trend and V_t is a random productivity shock. I will model growth as a stationary process by making the following assumptions about the properties of A_t and V_t;

$$(6) \qquad A_t = \gamma^t A_0,$$

$$(7) \qquad V_t = V_{t-1}^\zeta \exp\left(u_t^1\right).$$

The parameter γ is the growth factor of per capita GDP, ζ measures the persistence of the technology shock and u_t^1 is its innovation. I assume that u_t^1 has zero expected value, is independently and identically distributed (i.i.d.) through time and has small bounded support. The assumption of small bounded support is required to ensure that a linear approximation to the equilibrium of the model will remain close to the actual equilibrium. Later in the paper I will introduce an additional fundamental shock u_t^2 that represents a policy disturbance and I will introduce two sunspot shocks that I will refer to as e_t^1 and e_t^2. All three of these additional shocks will also be assumed to be i.i.d. stochastic processes with small bounded support although I will allow for the possibility that they may be contemporaneously correlated.

(3) Modeling Preferences

In this section I discuss the class of utility functions that I will use in my quantitative analysis. The choice of functional form is important since it will play a key role in allowing me to solve for linear approximations to the equilibria of the model and for allowing these equilibria to mimic both low frequency and high frequency features of the data. My choice of functional form was guided by two considerations; (i) the fact that consumption, investment and GDP are cointegrated (growth is balanced) and (ii) the velocity of circulation and the rate of interest are cointegrated.[3] These facts require that there exist a representation of the equations of the model in terms of transformed variables that are independent of time. The class of utility functions that I will introduce below, when combined with a technology that displays labor augmenting technical progress such as the Cobb-Douglas technology in equation (5), satisfies this condition.

It is well known[4] in the case of the non-monetary growth model that the period utility function must be of the form:

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[3] King, Plosser, Stock and Watson (1991).

[4] King, Plosser Rebelo (1988).

$$(8) \qquad U(C,L) = \frac{C^{1-\rho}}{1-\rho} V(L), \quad \rho > 0,$$

in order for the model to admit the existence of a balanced growth path.[5] In the case of a monetary model, the restriction to utility functions that admit the possibility of balanced growth implies that the period utility function U(C,M/P,L) should be homogenous in C and M/P. This homogeneity requirement places a number of restrictions on the class of admissible utility functions U(C,M/P,L) that are discussed in more depth below.

(4) Restrictions on the Parameter Space

Since I will be concerned with a linearized version of the model, it will be helpful to begin by defining the parameters of the model in terms of the elasticities of the utility function and of its partial derivatives. These parameters, defined in equations (9) – (12), together with the parameters of the production function, completely characterize the behavior of the linearized model around its balanced growth path:

$$(9) \qquad U_c = \frac{\partial U}{\partial C}, \qquad U_m = \frac{\partial U}{\partial (M/P)}, \qquad U_L = -\frac{\partial U}{\partial L},$$

$$(10) \qquad \delta_c = \frac{C}{U}U_c, \qquad \delta_m = \frac{(M/P)}{U}U_m, \qquad \delta_L = \frac{L}{U}U_L.$$

$$(10) \qquad \delta_{cc} = \frac{C}{U_c}\frac{\partial U_c}{\partial C}, \qquad \delta_{cm} = \frac{(M/P)}{U_c}\frac{\partial U_c}{\partial (M/P)}, \qquad \delta_{cL} = \frac{L}{U_c}\frac{\partial U_c}{\partial L},$$

$$(11) \qquad \delta_{mc} = \frac{C}{U_m}\frac{\partial U_m}{\partial C}, \qquad \delta_{mm} = \frac{(M/P)}{U_m}\frac{\partial U_m}{\partial (M/P)}, \qquad \delta_{mL} = \frac{L}{U_m}\frac{\partial U_m}{\partial L},$$

$$(12) \qquad \delta_{Lc} = \frac{C}{U_L}\frac{\partial U_L}{\partial C}, \qquad \delta_{Lm} = \frac{(M/P)}{U_L}\frac{\partial U_L}{\partial (M/P)}, \qquad \delta_{LL} = \frac{L}{U_L}\frac{\partial U_L}{\partial L}.$$

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[5] In the post war annual data, there is some evidence that employment has been trending up and hence, a model that displays a balanced growth path in which consumption, capital and GDP grow at the same rate and in which employment is stationary may not be the best description of the data. Nevertheless, I have chosen to begin an exploration of a monetary model by making a minimal departure from the existing literature and for this reason I will use a utility function that will reduce to equation (8) in the absence of money.

Notice that U_L is defined as the *negative* of the marginal disutility of labor supply. In matrix form one can describe the parameters of the utility function as a vector of three elements, δ:

$$(13) \qquad \delta = \{\delta_c, \delta_m, \delta_L\},$$

and a matrix D of nine elements:

$$(14) \qquad D = \begin{bmatrix} \delta_{cc} & \delta_{cm} & \delta_{cL} \\ \delta_{mc} & \delta_{mm} & \delta_{mL} \\ \delta_{Lc} & \delta_{Lm} & \delta_{LL} \end{bmatrix}.$$

Although D has nine elements, only six of these elements are unrestricted since D is related to the Hessian matrix of the utility function which is symmetric. The symmetry of the Hessian imposes the following three restrictions on D:

$$(15) \qquad \delta_{cm} = \delta_{mc} \frac{\delta_m}{\delta_c}, \qquad \delta_{cL} = -\delta_{Lc} \frac{\delta L}{\delta_c}, \qquad \delta_{mL} = -\delta_{Lm} \frac{\delta_L}{\delta_m}.$$

If one were concerned with the properties of a general utility function, U(C,M/P,L), and if one hoped to recover the parameters of the utility function by studying data on a linearized version of an economic model then one would need nine parameters to describe the class of utility functions consistent with the assumption that the data is described by the maximizing choices of a representative agent. These nine parameters would be described by the three elements of δ and by the six free parameters of the matrix D. The fact that the data displays (approximately) balanced growth places an additional three restrictions on the parameter space that are dictated by the requirement that U should be homogenous. If we introduce the assumption that the utility function is homogenous of degree $(1-\rho)$ in C and M/P then the set of free parameters can be described by $\{\rho,\delta,D\}$ together with the symmetry restrictions, (15) and the homogeneity restrictions:

$$(16) \qquad \delta_c + \delta_m = 1 - \rho, \qquad \delta_{cc} + \delta_{cm} = -\rho, \qquad \delta_{mc} + \delta_{mm} = -\rho, \qquad \delta_{Lc} + \delta_{Lm} = 1 - \rho.$$

Imposing these restrictions directly on D, one has a set of parameters described by the vector δ and the elements of the matrix D restricted in the way described in equation (18):

$$(17) \qquad \delta = \{\delta_c, \delta_m, \delta_L\}$$

$$(18) \qquad D = \begin{bmatrix} -\rho - \delta_{cm} & \delta_{cm} & \delta_{cL} \\ \delta_{cm}\dfrac{\delta_c}{\delta_m} & -\rho - \delta_{cm}\dfrac{\delta_c}{\delta_m} & -\left(1 - \rho + \delta_{cL}\dfrac{\delta_c}{\delta_L}\right)\dfrac{\delta_L}{\delta_m} \\ -\delta_{cL}\dfrac{\delta_c}{\delta_L} & \left(1 - \rho + \delta_{cL}\dfrac{\delta_c}{\delta_L}\right)\dfrac{\delta_L}{\delta_m} & \delta_{LL} \end{bmatrix} .$$

The complete class of preferences that is consistent with utility maximization and balanced growth allows for relatively flexible functional forms. In this paper I will exploit the flexibility of the utility function to parameterize the model in a way that is consistent with the existence of an indeterminate equilibrium. It is the fact that the equilibrium of the model is indeterminate that allows me to construct equilibria that mimic the propagation mechanisms for business cycles that we observe in time series data.

(5) A Parametric Utility Function

In the quantitative section of this paper I will calibrate some of the parameters of the utility function to capture the first moments of the data. If one restricts oneself to simple logarithmic or CES utility functions, the calibration of the first moments also restricts the elements of D in such a way that the behavior of the equilibria of a parameterized model may be relatively uninteresting because the choice of the elements of δ completely determines the elements of D. The following class of functions allows for a much richer variety of possible equilibrium behaviors:

$$(19) \qquad U\left(C, \frac{M}{P}, L\right) = \frac{X\left(C, \frac{M}{P}\right)^{1-\rho}}{1-\rho} - W\left(C, \frac{M}{P}\right)^{1-\rho} V(L), \qquad \rho > 1$$

where:

$$(20) \qquad V(L) = L^{1+\chi}, \qquad \chi > 0,$$

$$(21) \qquad X\left(C, \frac{M}{P}\right) = \left[(1-a)C^{1-\lambda} + a\left(\frac{M}{P}\right)^{1-\lambda}\right]^{\frac{1}{1-\lambda}}, \qquad \lambda > 0, \ \lambda \neq 1$$

$$(22) \qquad W\left(C, \frac{M}{P}\right) = \left[(1-b)C^{1-\lambda} + b\left(\frac{M}{P}\right)^{1-\lambda}\right]^{\frac{1}{1-\lambda}}, \qquad \lambda > 0, \ \lambda \neq 1.$$

The utility function in (19) is weighted sum of CES aggregators in which each individual CES aggregator combines consumption and real balances; the weight between the two functions W and X is determined by

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the use of labor as described by the disutility of effort, V(L). In my quantitative analysis I will restrict attention to the class of utility functions in which the homogeneity parameter, ρ, is strictly greater than unity mainly because in versions of this utility function in which money plays no role the parameter ρ would have the interpretation of the "coefficient of relative risk aversion"; there is a consensus in much of the literature that this parameter should be in the range of 1 to 3. The parameter λ is the elasticity of substitution in each of the two aggregator functions W and X. I have restricted this parameter to be the same in each case simply because I am able to solve this special case for the balanced growth path and the ability to solve the model analytically was important in suggesting where in the parameter space, to search for a functional form that can mimic the dynamic responses of data. The parameter λ plays an important role since it allows me to calibrate the model in a way that captures the interest coefficient that one would expect to observe in estimates of the "demand for money" that would be obtained from data generated by the model. The parameters a and b represent the relative importance of money and they play an important role in allowing the model to capture the fact that the interest cost of holding money is small in observed data. By making λ relatively large, but a and b very small, one can capture the fact that the "direct effect" of money is small but its "indirect effect" is big. By the direct effect, I mean the marginal utility of money evaluated at the steady state and by the indirect effect I mean the cross partial of money with consumption. The idea that one captures with this added flexibility in the utility function is that additional units of real balances do not yield much utility in themselves but they may nevertheless be highly complementary with other commodities and with labor supply.

(6) The Solution to the Individual Problem

In this section I will describe the solution to the household's optimizing problem. The household chooses sequences of state contingent money, debt, capital and labor supply to maximize (1) subject to the constraints (2), (3) and (4) and the initial conditions:

(23) $M_0 = \overline{M}_0, \quad B_0 = \overline{B}_0, \quad K_1 = \overline{K},$

The solution to this problem is given by the first order conditions:

(24) $$\frac{U_L\left(C_t, \dfrac{M_t}{P_t}, L_t\right)}{U_c\left(C_t, \dfrac{M_t}{P_t}, L_t\right)} = \frac{\alpha Y_t}{L_t},$$ Labor,

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$$
(25) \quad U_c\!\left(C_t, \frac{M_t}{P_t}, L_t\right)\left[1 - \frac{U_m\!\left(C_t, \frac{M_t}{P_t}, L_t\right)}{U_c\!\left(C_t, \frac{M_t}{P_t}, L_t\right)}\right] = E_t\!\left\{\beta U_c\!\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, L_{t+1}\right)\frac{P_t}{P_{t+1}}\right\}, \quad \text{Money,}
$$

$$
(26) \quad U_c\!\left(C_t, \frac{M_t}{P_t}, L_t\right) = E_t\!\left\{\beta U_c\!\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, L_{t+1}\right)\frac{P_t}{P_{t+1}}(1 + i_t)\right\}, \qquad \text{Bonds,}
$$

$$
(27) \quad U_c\!\left(C_t, \frac{M_t}{P_t}, L_t\right) = E_t\!\left\{\beta U_c\!\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, L_{t+1}\right)\left[1 - \delta + \frac{(1-\alpha)Y_{t+1}}{K_{t+1}}\right]\right\}, \qquad \text{Capital.}
$$

Equation (24) is the static first order condition for choice of L_t, and equations (25), (26) and (27) are the dynamic first order conditions for choice of M_t, B_t, and K_{t+1}.

(7) Government Policy

Since my main focus in this paper will be the conduct of monetary policy, I am going to make a particularly simple assumption about the conduct of fiscal policy; I will model fiscal policy with the assumption:

$$
(28) \quad B_t = 0, \qquad \text{for all } t\,.
$$

To model monetary policy I assume that government follows the feedback rule:

$$
(29) \quad \bar{I}\!\left(x_t, x_{t-1}, u_t^2\right) = 0
$$

where x is a vector of the endogenous variables of the model, u^2 is a policy shock and $\bar{I}$ is a function that represents the reaction of the Fed to current and past variables in the economy. A particularly simple policy is given by the fixed interest rate rule:

$$
(30) \quad I_t = \frac{i_t}{1 + i_t} = \bar{I}\,.
$$

I will use the rule in (30) in my subsequent analysis to evaluate the stability of the model since the fixed interest rate rule leads to a model with relatively simple dynamics. Equation (29) is, however, much more general. For example, the fixed money growth rate suggested by Friedman can be fitted into this framework, as can most of the stabilization rules suggested in recent literature, by adjusting the function

I( ).[6] In the analysis in the paper I will linearize (29) in the neighborhood of a fixed interest rate and in this linear analysis the choice of different monetary rules can be analyzed as the choice of the coefficients of a linear equation.

Given the fiscal rule, (28), the budget equation (2) and the capital accumulation definition (3), the transfer variable, T_t, is defined by the equation;

$$(31) \quad T_t = \frac{M_t - M_{t-1}}{P_t} .$$

(8) Equilibria

Using the policy rule and market clearing assumptions one can define an equilibrium as a set of stochastic processes for the variables, $\{C,K,V,M,P,Y,L\}$ that obey the following equations;

$$(32) \quad K_{t+1} = K_t(1 - \delta) + Y_t - C_t, \qquad \text{(Capital accumulation)},$$

$$(33) \quad Y_t = (A_t L_t)^\alpha K_t^{1-\alpha} V_t, \qquad \text{(Production function)},$$

$$(34) \quad V_t = V_{t-1}^\varsigma \exp(u_t^1) \qquad \text{(Productivity shock)},$$

$$(35) \quad \frac{U_L\left(C_t, \frac{M_t}{P_t}, L_t\right)}{U_c\left(C_t, \frac{M_t}{P_t}, L_t\right)} = \alpha \frac{Y_t}{L_t}, \qquad \text{(Labor market equilibrium)},$$

$$(36) \quad U_c\left(C_t, \frac{M_t}{P_t}, L_t\right)(1 - I_t) = \beta E_t\left\{U_c\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, L_{t+1}\right)\frac{P_t}{P_{t+1}}\right\}, \qquad \text{(Bond Euler equation)},$$

$$(37) \quad U_c\left(C_t, \frac{M_t}{P_t}, L_t\right) = \beta E_t\left\{U_c\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, L_{t+1}\right)\left[1 - \delta + (1-\alpha)\frac{Y_{t+1}}{K_{t+1}}\right]\right\}, \qquad \text{(Capital Euler equation)},$$

$$(38) \quad \frac{U_m\left(C_t, \frac{M_t}{P_t}, L_t\right)}{U_c\left(C_t, \frac{M_t}{P_t}, L_t\right)} = I_t, \qquad \text{(Monetary equilibrium)}.$$

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[6] For an analysis of the effect on the equilibrium of alternative monetary policies in a model that is closely related to this one, see Bennett (1996).

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Equation (38) combines the Euler equations for money and bonds (25) and (26). An equilibrium also requires that the sequences {C,K,Y,M/P} should not grow "too fast" in order to satisfy a transversality condition that is necessary for a candidate sequence to be the solution to an individual's problem. All of the equilibria that I will study will be described as steady states in transformed variables and one may show that boundedness of these transformed variables is sufficient to guarantee that the transversality condition holds.

(9) Transformed Variables

The following sections of the paper show how to reduce the equations of the model to a more manageable system that can be handled by linear methods. A necessary condition for the use of linear methods is that the model should display a fixed point around which linearization makes sense. It is partly for this reason that I have chosen functional forms for the production function and for the utility function that are consistent with the existence of a balanced growth path. To illustrate the existence of a balanced growth path, I will demonstrate that the equations of the model can be rewritten in terms of a set of *transformed* variables each of which is independent of time. Since the transformed equations are autonomous, it is possible to search for the existence of a steady state solution to these equations. If such a solution exists, it will define a balanced growth path.

The following equations define the transformed variables that I will use in this study:

$$(39) \quad y_t = \frac{Y_t}{A_t}, \quad k_t = \frac{K_t}{A_t}, \quad c_t = \frac{C_t}{A_t}, \quad m_t = \frac{M_{t-1}}{P_t A_t}, \quad \mu_t = \frac{M_t}{M_{t-1}}, \quad \gamma = \frac{A_t}{A_{t-1}}.$$

Lower case y, k and c are the ratios of consumption, GDP and capital to the productivity trend, A_t. The variable, lowercase m, is the ratio of real balances to the productivity trend where real balances are defined relative to last period's nominal stock of money. This choice of state variable will be convenient for a study of monetary dynamics since it will enable me to monitor innovations to the price level independently of innovations to the nominal stock of money. I will keep track of innovations to the stock of money with the variable μ (the money growth factor) and innovations to the price level with the variable m. An innovation to the price level will cause m to fall and an innovation to the money stock will cause μ to rise.

Using the definitions in equations (39) one may rewrite the equations of the model:

$$(40) \quad y_t = (L_t)^\alpha (k_t)^{1-\alpha} V_t, \qquad \text{(Production function)},$$

$$(41) \quad c_t \frac{U_L\left(1, \frac{\mu_t m_t}{c_t}, L_t\right)}{U_c\left(1, \frac{\mu_t m_t}{c_t}, L_t\right)} = \alpha \frac{y_t}{L_t}, \qquad \text{(Labor market equilibrium)},$$

$$(42) \quad \frac{U_m\left(1, \frac{\mu_t m_t}{c_t}, L_t\right)}{U_c\left(1, \frac{\mu_t m_t}{c_t}, L_t\right)} = I_t, \qquad \text{(Monetary equilibrium)},$$

$$(43) \quad \gamma k_{t+1} = k_t(1-\delta) + y_t - c_t, \qquad \text{(Capital accumulation)},$$

$$(44) \quad
\begin{aligned}
& c_t^{-\rho} U_c\left(1, \frac{\mu_t m_t}{c_t}, L_t\right)(1-I_t) = \\
& \frac{\beta}{\gamma^{-\rho}} E_t\left\{ c_{t+1}^{-\rho} U_c\left(1, \frac{\mu_{t+1} m_{t+1}}{c_{t+1}}, L_{t+1}\right) \frac{m_{t+1}}{m_t} \frac{1}{\mu_t} \right\}
\end{aligned}, \qquad \text{(Bond Euler equation)},$$

$$(45) \quad
\begin{aligned}
& c_t^{-\rho} U_c\left(1, \frac{\mu_t m_t}{c_t}, L_t\right) = \\
& \frac{\beta}{\gamma^{-\rho}} E_t\left\{ c_{t+1}^{-\rho} U_c\left(1, \frac{\mu_{t+1} m_{t+1}}{c_{t+1}}, L_{t+1}\right)\left[1-\delta+(1-\alpha)\frac{y_{t+1}}{k_{t+1}}\right] \right\}
\end{aligned}, \qquad \text{(Capital Euler equation)},$$

$$(46) \quad V_t = V_{t-1}^\varsigma \exp\left(u_1^t\right), \qquad \text{(Productivity shock)},$$

$$(47) \quad I_t = \bar{I}\left(x_t, x_{t-1}, u_t^2\right), \qquad \text{(Policy rule)}.$$

Equations (41), (42), (44) and (45) exploit the fact that U_L is homogenous of degree 1-ρ and U_c and U_m are homogenous of degree ρ.

(10) Balanced Growth

In this section I am going to search for a point around which to linearize equations (40)–(47). A natural candidate for linearization is the balanced growth path of a non-stochastic version of the model. To find such a path one must first shut down the shocks by setting $u_t^i = 0$ for i=1,2. The fact that the productivity shock is stationary then implies that $V_t = 1$. Even when there are no shocks to the model, there may still be no balanced growth path if policy is non-stationary; for some policy rules a steady state in transformed variables will exist, for others it will not. In the following analysis I will analyze the existence of steady state when the policy rule take the form of a fixed interest rate; the rule described in equation (30). Given

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such a policy one may define the non-stochastic balanced growth path of the model, $\{c^*, y^*, m^*, L^*, \mu^*, k^*\}$ to be a solution to the equations:

$$(48) \qquad y^* = \left(L^*\right)^\alpha \left(k^*\right)^{1-\alpha},$$

$$(49) \qquad c^* \frac{U_L\left(1, \frac{\mu^* m^*}{c^*}, L^*\right)}{U_c\left(1, \frac{\mu^* m^*}{c^*}, L^*\right)} = \alpha \frac{y^*}{L^*},$$

$$(50) \qquad \frac{U_m\left(1, \frac{\mu^* m^*}{c^*}, L^*\right)}{U_c\left(1, \frac{\mu^* m^*}{c^*}, L^*\right)} = \bar{I},$$

$$(51) \qquad k^*\left(\gamma - 1 + \delta\right) \ - \ y^* \ + \ c^* \ = 0,$$

$$(52) \qquad \left(1 - \bar{I}\right) = \frac{\beta}{\gamma^{-\rho}} \frac{1}{\mu^*},$$

$$(53) \qquad 1 = \frac{\beta}{\gamma^{-\rho}}\left[1 - \delta + \left(1 - \alpha\right)\frac{y^*}{k^*}\right].$$

In appendix A I show that when the utility function is given by the parametric class described by equations (19) – (22) there exists a unique solution to equations (48)–(53). One may also show that there is an open set of the parameter space for which this solution is indeterminate in the sense that close to the balanced growth path there exists a continuum of rational expectations equilibria.

(11) Vector Notation

In this section I will introduce a vector notation that will enable me to write the equations of the model in a more compact form. In the body of the paper I will work with these vector equations and in appendix A I define the elements of each of the various coefficient matrices.[7] My goal is to demonstrate that business cycles in this model can be described as solutions to a linear stochastic model in which the variables are deviations of each of the state variables from their balanced growth path. The stochastic elements that drive these equations will be of two kinds. First there are *fundamental* disturbances. These are the variables $\{u^1, u^2\}$ that represent the innovation to the productivity shock and the policy disturbance.

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[7] The Gauss code for computing equilibria of the model analytically is available from the author on request.

Secondly, in models in which there is an indeterminacy in the equations that describe the equilibrium of the non-stochastic model, there may be a role for *non-fundamental* disturbances, also known as sunspots or animal spirits. I will introduce two sunspot disturbances that I will refer to as e_t^1 and e_t^2.

To reduce the complexity of the notation I will employ the following definitions of vectors of variables, each expressed in the form of proportional deviations from the balanced growth path. These variables, in deviation form, are defined below:

$$(54) \qquad \tilde{c}_t = \frac{(c_t - c^*)}{c^*}, \quad \tilde{k}_t = \frac{(k_t - k^*)}{k^*}, \quad \tilde{I}_t = \frac{(I_t - \bar{I})}{\bar{I}}, \quad \tilde{v}_t = \frac{(V_t - 1)}{1}, \quad \tilde{\mu}_t = \frac{(\mu_t - \mu^*)}{\mu^*},$$

$$(55) \qquad \tilde{y}_t = \frac{(y_t - y^*)}{y^*}, \quad \tilde{L}_t = \frac{(L_t - L^*)}{L^*}, \quad \tilde{m}_t = \frac{(m_t - m^*)}{m^*}.$$

The five variables consumption, capital, the interest rate, the productivity shock and the money growth factor are the *state variables* and I will refer to the vector of state variables with the notation upper case Z. The three variables, GDP, employment and real balances (the ratio of *lagged* money to price) are *subsidiary variables* that can be written as functions of the state and I will refer to the vector of these three subsidiary variables with the notation upper case X. The definitions of Z and X are given in equations (56) and (57). The definition of the vector u in equation (58) collects the fundamental disturbances together into a single term.

$$(56) \qquad Z_t = \begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{I}_t \\ \tilde{V}_t \\ \tilde{\mu}_t \end{bmatrix}, \qquad (57) \qquad X_t = \begin{bmatrix} \tilde{y}_t \\ \tilde{L}_t \\ \tilde{m}_t \end{bmatrix}, \qquad (58) \qquad u_t = \begin{bmatrix} u_t^1 \\ u_t^2 \end{bmatrix}.$$

In addition to the fundamental disturbances $\{u^1, u^2\}$ it will also be useful to have a notation to refer to the conditional forecast errors of each of the variables of the model and to two linear combinations of these forecast errors that represent the sunspot variables e^1 and e^2. The conditional forecast errors are defined in equation (59). I have used the index variable x in this definition to represent an element of the vector $\{c,k,I,V,\mu,y,L,m\}$.

$$(59) \qquad \varepsilon_t^x = \left[x_t - E_{t-1}\{x_t\} \right].$$

Using this definition one can collect together the vectors of conditional forecast errors associated with each of the vectors variables, Z and X;

$$(60) \quad \varepsilon_t^Z = \begin{bmatrix} \varepsilon_t^c \\ \varepsilon_t^k \\ \varepsilon_t^I \\ \varepsilon_t^V \\ \varepsilon_t^\mu \end{bmatrix}, \qquad (61) \quad \varepsilon_t^X = \begin{bmatrix} \varepsilon_t^y \\ \varepsilon_t^L \\ \varepsilon_t^m \end{bmatrix}.$$

I will show in the following analysis that the equilibria of the model can be represented as a vector difference equation in the five variables c, k, I, V and μ. But although the dynamical system that describes the equilibria of this system has dimension five, only three of the variables, capital, the productivity shock and the rate of interest are predetermined. Consumption and real balances are free to move each period in a way that depends on the forward looking expectations of the families in the model. When the steady state of the system is locally stable – the case that I will study in this paper – there are two dimensions of indeterminacy. These two dimensions of indeterminacy imply that agents are free to form beliefs in which consumption and real balances are adjusted each period in line with extraneous variables that I will refer to as "sunspots". By modeling the process for the sunspot shocks, and by restricting agents to form beliefs in the same way each period, one can resolve the indeterminacy and generate an economic model in which the covariance properties of the data are uniquely determined. To represent the two sunspot shocks in the model I will use the notation e^1 and e^2. Equation (62) collects these two shocks into a single vector, e;

$$(62) \quad e_t = \begin{bmatrix} e_t^1 \\ e_t^2 \end{bmatrix}.$$

(12) Linearizing the Model

In this section I will write down a linear form of the equations of the model by taking a Taylor series expansion around the balanced growth path. Using the notation that I developed in the previous section one can write the equations of the model, (40) – (47) as approximate linear equations:

$$(63) \quad \tilde{y}_t - \alpha \tilde{L}_t - (1-\alpha)\tilde{k}_t - \tilde{V}_t = 0, \qquad \text{(Production function)},$$

$$(64) \quad a_1\tilde{m}_t + a_1\tilde{\mu}_t + a_2\tilde{c}_t + a_3\tilde{L}_t - \tilde{y}_t = 0, \qquad \text{(Labor market)},$$

$$(65) \quad a_4\tilde{m}_t + a_4\tilde{\mu}_t - a_4\tilde{c}_t + a_5\tilde{L}_t - \tilde{I}_t = 0, \qquad \text{(Monetary equilibrium)},$$

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(66) $\quad \tilde{k}_{t+1} + a_6\tilde{c}_t - a_7\tilde{k}_t + a_8\tilde{y}_t = 0,$ (Capital accumulation),

(67) $\quad \begin{aligned} & a_9\tilde{c}_{t+1} + a_{10}\tilde{m}_{t+1} + a_{11}\tilde{\mu}_{t+1} + a_{12}\tilde{L}_{t+1} - a_9\tilde{c}_t - a_{10}\tilde{m}_t + a_{13}\tilde{\mu}_t \\ & - a_{12}\tilde{L}_t - a_{10}\varepsilon^m_{t+1} - a_{11}\varepsilon^\mu_{t+1} - a_{12}\varepsilon^L_{t+1} - a_9\varepsilon^c_{t+1} + a_{14}\tilde{I}_t = 0 \end{aligned}$, (Bond Euler equation),

(68) $\quad \begin{aligned} & a_{15}\tilde{c}_{t+1} + a_{16}\tilde{m}_{t+1} + a_{16}\tilde{\mu}_{t+1} + a_{17}\tilde{L}_{t+1} - a_{15}\tilde{c}_t - a_{16}\tilde{m}_t \\ & - a_{16}\tilde{\mu}_t - a_{17}\tilde{L}_t + a_{18}\tilde{y}_{t+1} - a_{18}\tilde{k}_{t+1} - a_{15}\varepsilon^c_{t+1} - a_{16}\varepsilon^m_{t+1} \\ & - a_{16}\varepsilon^\mu_{t+1} - a_{17}\varepsilon^L_{t+1} - a_{18}\varepsilon^y_{t+1} + a_{18}\varepsilon^k_{t+1} = 0 \end{aligned}$ (Capital Euler equation),

(69) $\quad \tilde{V}_{t+1} - \zeta\tilde{V}_t - u^1_{t+1} = 0,$ (Productivity shock),

(70) $\quad \begin{aligned} & \tilde{I}_{t+1} + a_{19}\tilde{y}_t + a_{20}\tilde{L}_t + a_{21}\tilde{m}_t + a_{22}\tilde{c}_t + a_{23}\tilde{k}_t \\ & + a_{24}\tilde{I}_t + a_{25}\tilde{V}_t + a_{26}\tilde{\mu}_t - u^2_{t+1} = 0 \end{aligned}$ (Policy rule).

where the coefficients a_1-a_{18} are functions of the steady state values of the variables c*, k*, $\bar{I}$, μ*,y*, L*, and m* that are derived by evaluating the derivatives of the expressions in equations (40)–(47) at the balanced growth path – these parameters are defined in appendix A. The coefficients a_{19}–a_{26} define the policy rule and each of these coefficients represents the elasticity of the policy reaction function to one of the variables of the model.

(13) The Model in Vector Form

In vector notation one may write the equations of the model in the following way. Equations (63)–(65) form a block of three static equations that I describe in vector notation in equation (71);

(71) $\quad A_1 X_t + A_2 Z_t = 0.$

The equations, (66) – (70) form a separate dynamic block:

(72) $\quad A_3 X_{t+1} + A_4 Z_{t+1} + A_5 X_t + A_6 Z_t + A_7 u_{t+1} + A_8 \varepsilon^Z_{t+1} + A_9 \varepsilon^X_{t+1} = 0.$

Since A_1 is of full rank, one may use equation (71) to write the vector X as a function of Z:

(73) $\quad X_t = MZ_t, \qquad M = -A_1^{-1}A_2.$

Similarly, one may obtain an expression that relates the errors e^X to the errors ε^Z.

(74) $\quad \varepsilon^X_t = M\varepsilon^Z_t.$

Using equations (73) and (74) one may replace X and ε^X in (72) and solve for a set of equations in the state variables:

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$$\text{(75)} \quad Z_{t+1} = J_1 Z_t + J_2 u_{t+1} + J_3 \varepsilon^Z_{t+1}$$

where the matrices J_1, J_2 and J_3 are defined in equations (76):

$$\text{(76)} \quad \begin{aligned} J_1 &= -(A_4 - A_3 A_1^{-1} A_2)^{-1}\left(A_6 - A_7 A_1^{-1} A_2\right), \\ J_2 &= -(A_4 - A_3 A_1^{-1} A_2)^{-1} A_7, \\ J_3 &= -(A_4 - A_3 A_1^{-1} A_2)^{-1}\left(A_9 - A_8 A_1^{-1} A_2\right). \end{aligned}$$

So far I have shown that the equilibria of the model must obey a set of stochastic equations in which the disturbances to these equations are of two kinds, fundamental disturbances and forecast errors. If the equilibrium of the model was unique then one would be able to solve for the forecast errors ε^Z as functions of the fundamentals by solving the unstable roots of the matrix J_1 forwards. The technique for solving rational expectations models in this way is, by now well known, and is described in some detail in the work by King Plosser and Rebelo (1984). In the case that I will be studying in this paper however, all roots of J_1 are inside the unit circle and the standard approach breaks down. Instead, one is free to pick two sunspot variables each period. These sunspot variables, defined by the equations:

$$\text{(77)} \quad a_9 \varepsilon^c_{t+1} + a_{10} \varepsilon^m_{t+1} + a_{11} \varepsilon^\mu_{t+1} + a_{12} \varepsilon^L_{t+1} = e^1_{t+1},$$

$$\text{(78)} \quad a_{15} \varepsilon^c_{t+1} + a_{18} \varepsilon^k_{t+1} + a_{18} \varepsilon^y_{t+1} + a_{16} \varepsilon^m_{t+1} + a_{17} \varepsilon^L_{t+1} + a_{16} \varepsilon^\mu_{t+1} = e^2_{t+1},$$

are equal to the forecast errors of the two expectational Euler equations, (44) and (45). In addition to the equations (77) and (78) one also has information about three of the elements of the vector ε^Z that must hold in a rational expectations equilibrium;

$$\text{(79)} \quad \varepsilon^k_{t+1} = 0, \qquad \text{(80)} \quad \varepsilon^V_{t+1} = u^1_{t+1}, \qquad \text{(81)} \quad \varepsilon^I_{t+1} = u^2_{t+1}.$$

Equation (79) says that the forecast error on capital is zero. This follows from the fact that capital at date t+1 is known at date t. Equations (80) and (81) state that the forecast errors for the productivity shock "V" and for the interest rate "I" are equal to the true disturbances; these equations are an implication of the rational expectations assumption which implies that agents know the probability distributions of the fundamental shocks. In matrix form, equations (77)–(81) can be written as a single equation that describes the conditional forecast errors ε^Z as a function of the fundamental errors u and of the two sunspot errors, e:

$$\text{(82)} \quad \varepsilon^Z_{t+1} = H_5 u_{t+1} + H_6 e_{t+1},$$

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where the matrices H_5 and H_6 are defined in appendix A. Using equation (82) one can rewrite the stochastic difference equation (75) as follows:

(83) $Z_{t+1} = J_1 Z_t + J_4 u_{t+1} + J_5 e_{t+1}$,

where the details of the algebra, together with the definitions of the matrices J_4 and J_5 are again explained in appendix A. In the following section I will discuss the role of indeterminacy in this model and I will show how one might compute a solution.

(14) Indeterminacy and Equilibrium

In this discussion I will focus on the case of a perfect foresight model in which the policy rule is to set the rate of interest equal to a constant. In this case the dynamic equations (83) can be reduced to a non-stochastic system of difference equations in the three state variables, c, k and μ. Let us write this system as

(84)
$$\begin{bmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \\ \tilde{\mu}_{t+1} \end{bmatrix} = Q \begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{\mu}_t \end{bmatrix},$$

where Q is the subset of J_1 associated with the variables c, k and μ. It is relatively easy to show that, under interest rate control, the matrix Q has the special structure:

(85) $Q = \begin{bmatrix} Q_1 & 0 \\ q_2 & 0 \end{bmatrix}$

where Q_1 is a 2x2 matrix associated with the real variables c and k and q_2 is a 1x2 vector that feeds the changes in the real economy back to the money growth rate. Since the matrix Q has a zero root, interest rate control leads to an indeterminacy of the price level. The real variables c and k are determined by the upper left block of the system (85), but the initial value of money growth rate is free. Since the value of real balances (the ratio of lagged money to price) can be described as a function of c and k, the indeterminacy of the money supply implies that the price level is also indeterminate.

The fact that interest rate control may lead to indeterminacy is relatively well understood and has been widely discussed by previous authors, although typically the literature on the indeterminacy of an interest rate peg has focused on the case in which the real part of the system is determinate; in other words, most existing literature has studied the case in which Q_1 has two roots that split around unity in absolute value. The fact that these roots spilt around unity enable one to use standard techniques to solve the unstable root of the system forward to find consumption as a function of capital. In this case, although the price level is

indeterminate, interest rate control leaves the real variables of the economy uniquely determined. The novelty in this paper is to parameterize the economy in such a way that the roots of Q_1 are both inside the unit circle and thus there is a *real* indeterminacy in addition to the nominal indeterminacy that follows from interest rate control.[8]

Some intuition for indeterminacy and a sketch of a proof can be gleaned by studying the static equations (63) – (65). In the following discussion I will focus on the special case of the utility function described by equation (86).[9]

$$(86) \qquad U = \frac{C^{1-\rho}}{1-\rho} - \left(\frac{M}{P}\right)^{1-\rho} L^{1+\chi}.$$

If one uses the production function, equation (63) to eliminate GDP from equations (64) and (65) one can think of the resulting two equations as demand and supply equations for labor and for money. Consider first, equation (64), the first order condition for labor, and suppose that the labor market is decentralized. In this case one can think of a household equating the slope of its indifference curve to the real wage and a firm equating the real wage to the marginal product of labor. The log linear form of these demand and supply equations, for the utility function given by (86), are given in equations (87) and (88):

$$(87) \qquad (1-\rho)\log(\overline{m}) + \rho\log(c) + \chi\log(L) = \log(\omega)$$

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[8] The reason for this real indeterminacy is somewhat different from the cases that have been studied in recent work by Benhabib and Farmer (1994), Gali (1994), Beaudry and Devereux (1993) and other authors who rely on increasing returns to scale. In the model that we are analyzing in this paper we assume that the technology satisfies constant returns to scale; the indeterminacy follows purely from an interaction between the real and nominal parts of the economy.

[9] The utility function described in equation (86) is rich enough to display indeterminate equilibria and to illustrate the mechanism by which it occurs. It is not rich enough to fit all of the features of the data however since, when the function in (86) is parameterized in a way that displays an indeterminate equilibrium it implies that the share of social resources lost by the use of money should be of the order of 50% of GDP. The actual number is closer to 1%. Equation (86) also implies that the interest elasticity of the demand for money should be equal to one whereas estimates by Hoffman, Rasche and Tieslau (1995), based on the cointegrating coefficient of the rate of interest and the velocity of circulation suggest that this parameter should be closer to one half. The complications introduced in equations (19)-(22) retain the possibility of indeterminacy but allow the model to capture these additional features of the data.

(88) $(1-\alpha)\log(k) + (\alpha-1)\log(L) = \log(\omega)$.

In equations (87) and (88) I have used the symbol $\overline{m}$ to mean real balances $(m + \mu)$ and I have dropped the constant terms. Equation (87) can be thought of as a labor supply curve that is shifted by consumption and by real balances. Equation (87) can be thought of as a labor demand curve that is shifted by the stock of capital. Benhabib and Farmer (1994) studied a real model that is very similar to the model in this paper and they showed that a necessary and sufficient condition for the roots of the corresponding dynamical system (the analog of Q_1) to be inside the unit circle was that the slope of the labor demand curve should be greater than the slope of the labor supply curve. They demonstrated that one way in which this condition could hold would be if the labor demand curve were to slope up because of externalities in production. In the present context this would be equivalent to assuming that the parameter α is greater than one.

The magnitude of the increasing returns required in the Benhabib Farmer paper has been criticized by a number of authors since returns to scale just don't seem to be that big.[10] However, a similar route to indeterminacy follows in a monetary model in which the production function satisfies constant returns to scale. To see this I am going to show that, if one uses equation (65) to eliminate real balances from equation (64), that the reduced form of the labor market equation (equations (87) and (88)) may satisfy exactly the Benhabib Farmer indeterminacy condition even if the technology satisfies constant returns to scale. From equation (66) it follows that the logs of money, consumption and employment are related to the rate of interest:

(89) $\log(\overline{m}) = \log(c) + \dfrac{(1+\chi)}{\rho}\log(L) - \dfrac{1}{\rho}\log(I)$.

Equation (89) is the analog in this model of a "demand-for-money" function. If one replaces (89) in (87) then the modified labor supply curve has the form:

(90) $\log(c) - \dfrac{(1-\rho)}{\rho}\log(I) + \left(\dfrac{1+\chi}{\rho} - 1\right)\log(L) = \log(\omega)$.

Equation (90) is a reduced form that combines the money demand and labor supply equations together. In this reduced form equation, the "labor supply curve" can slope down instead of up because of the effect of real balances on employment. If one solves for real balances from the block of static equations (63)-(65) to leave two "reduced form equations" in GDP and labor supply, these two reduced from equations have

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[10] See for example the papers by Basu and Fernald (1995) and Burnside, Eichenbaum and Rebelo (1995).

exactly the same structure as the Benhabib Farmer real model. Furthermore, when the utility function is parameterized as in equation (86), the dynamic equations of the model also have the same structure as the Benhabib Farmer real model. It follows that, the condition for indeterminacy remains the same; that is, the labor demand and supply curves should cross with the "wrong slopes" and the labor supply curve should be steeper. Whereas in the real model this occurs when the labor demand curve slopes up, because of externalities, in the monetary model demand and supply can cross with the wrong slopes because the reduced form labor supply curve *slopes down* . The parameter configuration that leads to this possibility is given by equation (91):

$$(91) \qquad 1-\alpha < \left(1 - \frac{1+\chi}{\rho}\right), \quad \left(\frac{1+\chi}{\rho}\right) > 1 .$$

When α is equal to 2/3, indeterminacy requires a small value of χ (elastic labor supply), and a value of ρ well within the range that is considered acceptable from estimates of the intertemporal elasticity of substitution reported in the literature on calibration. For example, if one sets $\chi = 0$, the value consistent with Gary Hansen and Richard Rogerson's model of indivisible labor, indeterminacy occurs if the parameter ρ is greater than 1.5.

(15) Calibrating the Model

I have argued that a model, close in form, to the RBC model can display indeterminate equilibria and I have suggested that a model, parameterized in this way, might help one to explain the data. In this section I am going to calibrate a model of this form and I am going to use the calibrated model to simulate data and to compare the simulated data with actual data from the US economy. The example that I will use in the

Table 1:	Value	Definition
α	0.66	Labor's Share
δ	0.06	Annual Depreciation
β	0.928	Annual Rate of Time Preference
γ	0.01	Per capita Growth Rate
ζ	0.7	Autocorrelation of productivity shock
ρ	1.42	Intertemporal elasticity of substitution
χ	0	Inverse labor supply elasticity

calibration exercise is more complicated than the functional form in equation (86) since one requires additional flexibility in order to capture the fact that the share of social resources lost by using money is

relatively low. Table 1 lists some of the parameters that appear in real models and lists the values of these parameters that I have used in the simulations that I will report in the paper. Most of these are relatively standard, and are similar to the values used in other calibrated models. The parameters α, δ, β and γ are chosen to match long run features of the data. The parameter χ is the inverse labor supply elasticity and the choice of $\chi = 0$ implies an infinite labor supply elasticity. It is well known that highly elastic labor supply is necessary in this class of models to match the observed employment volatility and the choice of $\chi = 0$ is typically justified by appealing to the arguments for indivisible labor in Hansen (1984) and Rogerson (1988). The choice of $\rho = 1.42$ corresponds to a model that is non separable in leisure and consumption – the value $\rho = 1$ would imply logarithmic preferences. A value of ρ somewhat higher than unity is necessary in this model to generate indeterminate equilibria. Table 2 lists some of the first moments of the data that are implied by the real parameterization described in table 1.

Table 2:	Model	Data	
c/y	0.81	0.80	Consumption income ratio
y/k	0.35	0.44	Income capital ratio
L	1.21		Labor supply
r	.06	.06	Real interest rate

The consumption income ratio is higher than is usually reported because the consumption variable used in this study includes government consumption. The time preference factor of 0.928 was chosen to match the real interest rate of 6% and, as a consequence, the model reports a lower ratio of GDP to capital (0.35 as opposed to 0.44) than exists in the annual data. I have not reported labor hours in this study because the value of 1.21 reported from the model is unit dependent. The function V(L) contains an arbitrary constant that can be chosen to fit any observed value of L and the choice of this constant is equivalent to picking the units of measurement of hours; by the same reasoning, the production function can contain a normalizing constant that I have omitted in the paper.

Table 3:	Value	Definition
a	0.0002	Share of money in X
b	0.00015	Share of money in W
λ	2.5	Substitutability of money for consumption and leisure

In addition to the real parameters of the model, the utility function contains three additional parameters that determine the properties of the monetary features of the model. These parameters are reported in table 3.

Table 4:	Model	
y/m	6	Velocity of circulation
ε_i	0.4	Interest elasticity of money
I	.01	Nominal interest rate
im/y	.0016	Resource cost of using money

To calibrate these parameters in practice I picked values for a and for λ in a manner described below and I restricted b so that the predicted values of the velocity of circulation and of the interest cost of holding money would fit the numbers reported in table 4. I have not reported the actual data on the rate of interest and the velocity of circulation in this table because the data is non-stationary and it makes no sense to average them to obtain a single statistic.
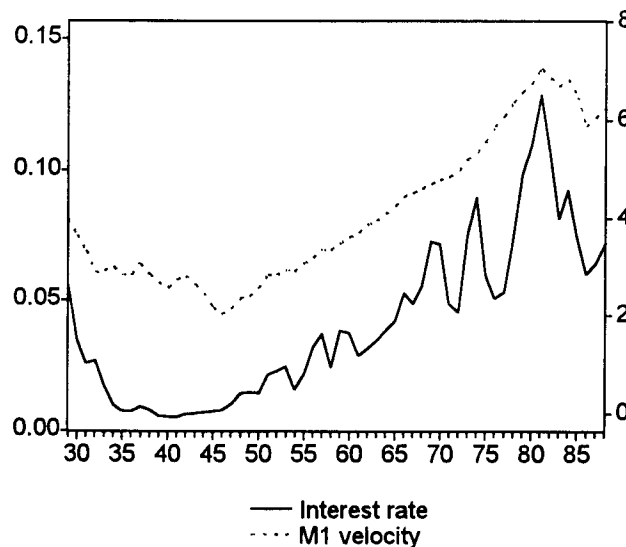
Figure 1: The Interest Rate and the Velocity of Circulation

Figure 1 graphs the data on the rate of interest (the six month commercial paper rate) and the velocity of circulation from US data from 1929 through 1988. The interest rate ranges from half of one percent to thirteen percent and the velocity of circulation from two to seven. The model contains a stationary balanced growth path only for a fixed interest rate policy and the fact that the interest rate was increasing over the sample period implies that the balanced growth of the model was itself shifting. I have handled this non-stationarity in the data by picking an arbitrary point around which to linearize the model and I have picked this point to lie within the range of observed interest rates in the data.

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A second complication arises in deciding the appropriate nominal rate of interest to use. The correct concept from the point of view of the model is the interest lost by holding M1 as opposed to some other interest bearing asset that has no use in exchange. Since a substantial component of M1 bears interest, the six month commercial paper rate undoubtedly overstates the opportunity cost of holding money. In calibrating the model I have erred on the side of caution by picking parameter values that imply that the first order effects of using money are relatively small. I chose a value for the opportunity cost of holding money of 1% and a value of the velocity of circulation of 6. Together, this parameterization implies that the resources lost by a representative agent through using money was equal to 0.16% of GDP.

Table 4 also reports a value for ε_i, the interest elasticity of the demand for money. To compute this statistic for the model one can show that, in the steady state, the velocity of circulation predicted by the model is a function of the rate of interest. The number ε_i reported in table 3 is the elasticity of this function evaluated along the balanced growth path. The corresponding statistic reported for the data comes for recent evidence of the demand for money that is based on the long properties of velocity and the rate of interest. Recent work, for example, by Hoffman, Rasche and Tieslau (1995) places the value of ε_i at around 0.5.

The process by which I calibrated the model parameters was to pick values for λ and a and to fix b so that money's share of GDP (iy/m) would be equal to 0.16% along the balanced growth path. I then used a computer to simulate impulse response functions for four variables in a manner described in the following section. Given the theoretical impulse response functions I experimented with different values of χ, ρ, a and λ, to match the observed impulse response functions as closely as possible whilst keeping the interest elasticity of the velocity of circulation, (the statistic ε_i) within the range 0.3–0.6. In theory ε_i depends on both a and λ, but in the relevant part of the parameter space it is determined mainly by the parameter λ that measures the elasticity of substitution between money and consumption and labor supply.

To fix the parameters of the interest rate rule I experimented with two different sets of assumptions. The first was to set all of the feedback coefficients, a_{19}– a_{26} equal to zero. The second was to estimate a regression of the rate of interest on lagged values of the endogenous variables in time series data and to use these estimated coefficients in the model. In practice, the only coefficient that made much of a difference to the dynamic properties of the simulated data was the coefficient a_{24} that represents the response of the interest rate to its own past values. In the simulations reported below I set a_{24} equal to its estimated value of -0.5 and I set all of the other coefficients of the policy rule to zero.

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In practice the shape of the reported impulse response functions was not overly sensitive to the choice of λ and I chose a value of $\lambda = 2.5$ to fix ε_i at 0.4. Given this choice of λ I found that the shape of the impulse response functions in the data was extremely sensitive to the choice of ρ and a. The parameter a is restricted to a relatively small range from around 0 to 0.0004 in order to maintain a positive value of b and to simultaneously match the money share statistic (im/y) of 0.16%. The following section describes the way that I computed impulse response functions from the model and it describes data that was generated for the parameter choice reported in tables 1 and 3.

(16) Computing Vector Auto-Regressions

This section describes how I picked the variance covariance matrix of the shocks to the model and it addresses the methods that I used to pick the key parameters ρ and a. The choice of these parameters was geared to address a particular question. Suppose that an econometrician were to be given a particular subset of the data generated by my model; how closely would the artificial data resemble the actual time series on the same subset of variables? The subset that I wish to study is one that was originally investigated by Chris Sims; the US data on price, GDP, the rate of interest and the stock of money. The analogues of these variables that I will use in this discussion are the variables, s_t, defined below:

$$(92) \qquad s_t = \begin{bmatrix} \tilde{y}_t \\ \tilde{m}_t \\ \tilde{I}_t \\ \tilde{\mu}_t \end{bmatrix}.$$

These variables can be described a linear function of the state vector;

$$(93) \qquad s_t = D_1 Z_t + D_2 X_t = (D_1 + D_2 m) Z_t = D_3 Z_t.$$

where the matrices D_1, D_2 and D_3 are defined in appendix A. The vector s_t consists of deviations from the balanced growth path of GDP, real balances (lagged money divided by current price), the rate of interest and the money growth rate.

Let us suppose that an econometrician were to run a vector autoregression using two lags of GDP, real balances (defined as in the model) the rate of interest, the rate of money growth and a time trend.[11] Let the

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[11] Since the set s_t does not contain observations on the auto-correlated productivity shock, a first order vector autoregression would not be sufficient to recover the dynamics of the underlying model. However, by rewriting the

estimated variance covariance matrix of the residuals from an autoregression of this form be denoted Ω_s. The estimated values of Ω_s from a VAR of this form on post war US data is given in table 5.

Table 5: Estimated VCV Matrix of Residuals x10⁴	GDP	Real Balances	Interest Rate	Money Growth
GDP	19.17	2.00	0.618	0.798
Real Balances	2.00	10.00	-0.33	-3.26
Interest Rate	0.618	-0.33	0.787	-0.602
Money Growth	0.798	-3.26	-0.602	14.39

If the data were generated by the model described in this paper then one could recover the VCV matrix of the underlying shocks from the VCV matrix of the residuals, since the true errors on the equations of the vector autoregression are a linear function of the error vector {u,e}. The relationship between the VCV matrix of {u,e} and the VCV matrix of the true residuals in a VAR is given by equation (94):

$$(94) \qquad \Omega = \left(D_4\right)\Omega_s\left(D_4{}^T\right)^{-1},$$

where the superscript T denotes transposition and the definition of the matrix D_4 is given in the appendix.

Table 6: VCV matrix of errors x10⁴	u^1 Productivity shock	u^2 Policy shock	e^1 First sunspot	e^2 Second Sunspot
u^1	17.11	-0.604	17.70	24.50
u^2	-0.604	0.787	-0.522	-0.837
e^1	17.70	-0.522	25.81	25.40
e^2	24.50	-0.837	25.40	35.09

Table 6 reports the VCV matrix of the shocks that is implied by my estimate of a vector autoregression for the parameterization reported in tables 1 and 3. In both cases tables 5 and 6 multiply the actual numbers by 10^4. This table can be used to interpret the residuals to the VAR in terms of the corresponding structural shocks. Notice that three of the diagonal elements of this matrix are relatively large and of a

unobservable variable V_t as a function of the first difference of the observable variables L_t, Y_t and K_t, one can show that a vector autoregression with two or more lags *would* recover the underlying dynamics.

comparable order of magnitude; these diagonal elements represent the productivity shock and the two sunspot shocks. The fact that these three elements are large implies that the model ascribes a relatively important role to the productivity shock, but it also implies an important role for sunspots.

In addition to relatively large diagonal elements for each of the sunspot shocks notice that these two errors are also highly correlated with the fundamental shock u^1. The fact that the off–diagonal elements that represent these shocks are also large implies that individuals "over react" to fundamentals.

(17) Simulated Data

This section reports some sample statistics from annual US data and it compares these moments with sample statistics from a single series of sixty observations simulated from the model. The simulated data was driven by a vector of four normal i.i.d. random variables with the variance covariance matrix Ω, described in table 6.

Table 7:	Simulation		Data	
	standard deviation	s.d. relative to GDP	standard deviation	s.d. relative to GDP
Consumption	0.09	0.47	0.09	0.75
Capital	0.22	1.15	0.08	0.66
Interest rate	0.01	0.05	0.02	0.16
Productivity shock	0.04	0.21	0.07	0.58
Money growth	0.04	0.21	0.06	0.5
GDP	0.19	1	0.12	1
Employment	0.20	1.05	0.08	0.66
Real balances	0.10	0.52	0.25	2.08

Table 7 reports the standard deviations of the variables used in the study and their standard deviations relative to GDP together with the standard deviations of a single simulation of the model. Figures 2 through 8 graph the US data and the model simulation. These graphs give an impression of the dimensions in which the model succeeds at generating business cycles, and the dimensions at which it fails. I have assumed implicitly that the data is generated by the same process for the entire period from 1929 through 1988. In fact there is considerable evidence that this assumption is false. The magnitude of the residuals in the prewar period is quite a bit greater than in the post war period. The simulated data behaves a lot more like the pre war business cycles, at least in magnitude. Consumption, in the simulations, is too smooth and it is not as highly correlated with GDP as the actual data. Employment, on the other hand, is too volatile.
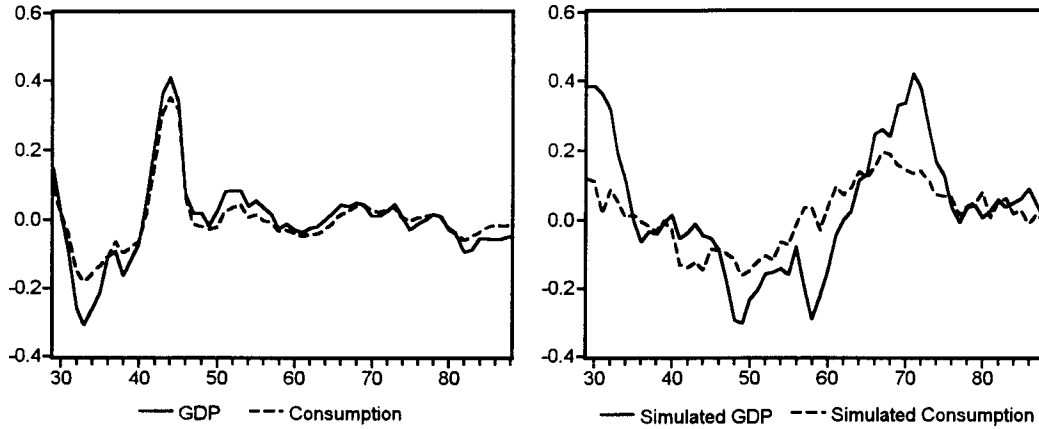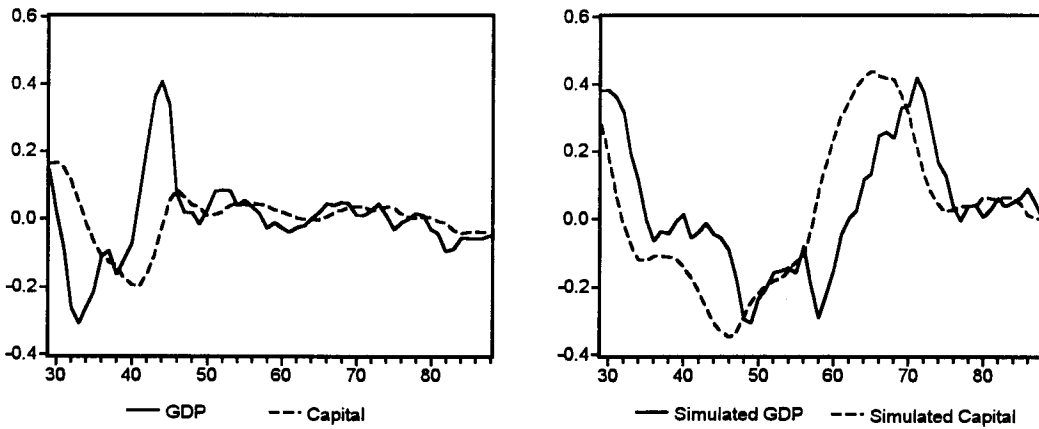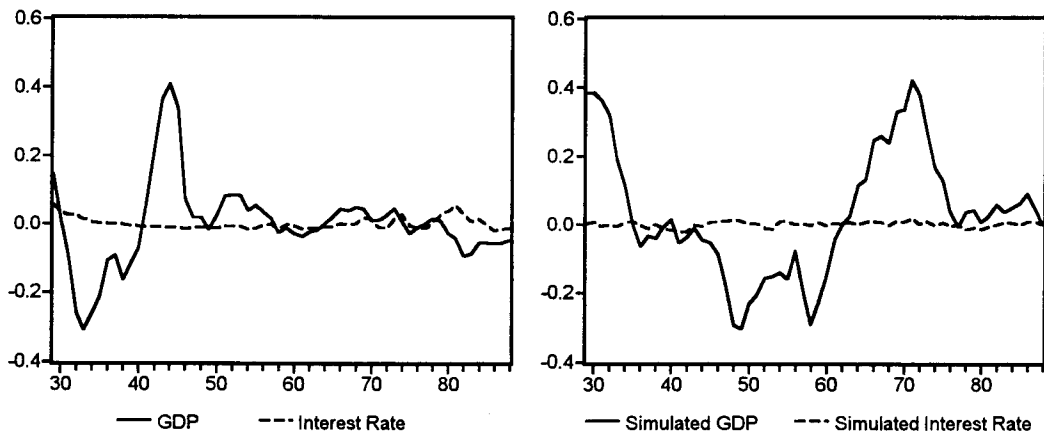
Figure 2: Consumption and GDP

Figure 3: Capital and GDP
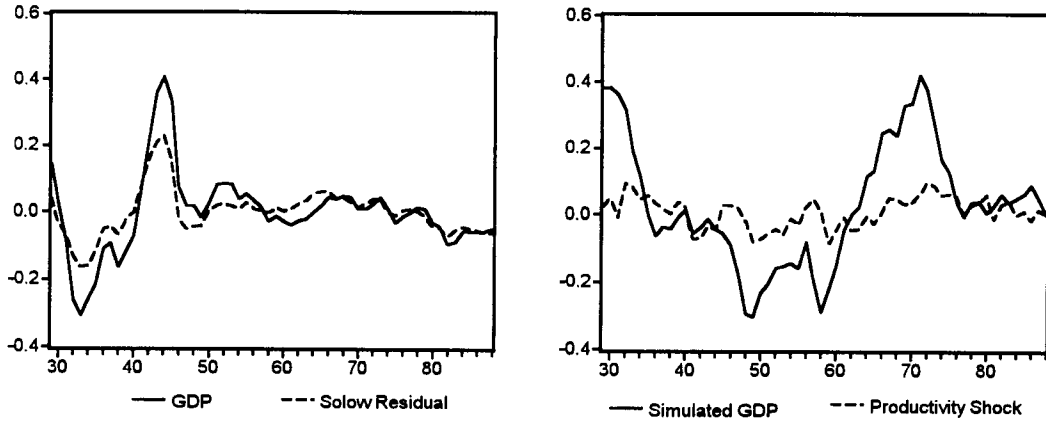

Figure 4: The Interest Rate and GDP

Figure 5: The Solow Residual, the Productivity Shock and GDP
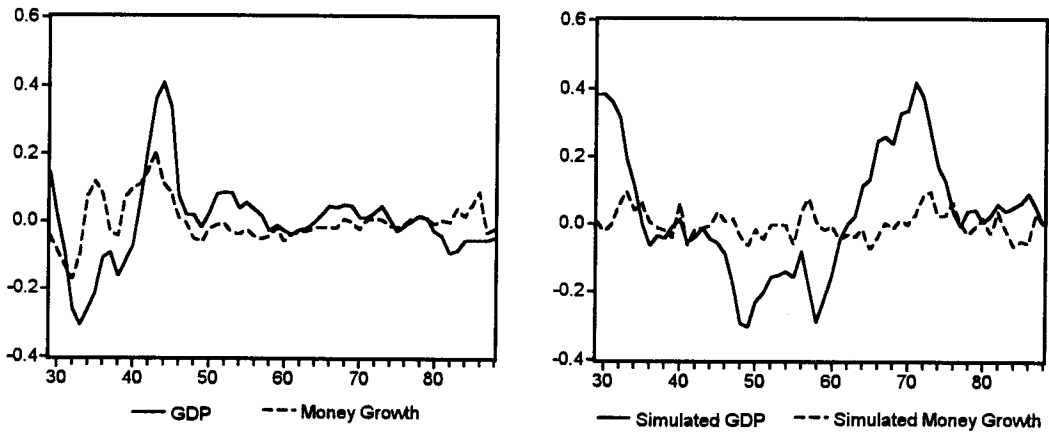

Figure 6: Money Growth and GDP
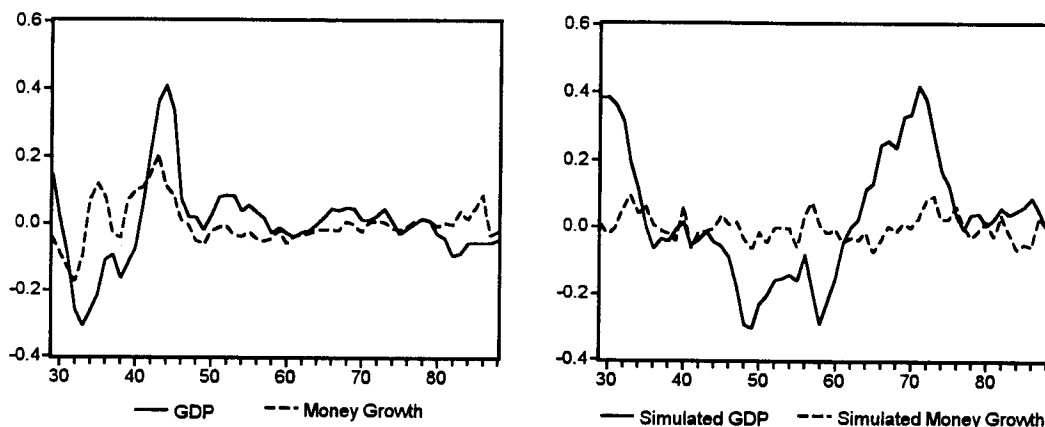

Figure 7: Employment and GDP

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Figure 8: Real Balances and GDP

Capital has the same long slow swings in the simulation that we see in the data, but it has the wrong phase and is much too volatile. Figure 3 illustrates that in the US data capital lags GDP whereas in the long swings in the simulated data it leads the cycle.

(18) Impulse Response Functions

To get a better understanding of how the dynamic properties of the simulation compare with the dynamic properties of the data I have computed the theoretical impulse response functions that would be observed in a four variable vector auto regression using data on GDP, real balances, the interest rate and money growth. I have picked one particular ordering, {Y,m,I,μ}, and I have plotted the theoretical impulse response functions computed from the model on the same figure as the estimated impulse response functions from US data. The VAR for the US used two lags of each of the variables together with a time trend and the estimated residuals from the VAR were used to compute the theoretical variance covariance matrix of the model shocks as reported in table 6.

Figures 9-12 report the results of this experiment. In each case the solid line is the estimated impulse response function and the short dashed lines represent two standard error bounds. The lines with long dashes are the theoretical impulse response functions from the model. Notice that, in every case, the theoretical impulse response functions and the estimated impulse response functions begin at the same point; this is by construction since I have chosen the variance covariance matrix of the shocks in such a way that the model generates residuals for the simulated data that has exactly the estimated variance covariance matrix as the data. Seven of the sixteen model impulse response functions lie within the two standard error bounds from the data in every year, a further four lie within these bounds seventy or eighty percent of the time. Even the impulse response functions that do relatively poorly have the right qualitative
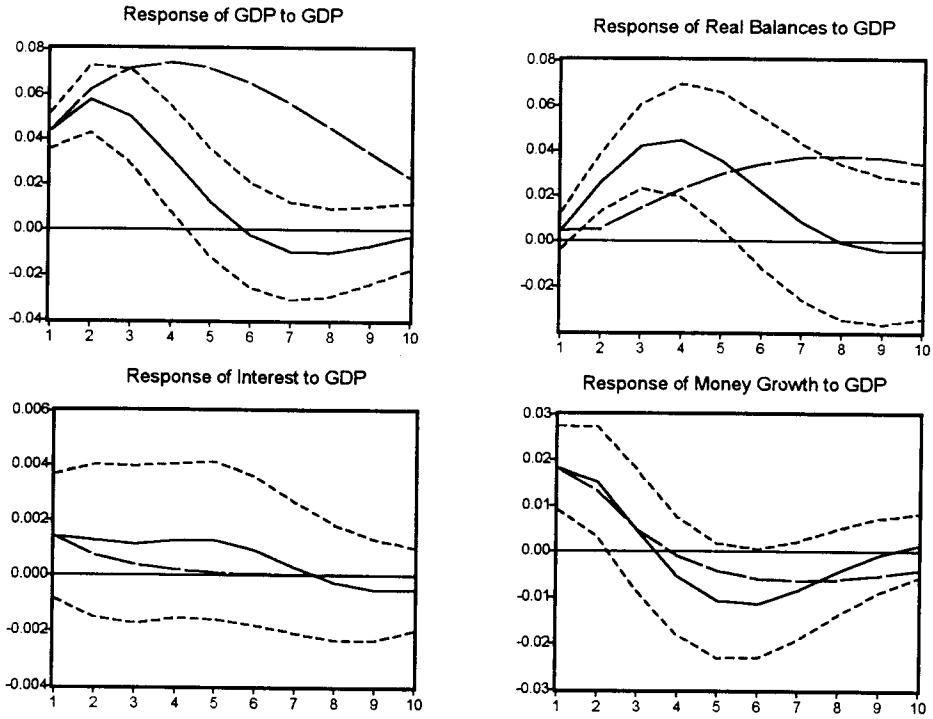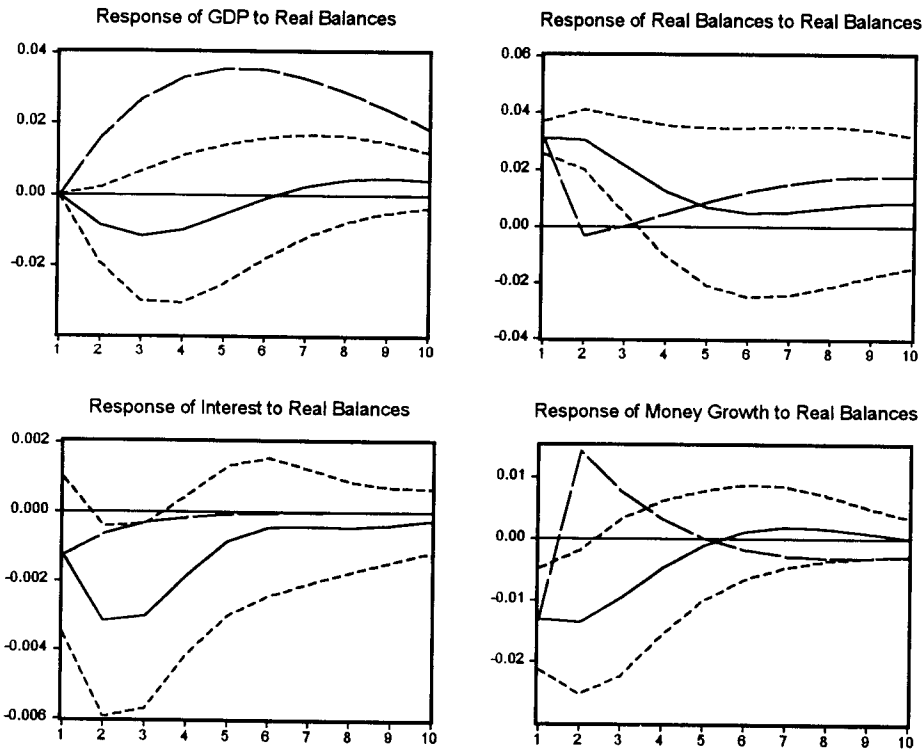
29

Response of GDP to GDP
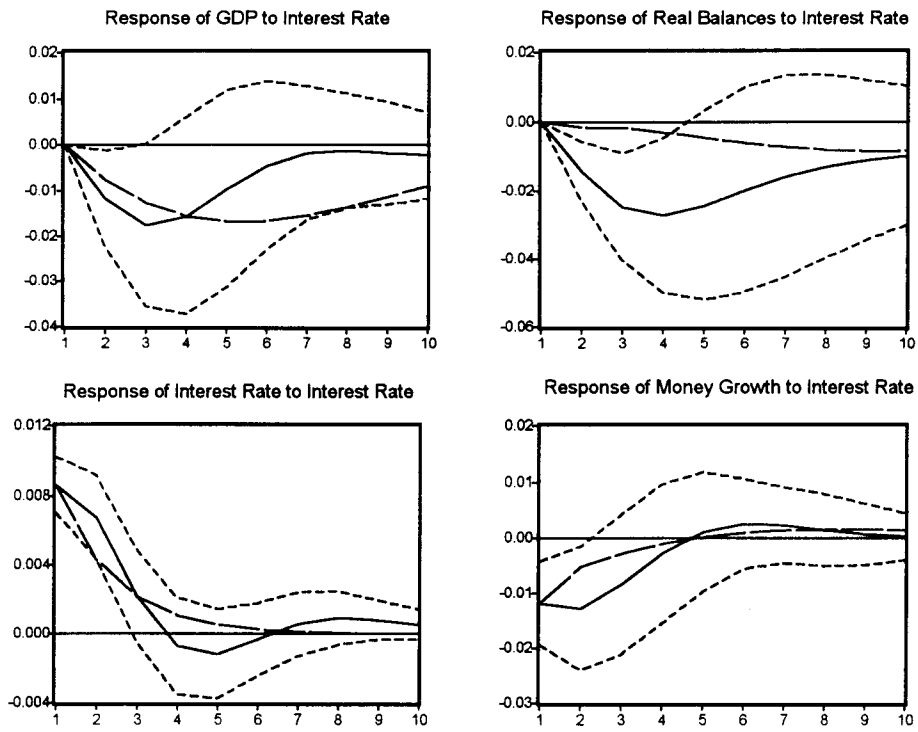
Response of Real Balances to GDP

Response of Interest to GDP

Response of Money Growth to GDP

Figure 9

Response of GDP to Real Balances

Response of Real Balances to Real Balances

Response of Interest to Real Balances

Response of Money Growth to Real Balances

Figure 10:

30

Figure 11:

Response of GDP to Interest Rate

Response of Real Balances to Interest Rate

Response of Interest Rate to Interest Rate

Response of Money Growth to Interest Rate

Figure 11:

Figure 12:

Response of GDP to Money Growth

Response of Real Balances to Money Growth

Response of Interest Rate to Money Growth
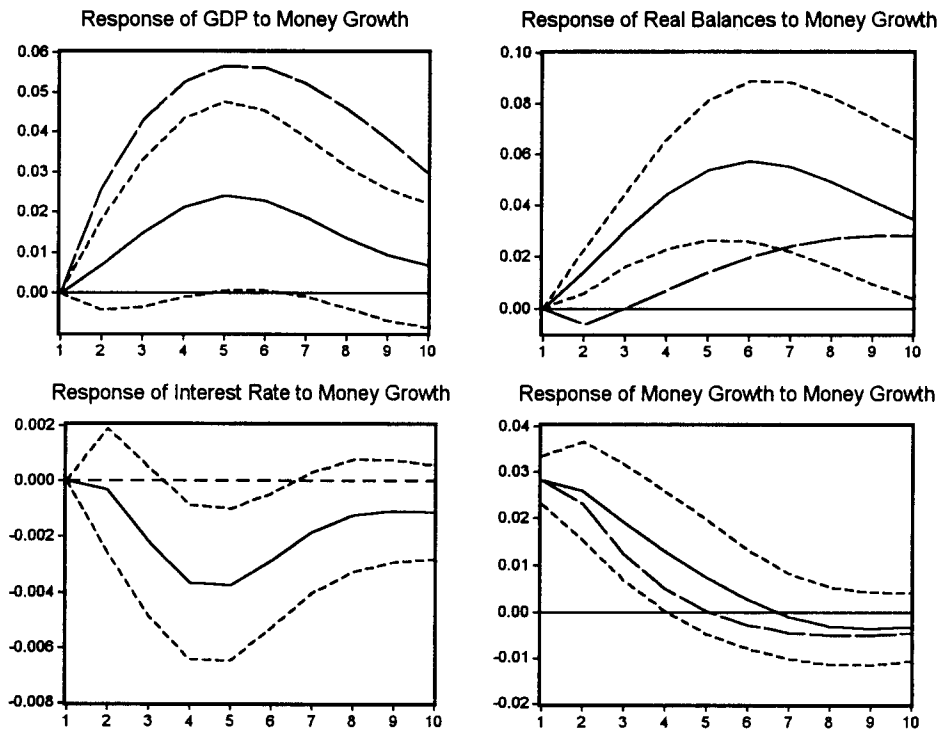
Response of Money Growth to Money Growth

Figure 12:

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shape. Consider for example, the response of GDP to a GDP shock; the top left panel of figure 9. In this graph the simulated impulse response function lies outside of the two standard error bounds in every year; Nevertheless; the broad qualitative picture is correct; GDP increases slowly to peak and then cycles back to the balanced growth path. In my experiments it proved possible to choose parameters in a way that would match the impulse response to a GDP shock perfectly by lowering the parameters "a" and "ρ". The cost of this, however, was to make the fit of the impulse response functions to money growth and to real balances much worse. The numbers that I reported in the final simulation were a compromise that was designed to minimize an implicit metric that paid attention to all four sets of graphs.

(19) Conclusion

In earlier work I have argued that general equilibrium models with indeterminate equilibria may be useful tools to help us to understand the mechanism by which real and nominal shocks are propagated over the business cycle. In this paper I have provided some quantitative support for this assertion by constructing a complete general equilibrium model and by comparing the equilibrium of the model with actual data. The model does a relatively good job of capturing the long slow humped shape responses that we see in actual data; and it accomplishes this task with a minimum of assumptions.

Alternative explanations of monetary dynamics all assume some artificial barrier that prevents agents from engaging in a mutually beneficial action. An example of the kind of model I have in mind is the nominal contracting model of Taylor (1979), in which the nature of the contracts remains unexplained, or the financial constraint models of Christiano and Eichenbaum (1992) in which agents are prevented from carrying out their transactions in an order that they would prefer. All of the models in this class suffer from two significant defects. First, they do not motivate the environment of the model in a way that explains *why* agents would choose to act in the ways that they do. Second, even if one accepts the assumptions that motivate the environment these models are *still* unable to explain the persistence of nominal shocks. Although the impulse response functions that I reported in this paper occasionally stray outside of the standard error bounds of the data, they are several orders of magnitude closer to the actual data than the monetary dynamics implied by simple contracting models or by cash in advance models with a determinate steady state.

Contrast the class of models that maintain determinacy, but add an artificial barrier to market clearing, with the approach that I have taken in this paper; an approach that uses standard general equilibrium theory amended to include a role for money in a way that was first suggested by Patinkin nearly fifty years ago. By parameterizing this model in a way that allows for the possibility that beliefs may influence outcomes I

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have shown that one is able to use standard equilibrium theory to understand monetary dynamics. It is unlikely that the model in this paper will be the last word on this topic; but I hope to have convinced the reader that the approach has considerable promise.

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Derivation of the Balanced Growth Path

This section contains an algorithm for computing a steady state of the model for the case when the utility function is parameterized as in equations (19) – (22). For this choice of utility function, the steady state equations (49) and (50) take the form:

a1)
$$\frac{(1+\chi)L^{\chi}f(q\,*)^{1-\rho}}{g(q\,*)^{1-\rho}+L^{1+\chi}(\rho-1)f(q\,*)^{\lambda-\rho}(1-b)} = \frac{\alpha y\,*}{L\,*c\,*},$$

a2)
$$\frac{ag(q\,*)^{\lambda-\rho}q\,*^{\lambda}+(\rho-1)L\,*^{1+\chi}f(q\,*)^{\lambda-\rho}q\,*^{\lambda}b}{g(q\,*)^{\lambda-\rho}(1-a)+L\,*^{1+\chi}(\rho-1)(1-b)f(q\,*)^{\lambda-\rho}} = \bar{I},$$

where the variable q* is defined as

a3)
$$q\,* = \frac{\mu\,*m\,*}{c\,*}$$

and the functions f( ) and g( ) are defined as:

a4)
$$\frac{W(c^{*},\mu\,*m\,*)}{c\,*} = f(q\,*) = \left[(1-b)+bq\,*^{\lambda-1}\right]^{\frac{1}{1-\lambda}},$$

a5)
$$\frac{X(c^{*},\mu\,*m\,*)}{c\,*} = g(q\,*) = \left[(1-a)+a\,*q\,*^{\lambda-1}\right]^{\frac{1}{1-\lambda}}.$$

The following steps exploit the structure of equations a1) and a2) to demonstrate the existence of a balanced growth path.

Step I: Use equation (52) to define $\mu*$

a6)
$$\mu\,* = \frac{\beta\gamma^{\rho}}{(1-\bar{I})}.$$

Step II: Define the steady state ratio of output to capital from equation (53);

a7)
$$\frac{y\,*}{k\,*} = \left(\frac{\gamma^{-\rho}}{\beta}-1+\delta\right)\frac{1}{(1-\alpha)}.$$

Step III: Use equation (51) to compute the steady state ratio of consumption to capital.

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a8) $\dfrac{c*}{k*} = \dfrac{y*}{k*} + 1 - \gamma - \delta$.

Equations a7) and a8) also imply a value for the steady state consumption-GDP ratio:

a9) $\dfrac{c*}{y*} = \left(\dfrac{c*}{k*}\right)\left(\dfrac{k*}{y*}\right)$.

Step IV: For given y* and c*, equations a1) and a2) are a pair of simultaneous equations in L* and q*. Rearranging a1) it follows that:

a10) $L* = \left\{ \dfrac{g(q*)^{\lambda-\rho}\alpha(y*/c*)}{\left[(1+\chi)f(q*)^{1-\lambda} - (\rho-1)(1-b)\alpha(y*/c*)\right]f(q*)^{\lambda-\rho}} \right\}^{\frac{1}{1+\chi}}$.

Substituting a10) in a2) and rearranging leads to:

a11) $H(q*) = q*^{\lambda}\dfrac{a}{1-a} + q*^{\lambda}f(q*)^{\lambda-1}\dfrac{\alpha(\rho-1)(y*/c*)}{(1+\chi)(1-a)}[b-a] = \bar{I}$.

In the following analysis we treat only the case $\lambda > 1$. For this case it follows from the properties of f( ) that $H(0) = 0$, $\lim_{q\to\infty} = \infty$ and further, H(q) is monotonically increasing for positive q. It follows that there exists a unique value q* for positive $\bar{I}$. The value of L* is then given by equation a10).

Step V: Since H(q*) is monotonic it is invertible and one can express q* as a function $H^{-1}(\bar{I})$. One can then solve for the velocity of circulation:

a12) $\dfrac{y*}{\mu*m*} = \dfrac{1}{H^{-1}(\bar{I})}\dfrac{y*}{c*}$.

Note that y*/μ*m* , rather than y*/m*, is equal to velocity since m* is the ratio of *lagged* money to the price.

Definition of the Co-efficients a_1–a_{18}

The following equations define the coefficients of the linearized equations (64) – (70) in terms of the parameters of the utility function. The elements of δ and D are all evaluated at the steady state $\{c*, y*, m*, L*, \mu*, k*\}$.

$a_1 = \delta_{Lm} - \delta_{cm}$ $\qquad a_2 = 1 - (\delta_{Lm} - \delta_{cm})$ $\qquad a_3 = 1 + (\delta_{LL} - \delta_{cL})$

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$$a_4 = \left(\delta_{mm} - \delta_{cm}\right) \qquad a_5 = \left(\delta_{mL} - \delta_{cL}\right) \qquad a_6 = \frac{c*}{\gamma k*}$$

$$a_7 = -\frac{(1-\delta)}{\gamma} \qquad a_8 = -\frac{y*}{\gamma k*} \qquad a_9 = \rho + \delta_{cm}$$

$$a_{10} = -\left(1+\delta_{cm}\right) \qquad a_{11} = -\delta_{cm} \qquad a_{12} = -\delta_{cL}$$

$$a_{13} = 1 + \delta_{cm} \qquad a_{14} = -\frac{\overline{I}}{1-\overline{I}} \qquad a_{15} = \rho + \delta_{cm}$$

$$a_{16} = -\delta_{cm} \qquad a_{17} = \delta_{cL} \qquad a_{18} = -\frac{\beta(1-\alpha)y*}{\gamma^{-\rho}k*}$$

Definitions of the Matrices A_1–A_9

This section writes out each of the terms in equations (71) and (72) in full. These definitions are to assist in constructing the Gauss code to compute an equilibrium numerically.

$$A_1 X_t = \begin{bmatrix} 1 & -\alpha & 0 \\ -1 & a_3 & a_1 \\ 0 & a_4 & a_4 \end{bmatrix} \begin{bmatrix} \tilde{y}_t \\ \tilde{L}_t \\ \tilde{m}_t \end{bmatrix}, \qquad A_2 Z_t = \begin{bmatrix} 0 & -(1-\alpha) & 0 & -1 & 0 \\ a_2 & 0 & 0 & 0 & a_1 \\ -a_4 & 0 & -1 & 0 & a_4 \end{bmatrix} \begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{I}_t \\ \tilde{V}_t \\ \tilde{\mu}_t \end{bmatrix},$$

$$A_3 X_{t+1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{12} & a_{10} \\ a_{18} & a_{17} & a_{16} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{y}_{t+1} \\ \tilde{L}_{t+1} \\ \tilde{m}_{t+1} \end{bmatrix}, \qquad A_4 Z_{t+1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ a_9 & 0 & 0 & 0 & a_{11} \\ a_{15} & -a_{18} & 0 & 0 & a_{16} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \\ \tilde{I}_{t+1} \\ \tilde{V}_{t+1} \\ \tilde{\mu}_{t+1} \end{bmatrix},$$

$$A_5 X_t = \begin{bmatrix} a_8 & 0 & 0 \\ 0 & -a_{12} & -a_{10} \\ 0 & -a_{17} & -a_{16} \\ 0 & 0 & 0 \\ a_{19} & a_{20} & a_{21} \end{bmatrix} \begin{bmatrix} \tilde{y}_t \\ \tilde{L}_t \\ \tilde{m}_t \end{bmatrix}, \qquad A_6 Z_t = \begin{bmatrix} a_6 & a_7 & 0 & 0 & 0 \\ -a_9 & 0 & a_{14} & 0 & a_{13} \\ -a_{15} & 0 & 0 & 0 & -a_{16} \\ 0 & 0 & 0 & -\zeta & 0 \\ a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \end{bmatrix} \begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{I}_t \\ \tilde{V}_t \\ \tilde{\mu}_t \end{bmatrix},$$

$$A_7 U_{t+1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u^1_{t+1} \\ u^2_{t+1} \end{bmatrix},$$

$$A_8 \varepsilon^X_{t+1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -a_{12} & -a_{10} \\ -a_{18} & -a_{17} & -a_{16} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon^y_{t+1} \\ \varepsilon^L_{t+1} \\ \varepsilon^m_{t+1} \end{bmatrix},$$

$$A_9 \varepsilon^Z_{t+1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -a_9 & 0 & 0 & 0 & -a_{11} \\ -a_{15} & a_{18} & 0 & 0 & -a_{16} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon^c_{t+1} \\ \varepsilon^k_{t+1} \\ \varepsilon^I_{t+1} \\ \varepsilon^V_{t+1} \\ \varepsilon^\mu_{t+1} \end{bmatrix}$$

Defining the Matrices J₄ and J₅.

The equations (60) – (65) in matrix form can be written as:

a13) $H_1 \varepsilon^Z_{t+1} + H_2 \varepsilon^X_{t+1} = H_3 u_{t+1} + H_4 e_{t+1}$.

Each of these terms is written explicitly below:

$$H_1 \varepsilon^Z_{t+1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ a_9 & 0 & 0 & 0 & a_{11} \\ a_{15} & a_{18} & 0 & 0 & a_{16} \end{bmatrix} \begin{bmatrix} \varepsilon^c_{t+1} \\ \varepsilon^k_{t+1} \\ \varepsilon^I_{t+1} \\ \varepsilon^V_{t+1} \\ \varepsilon^\mu_{t+1} \end{bmatrix}$$

$$H_2 \varepsilon^X_{t+1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{12} & a_{10} \\ a_{18} & a_{17} & a_{16} \end{bmatrix} \begin{bmatrix} \varepsilon^y_{t+1} \\ \varepsilon^L_{t+1} \\ \varepsilon^m_{t+1} \end{bmatrix}$$

$$H_3 U_{t+1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u^1_{t+1} \\ u^2_{t+1} \end{bmatrix},$$

$$H_4 e_{t+1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^1_{t+1} \\ e^2_{t+1} \end{bmatrix}.$$

It follows that:

a14) $\varepsilon^Z_{t+1} = H_5 U_{t+1} + H_6 e_{t+1}$,   where H₅ and H₆ are defined below:

a15) $H_5 = (H_1 + H_2 m)^{-1} H_3$,   a16) $H_6 = (H_1 + H_2 m)^{-1} H_4$.

Replacing a14) in (75) one arrives at:

a17) $Z_{t+1} = J_1 Z_t + J_4 u_{t+1} + J_5 e_{t+1}$

which is equation (83) on page 17. The matrices J_4 and J_5 are given by equations a18) and a19)

a18) $J_4 = J_2 + J_3 H_5$,   a19) $J_5 = J_3 H_6$.

Definition of the Matrices D_1, D_2, D_3 and D_4

This section defines the matrices D_1-D_4 and derives the relationship between the VCV matrix of the residuals of a VAR on the variables Y,i,μ and M/P and the VCV matrix of the underlying disturbances vector $\{u,e\}$. First define the matrices D_1 and D_2 as follows. These matrices select the variables for the VAR according to the linear equation:

a20) $s_t = D_1 Z_t + D_2 X_t$.

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

But since X is a linear function of Z we can use the definition of the matrix M from equation (73) to write s_t as a function solely of Z_t:

a21) $s_t = D_3 Z_t$, $D_3 = D_1 + D_2 M$.

Define the VCV matrix of $\{u,e\}$ as Ω. The innovations in Z are equal to:

a22) $Z_t - E[Z_t] = J_6 \begin{bmatrix} u_t \\ e_t \end{bmatrix}$

where the matrix J_6 is the 4x4 matrix obtained by concatenating the matrices J_4 and J_5 given in a18) and a19). Using this notation, the innovations in s_t can be written as a linear function of the disturbances $\{u,e\}$:

a23) $s_t - E[s_t] = D_3 \left(Z_t - E[Z_t] \right) = D_3 J_6 \begin{bmatrix} u_t \\ e_t \end{bmatrix}$.

The matrix D_4 referred to in equation (94) is then defined as: $D_4 = D_3 J_6$.