

Clubs and the Market: Continuum Economies

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Abstract

This paper defines a general equilibrium model with exchange and club formation. Agents trade multiple private goods widely in the market, can belong to several clubs, and care about the characteristics of the other members of their clubs. The space of agents is a continuum, but clubs are finite. It is shown that (i) competitive equilibria exist, and (ii) the core coincides with the set of equilibrium states. The central subtlety is in modeling club memberships and expressing the notion that membership choices are consistent across the population.

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1 Introduction

Consumption is typically a social activity. The company we keep affects our demand for private goods, and our consumption of private goods affects the company we seek. General equilibrium theory in the tradition of Arrow and Debreu focuses on the anonymous interactions of consumers with the market, largely ignoring the social aspect of consumption. Club theory in the tradition of Buchanan, on the other hand, focuses on the social activity of consumption, largely ignoring the anonymous interactions of individuals with the market. The principal purpose of this paper, and also of our (1997) companion paper, is to integrate club theory and general equilibrium theory, constructing a framework which incorporates widespread trading of private goods in competitive markets and individual consumption in small groups chosen voluntarily in equilibrium. This paper treats continuum economies and the companion paper treats large finite economies.

Cornes and Sandler (1986, p. 159) define a club as “... a voluntary group deriving mutual benefit from sharing ... production costs, the members’ characteristics, or a good characterized by excludable benefits.” Following Tiebout (1956), one tradition in the literature focuses on clubs as political jurisdictions, assumes that each agent can belong to at most one jurisdiction and takes a partition into jurisdictions as part of the basic description of a feasible state of the economy.¹ A different tradition, following an idea of Buchanan (1965), focuses on small clubs: a marriage, a gym, an academic department, a golfing foursome, or the clientele of a restaurant. When clubs are not to be thought of as political jurisdictions, we see no reason why each agent should belong to a single club nor why the club structure should be (or induce) a partition. In keeping with this view, we build here a framework in which each agent may belong to several clubs (partitions are a special case).

¹See for example Berglas (1976), Berglas and Pines (1981), Bewley (1981), Brueckner (1994), Conley and Wooders (1994), Gilles and Scotchmer (1997), Greenberg and Weber (1986), Greenberg and Shitovitz (1988) Ellickson (1973, 1979), Konishi (1996), Scotchmer (1985a, 1985b, 1994, 1996), Scotchmer and Wooders (1987a, 1987b), Wooders (1978, 1989).

Our work builds on a long tradition in the club literature, beginning with Buchanan (1965), that seeks to demonstrate that club activities can be interpreted as competitive, for example, Gilles and Scotchmer (1997), who studied replica economies. In keeping with the tradition in general equilibrium theory that perfect competition is best demonstrated in the continuum, we build a model in which the space of agents is a continuum, but we restrict clubs to be finite. Thus, as suggested by Buchanan (1965), clubs are “small” compared to society. In the continuum framework, the “integer problem” and other non-convexities disappear. As a result, we establish the existence of equilibrium and verify a fundamental test of perfect competition, the coincidence of the core with the set of equilibrium states. Central to our work is that we view clubs and club memberships as primitives on equal footing with more conventional primitives of general equilibrium theory. This view leads to a fuller integration of club theory into general equilibrium theory, and to a more general interpretation of clubs. Other papers in the same spirit include Makowski (1978) and Cole and Prescott (1994).

We describe a (type of) club as a pair consisting of a description of the external characteristics of its members and a specified activity; thus we follow Ellickson (1979) and Mas-Colell (1981) as viewing the activity of a club as a *public project* rather than as provision of some level of a public good. A club membership is an opening in a club available to agents with specified characteristics. Agents choose both private goods and club memberships, and private goods and club memberships are treated and priced in parallel fashion.

Despite the parallel treatment of club memberships and private goods, there are important differences from exchange economies. First, club memberships are indivisible. Cole and Prescott (1994) deal with this indivisibility by viewing the objects of choice as lotteries on private goods and club memberships. It seems to us that the indivisibility of club memberships is central to understanding clubs, and we prefer to address it directly.

Second, club membership choices must be consistent across the population. If a third of the population are women married to men, for example,

then a third of the population must be men married to women. Consistency must hold simultaneously for all types of clubs, and allow for the possibility that every individual may belong to several clubs.

Finally, there is an important difference in the pricing of private goods and of club memberships: private good prices must be positive, but club membership prices may be positive, negative or zero.

Our proofs follow lines that are typical of general equilibrium theory, but there are many subtleties. The central subtlety is in accommodating the club consistency condition, which has no analog in exchange economies.

Our proof of equivalence of the core with the set of equilibrium states follows an outline parallel to Schmeidler's (1969) proof of Aumann's (1964) core equivalence theorem: Begin with a core state, construct individual net preferred sets in the space of private goods and club memberships, and then an aggregate net preferred set. Use the Lyapunov convexity theorem to show that the aggregate net preferred set is convex and the core property of the given allocation to show that the aggregate net preferred set is disjoint from an appropriate cone. Obtain equilibrium prices by separating this aggregate net preferred set from an appropriate cone. Because we work in the space of private goods and club memberships, however, our argument differs from Schmeidler's, and we must work much harder to be certain that the private good prices we construct are not zero. And we must restrict the space of club memberships to accommodate the matching property.²

Our proof that equilibrium exists also follows a familiar outline: Construct an excess demand correspondence in the space of private goods and club memberships. Use the Lyapunov convexity theorem to show that this correspondence is convex valued. Apply Kakutani's fixed point theorem to find a zero. Because we work in the space of private goods and club memberships, however, and club membership prices may be positive, negative or

²In contrast to the proof outlined above, decentralization is usually accomplished in the club literature by first constructing prices for private goods and then defining prices for club memberships in terms of willingness to pay. Because we allow agents to belong to more than one club, the sequential construction does not work.

zero, there is no natural price domain on which to work. We must therefore work in perturbations of the original economy which have the property that equilibrium prices are known *a priori* to lie in some compact set, construct equilibria for these perturbed economies, show that equilibrium prices for the perturbed economies can be chosen to be bounded, and take limits as we relax the perturbations.

Following this Introduction, Section 2 provides some motivating examples. The formal model is described in Section 3. Section 4 establishes the first welfare theorem and Section 5 shows that the second welfare theorem may fail. This is not surprising in finite economies, but our examples show that it may fail even in atomless economies. Section 6 establishes the equivalence of the core and the set of equilibrium states and Section 7 establishes the existence of equilibrium. The text outlines the main proofs; details are collected in Section 8.

2 Examples

In this section we present four examples illustrating various aspects of competitive equilibrium in a club economy with a continuum of agents. The first example, a version of a familiar crowding story, illustrates the nature of competitive equilibrium in a setting where the composition of club memberships does not matter.

Example 2.1 Crowding

Consider an economy with a continuum of consumers uniformly distributed on $[0, 10]$ and a single private good. The endowment of consumer k is $e_k = k$. In addition to the private good, consumers have the option of using a swimming pool which they can enjoy alone or in a club. All consumers have the same preferences: a consumer who consumes x units of the private good derives utility $u(x; 0) = x$ if using no pool and $u(x; n) = 4x/n$ if she belongs to a swimming pool club with n members. (We assume a consumer can belong to at most one such club.) Building a swimming pool requires an input of 6 units of the private good.

Although swimming pool clubs could in principle be arbitrarily large, in equilibrium there will be no clubs of size greater than 4. Since consumers care only about the number and not other characteristics of fellow pool club members, all consumers belonging to the same club share equally in its cost. Normalizing the price of the private good to one, the price of a membership is $q_n = 6/n$ for $n = 2, 3$ or 4 . (Consistent with this formula, a swimming pool costs 6, but we prefer to treat singleton “clubs” separately.) The normalization also implies that consumer k has wealth k . Choosing no pool yields utility k and enjoying a pool by herself yields utility $4(k - 6)$. After paying her share of the cost, sharing a pool in a club with n members yields utility

$$u(k - q_n; n) = \begin{cases} 2(k - 3) & \text{if } n = 2; \\ \frac{4}{3}(k - 2) & \text{if } n = 3; \\ k - \frac{3}{2} & \text{if } n = 4. \end{cases}$$

Solving for the equilibrium choices of individuals is easy: the wealthiest consumers, with wealth $k \in (9, 10]$, have a pool of their own; consumers with wealth $k \in (6, 9]$ share a pool with one other person; and the poorest consumers, with wealth $k \in [0, 6]$, consume the private good but do not enjoy the use of a pool. Clubs of size greater than two do not form in equilibrium.



The second example, motivated by the commentary by Arrow (1972) on Becker (1957), illustrates the importance of allowing for membership prices which discriminate among types of membership (i.e., on external characteristics).

Example 2.2 Segregation

Consider an economy with a continuum of consumers uniformly distributed on $[0, 1]$: consumers in $[0, .3)$ are blue, consumers in $[\.3, 1]$ are green. There is a single private good. All consumers have endowment $e_a = 2$. In addition to the private good, duplex apartments are available. The utility of a consumer depends on his *external characteristic* (blue or green), on consumption of the private good, on whether or not housing is consumed, and on the external characteristic of the consumer with whom the housing is shared. (We assume that no consumer desires more than one unit of housing.) A blue or green consumer who consumes no housing and x units of the private good derives utility

$$u_B(x; 0) = u_G(x; 0) = x$$

Using the obvious notation for the external characteristics of the occupants of a duplex, a consumer who lives in a duplex and consumes x units of the private good derives utility

$$u_B(x; BB) = 4x \quad \text{and} \quad u_B(x; BG) = 6x$$

if blue and

$$u_G(x; GG) = 6x \quad \text{and} \quad u_G(x; BG) = 4x$$

if green. Note that a blue (respectively, green) consumer cannot consume housing in a duplex with two green (respectively, blue) consumers because there would be no space for her.

Assuming that a duplex can be produced using two units of the private good and that race-discriminatory pricing is possible, the prices blue or green consumers pay for segregated and integrated duplexes must satisfy:

$$\begin{aligned} 2q_B(BB) &= 2 \\ 2q_G(GG) &= 2 \\ q_B(BG) + q_G(BG) &= 2 \end{aligned}$$

(Again the notation should be self explanatory.) At these prices, a blue or a green consumer can obtain utility 2 by choosing no housing. Alternatively, a blue consumer can obtain utility 4 by choosing a segregated duplex at price 1 or utility $6(2 - q_B(BG))$ by choosing an integrated duplex at price $q_B(BG)$. Green consumers can obtain utility 6 by choosing a segregated duplex for price 1 or utility $4(2 - q_G(BG))$ by choosing an integrated duplex for price $q_G(BG)$. In order that integrated housing be chosen at equilibrium it is necessary that $6(2 - q_B(BG)) \geq 4$ and $4(2 - q_G(BG)) \geq 6$ or, equivalently, $q_B(BG) \leq 4/3$ and $q_G(BG) \leq 1/2$ whence $q_B(BG) + q_G(BG) \leq \frac{11}{6}$. However, we already know that $q_B(BG) + q_G(BG) = 2$, so no integrated housing will be chosen at equilibrium. Equilibrium prices for segregated housing are $q_B(BB) = q_G(GG) = 1$ while equilibrium prices for integrated housing are indeterminate, constrained only by the requirements

$$q_B(BG) \geq \frac{4}{3}, \quad q_G(BG) \geq \frac{1}{2}, \quad q_B(BG) + q_G(BG) = 2$$

At equilibrium, all consumers choose segregated housing.

Suppose the government offers a subsidy $s > 0$ for integrated housing, reducing its price to $2 - s$. Equilibrium prices must then satisfy

$$\begin{aligned} 2q_B(BB) &= 2 \\ 2q_G(GG) &= 2 \\ q_B(BG) + q_G(BG) &= 2 - s \end{aligned}$$

In order that integrated housing be chosen at equilibrium it remains necessary that $q_B(BG) \leq 4/3$ and $q_G(BG) \leq 1/2$. Because $q_B(BG) + q_G(BG) = 2 - s$, integration is possible only when $s \geq 1/6$.

Suppose that $1/6 \leq s < 1$. The number of green consumers choosing integrated housing must equal the number of blue consumers choosing integrated housing. Because there are more green consumers than blue consumers, some green consumers *must* choose segregated housing. Because all green consumers must enjoy the same equilibrium utility, it follows that $q_G(BG) = 1/2$ and hence $q_B(BG) = 3/2 - s > 1/2$. In equilibrium all blue consumers and $3/7$ of the green consumers will choose integrated housing; the remaining green consumers will choose segregated housing.³

Thus the government subsidy achieves integration — *but only if housing prices are discriminatory*: $q_B(BG) > 1/2 = q_G(BG)$. In order to achieve integration with *non-discriminatory prices*, the government must raise the subsidy to $s = 1$.⁴ ♣

Our third example illustrates some of the subtleties inherent in allowing for membership in several clubs.

Example 2.3 Monogamy, Polygamy and Group Marriage

Consider an economy comprised of a continuum of consumers uniformly distributed on $[0, 11]$; consumers in $[0, 6)$ are male, consumers in $[6, 11]$ are female. There is a single consumption good; endowments are

$$e_a = \begin{cases} a & \text{if } 0 \leq a < 6 \\ 1 & \text{if } 6 \leq a \leq 11 \end{cases}$$

(Thus, male endowments are uniformly distributed between 0 and 6; female endowments are identically 1.) In addition to consuming the private good, individuals may enter into several kinds of marriage: exclusive monogamy (1

³If $s = 1/6$ equilibrium choices are indeterminate, constrained only by the requirement that the same number of blue consumers and green consumers choose integrated housing.

⁴We leave it to the reader to examine the welfare consequences of various methods by which the government could tax individuals to provide the necessary subsidy.

male and 1 female, symbolized m_e); non-exclusive monogamy (1 male and 1 female, symbolized m_{ne}); and a group marriage (1 male and 2 females, symbolized m_g). Males have the option of belonging to one or two non-exclusive marriages; females can be in only one. (We could incorporate these restrictions into consumption sets or into preferences.) A consumer consuming x units of the private good derives utility according to sex and marital status shown in the following table:

	single	m_e	m_{ne}	$2m_{ne}$	m_g
M	x	$3x/2$	$3x/2$	$3x$	$15x/4$
F	x	$9x$	$8x$		$36x/5$

Note that the choice of two non-exclusive marriages and the choice of a group marriage are distinct and males prefer the latter (holding consumption fixed).

We assume that all marriages are costless activities. Hence, if $q_M(m_e)$, $q_M(m_{ne})$, $q_M(m_g)$ are the prices paid by males to enter an exclusive, non-exclusive or group marriage, respectively, and $q_F(m_e)$, $q_F(m_{ne})$, $q_F(m_g)$ are the corresponding prices paid by females, it follows that:

$$q_M(m_e) + q_F(m_e) = 0 \quad (1)$$

$$q_M(m_{ne}) + q_F(m_{ne}) = 0 \quad (2)$$

$$q_M(m_g) + 2q_F(m_g) = 0 \quad (3)$$

To determine the equilibrium, note first that, because both males and females find any form of marriage preferable to being single, having both unmarried males and unmarried females would contradict the Pareto optimality of an equilibrium. Because males outnumber females, some males must necessarily be single and hence all females must be married at equilibrium. Write a, b, c, d for the fraction of males choosing 1 exclusive marriage, 1 non-exclusive marriage, 2 non-exclusive marriages and 1 group marriage, respectively. Keeping in mind that males who choose two non-exclusive marriages or one group marriage are involved with two females, it follows that

$$a + b + 2c + 2d = 5 \quad (4)$$

Because non-exclusive marriage is less desirable than exclusive marriage for females, non-exclusive marriage will be more expensive at equilibrium. Because males find one non-exclusive or one exclusive marriage to be perfect substitutes, no men will choose one non-exclusive marriage. Therefore,

$$b = 0 \quad (5)$$

It is evident that wealthier males choose more desirable marriage arrangements. Thus, males in $[6 - d, 6]$ choose 1 group marriage, males in $[6 - c - d, 6 - d]$ choose 2 non-exclusive marriages, males in $[6 - a - c - d, 6 - c - d]$ choose 1 exclusive marriage, and the remaining males choose no marriage at all. Because male $6 - d$ must be exactly indifferent between choosing 1 group marriage or 2 non-exclusive marriages, it follows that:

$$3(6 - d - 2q_M(m_{ne})) = \frac{15}{4}(6 - d - q_M(m_g)) \quad (6)$$

Similarly, the indifference of male $6 - c - d$ between choosing 2 non-exclusive marriages or 1 exclusive marriage implies

$$\frac{3}{2}(6 - c - d - q_M(m_e)) = 3(6 - c - d - 2q_M(m_{ne})) \quad (7)$$

and the indifference of male $6 - a - c - d$ between choosing 1 exclusive marriage or remaining single implies

$$6 - a - c - d = \frac{3}{2}(6 - a - c - d - q_M(m_e)) \quad (8)$$

Because all females are identical, their equilibrium utility is independent of marital state. Consequently,

$$9(1 - q_F(m_e)) = 8(1 - q_F(m_{ne})) = \frac{36}{5}(1 - q_F(m_g)) \quad (9)$$

Solving equations (1)–(9) yields

$$a = 1 \quad b = 0 \quad c = 1 \quad d = 1$$

$$q_M(m_e) = 1 \quad q_M(m_{ne}) = \frac{5}{4} \quad q_M(m_g) = 3$$

and

$$q_F(m_e) = -1 \quad q_F(m_{ne}) = -\frac{5}{4} \quad q_F(m_g) = -\frac{3}{2}$$

Thus, at equilibrium, the wealthiest $\frac{1}{6}$ of males enter into group marriage, the next wealthiest $\frac{1}{6}$ enter into 2 non-exclusive marriages, the next wealthiest $\frac{1}{6}$ enter into 1 exclusive marriage, and the poorest $\frac{1}{2}$ of males remain single. ♣

To this point, all of our examples have involved a single private good, so that there is no trade between clubs. Our final example shows that the interaction between the demand for club memberships and the demand for private goods can have profound and unexpected consequences when there are multiple private goods.

Example 2.4 Marriage and the Market

Consider an economy with a continuum of consumers uniformly distributed on $[0, 1]$; consumers in $[0, \beta)$ are male, consumers in $[\beta, 1]$ are female. There are 2 private goods and each consumer has endowment $e_a = (10, 10)$. For consumers remaining single,

$$u_M(x_1, x_2; 0) = x_1 \quad \text{and} \quad u_F(x_1, x_2; 0) = x_2$$

while for those who marry

$$u_M(x_1, x_2; m) = u_F(x_1, x_2; m) = \frac{5}{2}\sqrt{x_1 x_2}$$

Write q_M, q_F for the marriage prices paid by males and females, respectively. (Because sex is the only characteristic that matters to others, we assume that all males pay the same price and all females pay the same price.) Because marriage is costless, marriage prices $q_M + q_F = 0$ (i.e., one sex bribes the other to be married).

To solve for equilibrium, it is convenient to work backwards from a hypothesized distribution of marriages and single individuals. To give the reader a flavor of the solution, consider a hypothetical distribution in which all males are married and some females remain unmarried; of course this requires $\beta < 1/2$.

Normalizing so that private good prices sum to 1, all individuals have wealth 10. Unmarried females spend all their wealth on good 2. Married females spend q_F and married males q_M to enter a marriage, each spending half of their remaining income on each of the private goods. Consequently, the market clearing conditions for x_1 and x_2 become:

$$\begin{aligned} \beta \left[\frac{10 - q_M}{2p_1} + \frac{10 - q_F}{2p_1} \right] &= 10 \\ (1 - 2\beta) \frac{10}{p_2} + \beta \left[\frac{10 - q_M}{2p_2} + \frac{10 - q_F}{2p_2} \right] &= 10 \end{aligned}$$

Solving yields $p_1 = \beta$ and $p_2 = 1 - \beta$.

To solve for marriage prices, keep in mind that unmarried females and married females must obtain the same equilibrium utilities and that males, all married, must obtain at least as much utility in the marriage as they would if they were single. These considerations lead to:

$$\frac{10}{1 - \beta} = \frac{5}{2} \sqrt{\left(\frac{10 - q_F}{2\beta} \right) \left(\frac{10 - q_F}{2(1 - \beta)} \right)}$$

and

$$\frac{10}{\beta} \leq \frac{5}{2} \sqrt{\left(\frac{10 - q_M}{2\beta} \right) \left(\frac{10 - q_M}{2(1 - \beta)} \right)}$$

Solving yields

$$q_F = 10 - 8\sqrt{\frac{\beta}{1 - \beta}}$$

and

$$q_M \leq 10 - 8\sqrt{\frac{1 - \beta}{\beta}}$$

Because $q_M + q_F = 0$, this entails $\beta \geq 1/5$ and hence $1/5 \leq \beta < 1/2$.

Proceeding in similar fashion for other possible distributions of marriages and single individuals, we can work out the equilibrium correspondence as β varies from 0 to 1. As in the case above, private good prices vary linearly with β :

$$p_1 = \beta \quad \text{and} \quad p_2 = 1 - \beta$$

The proportion of married males is somewhat more complex:

$$m = \begin{cases} 0 & \text{if } 0 \leq \beta < 1/2; \\ [0, \beta] & \text{if } \beta = 1/5; \\ \beta & \text{if } 1/5 < \beta \leq 1/2; \\ 1 - \beta & \text{if } 1/2 < \beta < 4/5; \\ [0, 1 - \beta] & \text{if } \beta = 4/5; \\ 0 & \text{if } 4/5 < \beta \leq 1. \end{cases}$$

(If $\beta = 1/5$ or $4/5$, the proportion of married males is indeterminate.) Note that, when the males and females are too far out of balance, marriage is priced out of existence! The price q_F females pay to be in a marriage varies with β as follows:

$$q_F = \begin{cases} 10 - 8\sqrt{\frac{\beta}{1-\beta}} & \text{if } 0 \leq \beta < 1/2; \\ [-2, 2] & \text{if } \beta = 1/2; \\ -10 + 8\sqrt{\frac{\beta}{1-\beta}} & \text{if } 1/2 \leq \beta \leq 1. \end{cases}$$

As the proportion of males increases from 0 to $1/2$, the price females pay for marriage decreases toward 2, and becomes indeterminate in the interval $[-2, 2]$ at $\beta = 1/2$. Once β exceeds $1/2$, the position of males and females is reversed: females receive the subsidy, and it increases as the proportion of males increase. ♣

3 Club Economies

In this section we describe a club economy and define Pareto optimality, the core and equilibrium for such economies.

3.1 Private Goods

We assume throughout that there are $N \geq 1$ private goods, each perfectly divisible and publicly traded; the space of private goods is therefore \mathbf{R}^N . For $x, x' \in \mathbf{R}^N$, we write $x \geq x'$ to mean $x_i \geq x'_i$ for each i , $x > x'$ to mean that $x \geq x'$ but $x \neq x'$, and $x \gg x'$ to mean that $x_i > x'_i$ for each i . We write $|x| = \sum_{n=1}^N |x_n|$.

3.2 Clubs

We will describe a *type of club* by the number and characteristics of its members and the activity in which the club is engaged.

Formally, we let Ω be a finite set⁵ of *external characteristics* (of potential members of a club). An element $\omega \in \Omega$ is (or encodes) a complete description of the characteristics of an individual that are relevant for the *other* members of a club. For further discussion of the interpretation of external characteristics, see Section 3.10.

A *profile* is a function $\pi : \Omega \rightarrow \mathbf{Z}_+ = \{0, 1, \dots\}$ describing the members of a club. For $\omega \in \Omega$, $\pi(\omega)$ represents the number of members of the club having external characteristic ω . For π a profile, write $|\pi| = \sum_{\omega \in \Omega} \pi(\omega)$ for the total number of members. We write 0 for the zero profile (representing the empty club).

The *activities* available to a profile of agents belong to a finite set Γ .

⁵We could relax the constraint that Ω be finite, and allow Ω to be a compact metric space. However, this would increase the complexity of the model and of the arguments.

We interpret the elements $\gamma \in \Gamma$ as public projects in the sense of Ellickson (1979) and Mas-Colell (1980), rather than as public goods in the sense of Samuelson.

A *club type* is a pair $c = (\pi, \gamma)$ consisting of a profile and an activity. We take as given a finite set of possible club types $\mathbf{Clubs} = \{(\pi, \gamma)\}$. We find it convenient to treat singleton clubs separately, so we assume that $|\pi| \geq 2$ for all $(\pi, \gamma) \in \mathbf{Clubs}$.⁶ Formation of the club (π, γ) requires a total input of private goods equal to $\mathbf{inp}(\pi, \gamma) \in \mathbf{R}_+^N$.⁷

A *club membership* is an opening in a particular type of club for an agent of a particular external characteristic; i.e., a triple $m = (\omega, \pi, \gamma)$ such that $(\pi, \gamma) \in \mathbf{Clubs}$ and $\pi(\omega) \geq 1$. (An agent can belong to a club only if the description of that club type includes one or more members of his/her external characteristics.) Write \mathcal{M} for the set of club memberships.

Each agent may choose to belong to many clubs or to none. A *list* is a function $\ell : \mathcal{M} \rightarrow \{0, 1, \dots\}$; $\ell(\omega, \pi, \gamma)$ specifies the number of memberships of type (ω, π, γ) chosen by an agent. Write:

$$\mathbf{Lists} = \{\ell : \ell \text{ is a list} \}$$

for the set of lists. We frequently find it convenient to view \mathbf{Lists} (which is a set of functions from \mathcal{M} to $\{0, 1, \dots\}$) as a subset of $\mathbf{R}^{\mathcal{M}}$ (which is the set of functions from \mathcal{M} to \mathbf{R}). For $m \in \mathcal{M}$ we write δ_m for the list defined by

$$\delta_m(m') = \begin{cases} 1 & \text{if } m = m' \\ 0 & \text{otherwise} \end{cases}$$

That is, δ_m is the list specifying 1 membership of type m and no others.

⁶Since activities are not traded, the choice of activities of singleton clubs can be incorporated into preferences.

⁷More generally, we could assume that each project could be produced from any input vector from some specified set and incorporate the choice of production technology into our notion of feasibility.

3.3 Agents

The set of agents is a nonatomic finite measure space $(A, \mathcal{F}, \lambda)$; that is, A is a set, \mathcal{F} is a σ -algebra of subsets of A and λ is a non-atomic measure on \mathcal{F} with $\lambda(A) < \infty$.⁸

A complete description of an agent $a \in A$ consists of his/her external characteristics, choice set, endowment of private goods and utility function.⁹ An external characteristic is an element $\omega_a \in \Omega$. The choice set X_a for an agent $a \in A$ specifies which bundles of private goods and which choices of club memberships are feasible, so $X_a \subset \mathbf{R}^N \times \mathbf{Lists}$. For simplicity, we assume that the only restriction on private good consumption is that it be non-negative, so that $X_a = \mathbf{R}_+^N \times \mathbf{Lists}(a)$ for some subset $\mathbf{Lists}(a) \subset \mathbf{Lists}$.¹⁰ We assume that $\ell(\omega, \pi, \gamma) = 0$ for every $(\omega, \pi, \gamma) \in \mathcal{M}$ for which $\omega \neq \omega_a$; that is, no individual may choose membership in any club type containing no members of his/her external characteristic. We also assume throughout that there is an exogenously given upper bound M on the number of memberships an individual may choose, so $|\ell| \leq M$ for each $\ell \in \mathbf{Lists}(a)$. The utility function for agent a is defined over private goods consumptions and club memberships and is thus a mapping $u_a : X_a \rightarrow \mathbf{R}$.

We assume throughout that utility functions are strictly monotone in private goods; i.e., $u_a(x, \ell) > u_a(x', \ell)$ for $a \in A, x, x' \in \mathbf{R}_+^N, x > x'$. However, we make no assumption that utility is monotone in the level of any activity; indeed, in our framework it is meaningless to talk about the level of an activity. The ranking of activities may be different for different individuals, and an individual's ranking of activities may depend on his/her

⁸There would be little loss of generality in assuming that $A = [0, 1]$, \mathcal{F} is the σ -algebra of Lebesgue measurable sets, and λ is Lebesgue measure.

⁹We use utility functions rather than preferences as a matter of convenience; under the assumptions made here, the two specifications are equivalent.

¹⁰Thus we incorporate into consumption sets various kinds of restrictions on club memberships. For instance, we may forbid membership in 2 marriages. More general specifications of consumption sets would be easily accommodated at the cost of complicating some definitions (of a linked state and of an irreducible economy) and proofs (of the coincidence of weak and strong Pareto optimality and of quasi-equilibrium and equilibrium).

consumption of private goods. We take the view that an agent's preferences for private goods and for club memberships are interdependent and cannot be disentangled (except for monotonicity in private goods).¹¹ Activities are not traded.

3.4 Club Economies

A *club economy* \mathcal{E} is a mapping $a \mapsto (\omega_a, X_a, e_a, u_a)$ for which:

- the external characteristic mapping $a \mapsto \omega_a$ is a measurable function
- the consumption set correspondence $a \mapsto X_a$ is a measurable correspondence
- the endowment mapping $a \mapsto e_a$ is an integrable function
- the utility mapping $(a, x, \ell) \mapsto u_a(x, \ell)$ is a (jointly) measurable function (of all its arguments)¹²

As above, we assume that utility functions are continuous and strictly monotone in private goods.

We assume that the *aggregate endowment*

$$\bar{e} = \int_A e_a d\lambda(a)$$

is strictly positive, so all private goods are represented in the aggregate.

3.5 States

A *state* of a club economy is a measurable mapping

$$f = (x, \mu) : A \rightarrow \mathbf{R}^N \times \mathbf{R}^{\mathcal{M}}$$

¹¹See Diamantaras and Gilles (1996), Gilles and Scotchmer (1997) and Diamantaras, Gilles and Scotchmer (1996) for further discussion on this point.

¹²It can be shown that this measurability requirement is equivalent to the usual requirement on measurability of preferences.

A state describes choices for each individual agent, ignoring feasibility at the level of the individual and at the level of society. *Individual feasibility* means that $(x_a, \mu_a) \in X_a$. *Social feasibility* entails market clearing for private goods and consistent matching of agents.

We define consistency as a property of choice functions $\mu : B \rightarrow \mathbf{Lists}$, and show that it is equivalent to a property of aggregate membership vectors $\int_B \mu_a d\lambda(a)$. For each integer j and each membership (ω, π, γ) , let

$$E_\mu^j(\omega, \pi, \gamma) = \{a \in B : \mu_a(\omega, \pi, \gamma) = j\}$$

be the set of all agents who choose j memberships (ω, π, γ) . If we interpret $\lambda(E)$ as the proportion of agents who belong to a set $E \subset A$, then $\lambda(E_\mu^j(\omega, \pi, \gamma))$ is the proportion of agents who choose j memberships of type (ω, π, γ) and $j\lambda(E_\mu^j(\omega, \pi, \gamma))$ is the proportion of memberships of type (ω, π, γ) chosen by these agents. Hence the sum

$$\sum_{j=1}^{\infty} j\lambda(E_\mu^j(\omega, \pi, \gamma))$$

is the proportion of memberships of type (ω, π, γ) chosen by agents in A . We therefore say that a function $\mu : B \rightarrow \mathbf{Lists}$ is *consistent for B* if

$$\frac{\sum_{j=1}^{\infty} j\lambda(E_\mu(\omega, \pi, \gamma))}{\sum_{j=1}^{\infty} j\lambda(E_\mu(\omega', \pi, \gamma))} = \frac{\pi(\omega)}{\pi(\omega')}$$

for each $(\pi, \gamma) \in \mathbf{Clubs}$ and each $\omega, \omega' \in \Omega$. Equivalently, μ is consistent for B if for each (π, γ) there is a real number $\alpha(\pi, \gamma)$ such that

$$\sum_{j=1}^{\infty} j\lambda(E_\mu(\omega, \pi, \gamma)) = \alpha(\pi, \gamma)\pi(\omega)$$

for each $\omega \in \Omega$. Thus, consistency means that the distribution of club membership choices in the population is the same as in the club itself.¹³

¹³Consider Example 2.3 for instance. Keeping in mind that some males may choose one non-exclusive marriage, and some may choose two, consistency entails that the number of non-exclusive marriages chosen by males is the same as the number of non-exclusive marriages chosen by females.

We say that a club membership vector $\bar{\mu} \in \mathbf{R}^{\mathcal{M}}$ is *consistent* if for every club type (π, γ) there is a real number $\alpha(\pi, \gamma)$ such that

$$\bar{\mu}(\omega, \pi, \gamma) = \alpha(\pi, \gamma) \pi(\omega)$$

for every $\omega \in \Omega$. Write

$$\mathbf{Cons} = \{ \bar{\mu} \in \mathbf{R}^{\mathcal{M}} : \bar{\mu} \text{ is consistent } \}$$

Note that \mathbf{Cons} is a subspace of $\mathbf{R}^{\mathcal{M}}$. Because agents will choose lists of memberships that are nonnegative, the feasible states will have membership vectors in the positive part of \mathbf{Cons} .

The following lemma, whose simple proof is left to the reader, states the relationship between these two notions.

Lemma 3.1 *Let \mathcal{E} be a non-atomic club economy, let $B \subset A$ be a measurable set, and let $\mu : B \rightarrow \mathbf{Lists}$ be an integrable function. Then the function μ is consistent for B if and only if the membership vector $\int_B \mu_a d\lambda(a)$ is consistent.*

We say that the state $f = (x, \mu)$ is *feasible* for the measurable subset $B \subset A$ if it satisfies the following requirements:

(i) **Individual Feasibility**

$$(x_a, \mu_a) \in X_a \text{ for each } a \in A$$

(ii) **Material Balance**¹⁴

$$\int_B x_a d\lambda(a) + \int_B \sum_{(\omega, \pi, \gamma) \in \mathcal{M}} \frac{1}{|\pi|} \text{inp}(\pi, \gamma) \mu_a(\omega, \pi, \gamma) d\lambda(a) = \int_B e_a d\lambda(a)$$

¹⁴Material balance means that the social consumption of private goods (within B) plus the quantity of private goods used as inputs to club activities (by members of B) is equal to the social endowment of private goods (within B).

(iii) Consistency

μ is consistent for B

We say the state f is *feasible* if it is feasible for the set A itself.

Our description of feasible states of the economy is different from the description of feasible states in most of club theory, where the analog of consistency is expressed by a requirement that clubs form a partition of the set of agents. Our description allows for the possibility that agents belong to many clubs, that different agents belong to different numbers of clubs, and that clubs have overlapping memberships. For instance, agents may be married, have employment in a firm, belong to a gym, attend movies and concerts, take meals in a restaurant, and so forth. In the special case that agents can belong to only one club ($M=1$), the consistency condition reduces to the assertion that clubs form a partition that is “measure consistent” in the sense of Hammond, Kaneko and Wooders (1989).

We do not keep track of which person belongs to which club, nor do we need to do so: every function $\mu : A \rightarrow \mathbf{Lists}$ that satisfies the consistency condition corresponds to a consistent assignment of individuals to clubs (and vice versa). Of course, a given μ may correspond to many consistent assignments, but we do not need to distinguish them, because we assume that individuals care only about the external characteristics of their consumption partners, not about their identities. (See Section 3.10.)

3.6 Pareto Optimality and the Core

As in the exchange setting, we distinguish two notions of Pareto optimality and the core; the stronger notion allows blocking if some agents (in the relevant group) are made better off and none are made worse off, the weaker notion requires that all agents be made better off. For exchange economies, strict monotonicity of preferences guarantees that the two notions coincide. Because choices of club memberships are indivisible, however, the notions

may be distinct, even if preferences are strictly monotone in private goods. In this subsection we define two notions of Pareto optimality and the core and give a natural condition that guarantees that they coincide.

Let f be a feasible state. We say that f is *weakly Pareto optimal* if there is no feasible state g such that $u_a(g(a)) > u_a(f(a))$ for almost all $a \in A$; f is *strongly Pareto optimal* if there is no feasible state h such that $u_a(h(a)) \geq u_a(f(a))$ for almost all $a \in A$ and $u_{a'}(h(a)) > u_{a'}(f(a))$ for all a in some subset $A' \subset A$ having positive measure. Note that strong Pareto optimality is a more restrictive notion than weak Pareto optimality. Similarly, f is in the *weak core* if there is no subset $B \subset A$ of positive measure and state g that is feasible for B such that $u_b(g(b)) > u_b(f(b))$ for almost every every $b \in B$; f is in the *strong core* if there is no subset $B \subset A$ of positive measure and state h that is feasible for B such that $u_b(h(b)) \geq u_b(f(b))$ for every $b \in B$ and $u_{b'}(h(b')) > u_{b'}(f(b'))$ for all b' in some subset $B' \subset B$ having positive measure. The strong core is a subset of the weak core.

In general, weakly Pareto optimal allocations may not be strongly Pareto optimal, and the weak core may be a proper superset of the strong core. The following assumption, adapted from Gilles and Scotchmer (1997), guarantees that weak and strong Pareto optimality coincide and that the weak and strong cores coincide.

We say that *endowments are desirable* if for every agent a and every list $\ell \in \mathbf{Lists}(a)$, $u_a(e_a, 0) > u_a(0, \ell)$. That is, each agent would prefer to remain single and consume his endowment rather than to belong to any feasible set of clubs and consume no private goods. Desirability of endowments is weaker than the assumption Mas-Colell (1980) refers to as *essentiality* of private goods, which in our framework would be:

$$u_a(0, \ell) = \min_{(x^*, \ell^*) \in X_a} u_a(x^*, \ell^*)$$

for every $\ell \in \mathbf{Lists}(a)$.

Proposition 3.2 *If endowments are desirable, then weak and strong Pareto*

optimality coincide and the weak and strong core coincide.

Proof Let f be a feasible state not in the strong core. By definition, there exists a subset $B \subset A$ of positive measure and a state g that is feasible for B such that $u_b(g(b)) \geq u_b(f(b))$ for every $b \in B$ and $u_b(g(b)) > u_b(f(b))$ for all b in some subset $B' \subset B$ having positive measure. Because endowments are desirable, in the state g all members of B' must be consuming strictly positive amounts of private goods. Making use of continuity and strict monotonicity of preferences, we can find a small transfer of private goods from members of $B \setminus B'$ that leads to a state g' which is feasible for B and which all members of B strictly prefer to f . That is, f is not in the weak core.

The same argument with $B = A$ establishes coincidence of weak and strong Pareto optimality. ■

When endowments are desirable, we omit the modifiers and refer unambiguously to Pareto optimality and the core.

3.7 Equilibrium

Our notion of equilibrium involves the pricing of private goods and of club memberships. Private goods prices p lie in \mathbf{R}^N ; prices for club memberships q lie in \mathbf{R}^M , so the vector of all prices lies in $\mathbf{R}_+^N \times \mathbf{R}^M$. Because we assume that preferences are monotone in private goods, we will require that private goods prices be non-negative. However, prices for club memberships may be positive, negative or zero; prices for club memberships include transfers between agents in a given club — some agents may subsidize others. For $(x, \bar{\mu}) \in \mathbf{R}^N \times \mathbf{R}^M$ a vector of private goods and club memberships and $(p, q) \in \mathbf{R}^N \times \mathbf{R}^M$ a vector of prices, write

$$(p, q) \cdot (\bar{x}, \bar{\mu}) = p \cdot \bar{x} + q \cdot \bar{\mu}$$

for the cost of $(\bar{x}, \bar{\mu})$.

A *club equilibrium* consists of a feasible state $f = (x, \mu)$, private good

prices $p \in \mathbf{R}_+^N \setminus \{0\}$ and club membership prices $q \in \mathbf{R}^M$, satisfying the conditions:

(1) Budget Feasibility for Individuals

For almost all $a \in A$:

$$p \cdot x_a + q \cdot \mu_a \leq p \cdot e_a$$

(2) Optimization

For almost all $a \in A$:

$$(x'_a, \mu'_a) \in X_a \text{ and } u_a(x'_a, \mu'_a) > u_a(x_a, \mu_a) \Rightarrow p \cdot x'_a + q \cdot \mu'_a > p \cdot e_a$$

(3) Budget Balance for Clubs

For each club type $(\pi, \gamma) \in \mathbf{Clubs}$:

$$\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) = p \cdot \mathbf{inp}(\pi, \gamma)$$

Thus, at an equilibrium, individuals optimize subject to their budget constraint and the total cost of memberships in a given club is just enough to pay for the inputs to the given activity.

A *club quasi-equilibrium* differs from a club equilibrium only in the optimization condition (2) above is replaced by the weaker quasi-optimization condition:

(2') Quasi-Optimization

For almost all $a \in A$:

$$(x'_a, \mu'_a) \in X_a \text{ and } u_a(x'_a, \mu'_a) > u_a(x_a, \mu_a) \Rightarrow p \cdot x'_a + q \cdot \mu'_a \geq p \cdot e_a$$

That is, nothing that is feasible and strictly preferred can cost strictly less than agent a 's wealth. An equilibrium is necessarily a quasi-equilibrium.

3.8 Equilibrium and Quasi-Equilibrium

In the exchange case, the possibility of a quasi-equilibrium that is not an equilibrium is frequently viewed as a mere technical problem; the combination of strictly monotone preferences and strictly positive aggregate endowments is enough to assure that this problem does not occur. However, the indivisibilities and activities in our setting make the issue more subtle. The following example illustrates the problems that may arise when private goods are used as inputs to club activities; see Gilles and Scotchmer (1997) for an example illustrating the problems that may arise when endowments are not desirable

Example 3.3 Consider an economy with two private goods, a single external characteristic ω and a single club $c = (2, \gamma)$ consisting of two people, requiring inputs $\mathbf{inp}(c) = (2, 0)$. We assume agents are constrained to choose at most one club membership. All agents are identical, with endowments $e_a = (1, 1)$ and utility functions:

$$\begin{aligned} u_a(x, 0) &= 1 - e^{-x_1 - x_2} \\ u_a(x, \delta_{(\omega, c)}) &= \sqrt{x_1} + \sqrt{x_2} \end{aligned}$$

Reminder: $\delta_{(\omega, c)}$ is the list specifying choice of the unique membership (ω, c) . Because endowments are desirable, the weak and strong cores coincide. Indeed, there is a unique state f in the core: all agents belong to clubs, consume none of good 1 and 1 unit of good 2, and the entire supply of good 1 is used to provide the input to the club activity. However, the state f cannot be supported as an equilibrium, because the marginal rate of substitution of good 1 for good 2 is infinite, so the equilibrium price ratio would have to be infinite also. On the other hand, f can be supported as a quasi-equilibrium: quasi-equilibrium prices are $p = (1, 0)$, $q(\omega, c) = 1$. (This is *not* an equilibrium, because good 2 is free and every agent desires more of it.) ♣

In the familiar exchange setting, a quasi-equilibrium may fail to be an equilibrium if some agents are in the “minimum expenditure situation,” that

is, when quasi-equilibrium consumptions require expenditures exactly equal to wealth and slightly smaller expenditures are not possible. As the example above illustrates, it is easier for this minimum expenditure situation to arise in club economies, because private goods are used as inputs to club activities and club choices are indivisible. Various assumptions would enable us to avoid the minimum expenditure setting and guarantee that a quasi-equilibrium is an equilibrium; we take a route parallel to the exchange setting.

Let \mathcal{E} be a club economy and let $f = (x, \mu)$ be a feasible state. Write δ_j for the consumption bundle consisting of one unit of good j and nothing else. Say that f is *club linked* if whenever

$$I \cup J = \{1, \dots, N\}$$

is a partition of the set of private goods and $x_{ai} = 0$ for all $i \in I$ and almost all $a \in A$, then for almost all $a \in A$ there exist $r \in \mathbf{R}_+$, $j \in J$ such that

$$u_a(e_a + r\delta_j, 0) > u_a(x_a, \mu_a)$$

That is, if (as in Example 3.3) the *entire* social endowment of the private goods in I is used in the production of club activities, then for almost all agents a , there is some good $j \notin I$ and some sufficiently large level of consumption of good j such that agent a would prefer consuming his endowment together with this large level of good j , and belong to no clubs, rather than consume the bundle x_a in the club memberships μ_a . Say that \mathcal{E} is *club irreducible* if every feasible allocation is club linked.¹⁵

Proposition 3.4 *Let \mathcal{E} be a club economy for which endowments are desirable. If (f, p, q) is a club quasi-equilibrium and f is club linked, then $p \gg 0$ and (f, p, q) is an equilibrium.*

¹⁵We use the terms “club linked” and “club irreducible” because these notions play the same role for us that linked allocations and irreducibility play in the exchange setting; see Mas-Colell (1985) for instance.

Proof We show first that all private good prices are strictly positive. If not, let I be the set of indices for which $p_i > 0$, and let $J \neq \emptyset$ be the complementary set of indices. Fix $i \in I$. If $x_{ai} \neq 0$ for some set of consumers having positive measure, then some of these consumers could sell a small amount of their consumption of x_i and buy an unlimited quantity of x_j (for any $j \in J$) and be strictly better off with a lower expenditure; this would contradict the quasi-equilibrium conditions. We conclude that, for each $i \in I$, $x_{ai} = 0$ for almost all $a \in A$. Club linkedness guarantees that all consumers would prefer to consume their endowments plus a large quantity of some commodity x_j rather than their quasi-equilibrium consumption. Since aggregate endowments of private goods are strictly positive, the endowments of some consumers have a strictly positive value and those consumers would (by continuity of preferences) prefer to consume a very large fraction of their endowment plus a large quantity of commodity x_j , rather than their quasi-equilibrium consumption. Again, this would contradict the quasi-equilibrium conditions, so we conclude that all private good prices are strictly positive.

If (f, p, q) is not an equilibrium, then there is an agent a who is quasi-optimizing, but not optimizing. Hence there is a choice $(x', \mu') \in X_a$ which is strictly preferred to agent a 's quasi-equilibrium choice and costs no more than his endowment. Desirability of endowments entails that $x' \neq 0$, so $p \cdot x' > 0$. Continuity of preferences entails that there is a bundle x'' such that $p \cdot x'' < p \cdot x'$, $(x'', \mu') \in X_a$ and (x'', μ') is strictly preferred to agent a 's quasi-equilibrium choice — but costs strictly less than his endowment. This is a contradiction, so the proof is complete. ■

3.9 Pure Transfers

Our formulation of equilibrium requires that the sum of membership prices in each club type be exactly sufficient to pay for the inputs required for production of the club activity. An equivalent notion makes clear the role of membership prices as taxes and subsidies (and will prove to be more convenient in proofs).

Say that $q \in \mathbf{R}^M$ is a *pure transfer* if $q \in \mathbf{Trans}$, defined as:

$$\mathbf{Trans} = \{q \in \mathbf{R}^M : q \cdot \mu = 0 \text{ for each } \mu \in \mathbf{Cons}\}$$

Thus for each club type (π, γ) and $q \in \mathbf{Trans}$,

$$\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) = 0$$

A *pure transfer equilibrium* is a triple (f, p, q) where f is a feasible state, $p \in \mathbf{R}_+^N \setminus \{0\}$ is a vector of private good prices and $q \in \mathbf{R}^M$ is a vector of membership prices satisfying the conditions:

(1) **Budget Feasibility**

For almost all $a \in A$,

$$p \cdot x_a + q \cdot \mu_a + \sum_{(\omega, \pi, \gamma)} p \cdot \frac{1}{|\pi|} \mathbf{inp}(\pi, \gamma) \mu_a(\omega, \pi, \gamma) \leq p \cdot e_a$$

(2) **Optimization**

For almost all $a \in A$, if $(x'_a, \mu'_a) \in X_a$ and

$$u_a(x'_a, \mu'_a) > u_a(x_a, \mu_a)$$

then

$$p \cdot x'_a + q \cdot \mu'_a + \sum_{(\omega, \pi, \gamma)} p \cdot \frac{1}{|\pi|} \mathbf{inp}(\pi, \gamma) \mu'_a(\omega, \pi, \gamma) > p \cdot e_a$$

(3) **Pure Transfers**

$$q \in \mathbf{Trans}$$

We define a pure transfer quasi-equilibrium in the obvious way.

The following lemma tells us that equilibrium (respectively quasi-equilibrium) and pure transfer equilibrium (respectively quasi-equilibrium) are equivalent notions; we leave the simple proof to the reader.

Lemma 3.5 *Let \mathcal{E} be a club economy. For $q^* \in \mathbf{Trans}$ define $q \in \mathbf{R}^M$ by*

$$q^*(\omega, \pi, \gamma) = q(\omega, \pi, \gamma) + p \cdot \frac{1}{|\pi|} \mathbf{inp}(\pi, \gamma)$$

Then: (f, p, q) is a pure transfer equilibrium (respectively, pure transfer quasi-equilibrium) if and only if (f, p, q^) is an equilibrium (respectively, quasi-equilibrium).*

3.10 Discussion

In our model, agents care about their own consumption and about the external characteristics of others in their clubs. The characteristics we have in mind should be observable to others in the club, which is why we call them external. Such characteristics might include sex, intelligence, appearance, even tastes and endowments, to the extent that such characteristics can be observed by others.¹⁶ On the other hand, we exclude private characteristics which are known only to the individual. Because we assume that memberships are priced according to external characteristics, our construction can be viewed as a compromise between the non-discriminatory pricing of competitive equilibrium and the personalized prices of Lindahl. To capture the essence of club theory, we regard as essential a certain degree of anonymity, but we also think it important to recognize that clubs offer different types of membership.¹⁷

One restriction in this model, which would be particularly desirable to eliminate in future work is that external characteristics are ascriptive, not

¹⁶But keep in mind that we assume in this paper that the set of external characteristics is finite.

¹⁷Much of the club literature indexes both the external characteristics and the tastes and endowments by a single “type;” see Berglas (1976), Gilles and Scotchmer (1997) for instance. Our use of external characteristics is closer in spirit to Conley and Wooders (1994), Engl and Scotchmer (1996) and Scotchmer (1996), where prices are understood as “externality prices.” However, these latter papers treat only finite TU economies with a single private good, restrict agents to belong to at most one club, and do not discuss existence.

acquired. Intelligence and endowments (if observable) are possible external characteristics, skill and consumption are not.

Of course we could formulate a model in which preferences for club memberships depend on various characteristics of club partners, but insist that prices be independent of those characteristics. In that case, however, and in contrast to the results proved here, core allocations might not be decentralizable by prices, and equilibria could fail to exist. (A similar comment applies to the possibility of preferences that depend on the *consumptions* of club partners.)

4 The First Welfare Theorem and the Core

In our club context, as in the exchange case, we easily obtain the first welfare theorem.

Theorem 4.1 *Every equilibrium state of a club economy belongs to the weak core and, in particular, is weakly Pareto optimal. If endowments are desirable, every equilibrium state belongs to the strong core and, in particular, is strongly Pareto optimal.*

Proof Let \mathcal{E} be a club economy and let $f = (x, \mu)$ be an equilibrium state, supported by the prices $p \in \mathbf{R}_+^N \setminus \{0\}, q \in \mathbf{R}^M$. If f is not in the weak core, there is a subset $B \subset A$ of positive measure and a state $g = (y, \nu)$ that is feasible for B and preferred to f by every member of B . Feasibility of g for the coalition B entails the material balance condition:

$$\int_B y_a d\lambda(a) + \int_B \sum_{(\omega, \pi, \gamma) \in \mathcal{M}} \frac{1}{|\pi|} \mathbf{inp}(\pi, \gamma) \nu_a(\omega, \pi, \gamma) d\lambda(a) = \int_B e_a d\lambda(a)$$

and the budget balance condition for each club type (π, γ) :

$$\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) = p \cdot \mathbf{inp}(\pi, \gamma)$$

Combining these with the consistency condition, we conclude that

$$\int_B (p, q) \cdot (y_a, \nu_a) d\lambda(a) \leq \int_B p \cdot e_a d\lambda(a)$$

Hence there is a set $B' \subset B$ having positive measure for which

$$(p, q) \cdot (y_b, \nu_b) \leq p \cdot e_b$$

for every $b \in B'$. Since g is unanimously preferred to f by members of B , this contradicts the equilibrium nature of f . We conclude that f is in the weak core, as desired.

That f is weakly Pareto optimal follows immediately by taking $B = A$ in the argument above.

If endowments are desirable, the weak and strong cores coincide and weak and strong Pareto optimality coincide, so the proof is complete. ■

5 Failure of the Second Welfare Theorem

For exchange economies, the second welfare theorem asserts that every Pareto optimal allocation can be realized as an equilibrium allocation after a suitable redistribution of endowments. For finite economies, the second welfare theorem depends on convexity of preferences; because the indivisibility of club memberships introduces an essential non-convexity in our context, it should come as no surprise that the second welfare theorem may fail for finite club economies. More surprising is that the second welfare theorem may fail even for *non-atomic* club economies. The following simple examples (see also Example 6.2) illustrate what may go wrong.

Example 5.1 We consider an economy with a single consumption good. Agents have one of two external characteristics, $\Omega = \{M, F\}$ (males and females); a single club c , a monogamous marriage, requiring no inputs, is possible, and agents are constrained to choose at most one club membership. In this case $\mathcal{M} = \{(M, c), (F, c)\}$. Agents in the interval $[0, 1/2)$ are male, agents in the interval $[1/2, 1]$ are female. Males love marriage and females hate it:

$$\begin{aligned} u_a(x, 0) &= x && \text{for all } a \in [0, 1] \\ u_a(x, \delta_{(M,c)}) &= 2x && \text{for all } a \in [0, 1/2) \quad (\text{males}) \\ u_a(x, \delta_{(F,c)}) &= 1 - e^{-x} && \text{for all } a \in [1/2, 1] \quad (\text{females}) \end{aligned}$$

Endowments are $e_a = 1$ for all a . Define x by

$$x_a = \begin{cases} \frac{1}{\sqrt{2a}} & \text{for all } a \in [0, 1/2) \quad (\text{males}) \\ 1 & \text{for all } a \in [1/2, 1] \quad (\text{females}) \end{cases}$$

and set $f = (x, 0)$. It is easily checked that f is a Pareto optimal feasible state, but cannot be supported as an equilibrium following any redistribution of endowments: Whatever the marriage price, some males will be rich enough to desire and afford marriage; those males will not be optimizing. ♣

Example 5.2 The economy is as described in Example 5.1. Consider the feasible state in which there is no exchange of the consumption good, but all agents are married. That is, $g = (\mathbf{1}, \nu)$, where ν is defined by:

$$\nu_a = \begin{cases} \delta_{(M,c)} & \text{for all } a \in [0, 1/2) \text{ (males)} \\ \delta_{(F,c)} & \text{for all } a \in [1/2, 1] \text{ (females)} \end{cases}$$

Again, it is easily checked that g is Pareto optimal, but cannot arise as an equilibrium state, no matter what the endowments: No matter what the prices, females — who hate marriage — cannot be optimizing when they are married. ♣

The role of unbounded consumption in the failure of the 2nd welfare theorem seen in Example 5.1 foreshadows the role unbounded endowments will play in the failure of core equivalence in Example 6.2. The failure of the 2nd welfare theorem seen in Example 5.2 reflects the fundamental asymmetry between initial states (in which agents choose no clubs) and other feasible states (in which agents may choose various clubs).¹⁸

¹⁸This problem might be “solved” by allowing for endowments of clubs, but it is not clear what endowments of clubs should mean. Here we follow tradition in the club literature and assume endowments consist of private goods only.

6 Core/Equilibrium Equivalence

In this section we establish that non-atomic club economies pass a familiar test of perfect competition: coincidence of the core with the set of equilibrium states.

Theorem 6.1 *Let \mathcal{E} be a non-atomic club economy in which endowments are desirable and uniformly bounded above. Then every core state can be supported as a quasi-equilibrium and every core state that is club linked can be supported as an equilibrium. In particular, if \mathcal{E} is club irreducible, then the core coincides with the set of equilibrium states.*

In the proof (which we defer to Section 8), we find it convenient to construct a pure transfer equilibrium. The argument parallels Schmeidler's (1969) proof of Aumann's core equivalence theorem for exchange economies:

- 1 Construct a preferred net trade correspondence and aggregate net preferred set.
- 2 Apply the Lyapunov convexity theorem to show that the aggregate net preferred set is convex.
- 3 Use the core property to show that the aggregate net preferred set is disjoint from a cone that represents feasible net trades (for all coalitions).
- 4 Construct a quasi-equilibrium price as a price that separates the net aggregate preferred set from this cone. Use linkedness to conclude that the quasi-equilibrium is an equilibrium.

The argument contains two surprises. The first is that we require endowments to be bounded; no such assumption is required in the familiar exchange case. This is not merely an artifact of the proof, however: if endowments are unbounded, the core may not coincide with the set of equilibrium states

and equilibrium may not exist. The following variant of Examples 5.1 and 5.2 illustrates the problem.

Example 6.2 We consider an economy with a single consumption good. Agents have one of two external characteristics, $\Omega = \{M, F\}$ (males and females); a single club c , a monogamous marriage requiring no inputs, is possible, and we assume agents are constrained to choose at most one membership. Agents in the interval $[0, 1/2)$ are male, agents in the interval $[1/2, 1]$ are female. Males love marriage and females hate it:

$$\begin{aligned} u_a(x, 0) &= x && \text{for all } a \in [0, 1] \\ u_a(x, \delta_{(M,c)}) &= 2x && \text{for all } a \in [0, 1/2) \text{ (males)} \\ u_a(x, \delta_{(F,c)}) &= 1 - e^{-x} && \text{for all } a \in [1/2, 1] \text{ (females)} \end{aligned}$$

Endowments are $e_a = \frac{1}{\sqrt{a}}$. It is easily checked that the initial state is the unique element of the core but cannot be supported as an equilibrium: there is no upper bound on the amount men would pay to enter a marriage (because males are willing to give up half their endowment to enter a marriage, and male endowments are unbounded), but no female is willing to enter a marriage at any price. ♣

The other surprise in the proof is that it will not be quite good enough to find prices (p, q) that separate the aggregate net preferred set from the cone representing feasible net trades; we must also be sure that $p \neq 0$. To achieve this we will separate the aggregate net preferred set from a cone that is larger than the cone representing feasible net trades. To show that the aggregate net preferred set is disjoint from this cone, we will need to show that if $g = (y, \nu)$ is a state, $B \subset A$ is a coalition, and ν nearly satisfies the consistency condition with respect to B , then there is a large subset $B' \subset B$ such that ν *exactly* satisfies the consistency condition with respect to B' .

We formalize this idea in the following Lemma, but we must first introduce a little notation. For $L \subset \mathbf{R}^M$ write $\text{conv}(L)$ for its convex hull. We have assumed that individuals are constrained to choose lists with no more than

M memberships; write

$$\mathbf{Lists}_M = \{\ell \in \mathbf{Lists} : |\ell| \leq M\}$$

(Recall that M is the given upper bound on the number of memberships that may be chosen by any individual.) Set

$$\mathcal{D} = \{L \subset \mathbf{Lists}_M, \text{conv}(L) \cap \mathbf{Cons} = \emptyset\}$$

and

$$D = \inf \left\{ \text{dist}(\text{conv}(L), \mathbf{Cons}) : L \in \mathcal{D} \right\}$$

Lemma 6.3 *Let $B \subset A$ be a measurable set of positive measure and let $\nu : B \rightarrow \mathbf{Lists}_M$ be a measurable function. Then there is a measurable subset $B' \subset B$ such that*

$$\int_{B'} \nu_b d\lambda(b) \in \mathbf{Cons}$$

and

$$\lambda(B') \geq \lambda(B) - \frac{1}{D} \text{dist} \left(\int_B \nu_b d\lambda(b), \mathbf{Cons} \right) \quad (10)$$

7 Existence of Equilibrium

In this Section we establish the existence of equilibrium for non-atomic economies.

Theorem 7.1 *Let \mathcal{E} be a non-atomic club economy. If endowments are desirable and uniformly bounded above, then a quasi-equilibrium exists. If in addition \mathcal{E} is club irreducible, then an equilibrium exists.*

The basic idea of the argument will be familiar: construct an excess demand correspondence, use a fixed point theorem to find a zero, and show that this zero is an equilibrium price. However, the club structure gives rise to many subtleties:

- The balance condition for private goods translates to the requirement that the excess demand for private goods be 0; the balance condition for club memberships translates to the requirement that the demand for club memberships be in **Cons**, a more subtle condition to verify.
- In equilibrium, prices for private goods must be positive, but prices for club membership prices may be positive, negative or 0. Hence the relevant space of all prices is not a proper cone, and the usual forms of the excess demand lemma will not apply.
- Private good prices can be normalized to sum to 1; club membership prices admit no obvious normalization or bound. Hence it is not clear how to construct a domain for prices on which to apply a fixed point theorem.
- We assume that all private goods are present in the aggregate, but we do not assume that all external characteristics are present in the aggregate, and some clubs may not be chosen at equilibrium. In effect, therefore, we must construct reservation prices for unavailable club memberships.

One reason that membership prices may be unbounded is that they may be indeterminate. The following variant on Example 5.1 illustrates the point.

Example 7.2 We consider an economy with a single consumption good. Agents have one of two external characteristics, $\Omega = \{M, F\}$ (males and females); a single club c , a monogamous marriage requiring no inputs, is possible. Agents are constrained to choose at most one membership. The agent space is $A = [0, 1]$; agents in $[0, 1/2]$ are male, agents in $[1/2, 1]$ are female. Endowments are $e_a = 1$ and utility functions for all agents are:

$$\begin{aligned} u_a(x, 0) &= x && \text{for all } a \\ u_a(x, \delta_{(M,c)}) &= 1 - e^{-x} && \text{for } a \in [0, 1/2] \\ u_a(x, \delta_{(F,c)}) &= 1 - e^{-x} && \text{for } a \in [1/2, 1] \end{aligned}$$

In this example both males and females hate marriage. The core consists of the single autarkic state, and is supported as an equilibrium by *any* prices p, q such that $p > 0$ and $q(M, c) + q(F, c) = 0$ (because agents will never choose to marry, no matter what the subsidy). ♣

With multiple club memberships the problem is more subtle, as the following example illustrates.

Example 7.3 We consider an economy with a single consumption good. Agents have one of two external characteristics, $\Omega = \{M, F\}$ (males and females); two club types c_1, c_2 , each consisting of one male and one female and requiring no inputs, are possible. Agents are allowed to choose at most 2 memberships. The agent space is $A = [0, 1]$; agents in $[0, 1/2]$ are male, agents in $[1/2, 1]$ are female. Endowments are $e_a = 1$. Utility functions for males $a \in [0, 1/2)$ are:

$$\begin{aligned} u_a(x, 0) &= x \\ u_a(x, \delta_{(M,c_1)}) &= u_a(x, \delta_{(M,c_2)}) = 1 - e^{-x} \\ u_a(x, 2\delta_{(M,c_1)}) &= u_a(x, 2\delta_{(M,c_2)}) = 1 - e^{-x} \\ u_a(x, \delta_{(M,c_1)} + \delta_{(M,c_2)}) &= 2x \end{aligned}$$

(Note that $\delta_{(M,c_1)} + \delta_{(M,c_2)}$ is the list representing choice of one membership of type (M, c_1) and one membership of type (M, c_2) .) Utility functions for females $a \in [1/2, 1]$ are:

$$\begin{aligned} u_a(x, 0) &= x \\ u_a(x, \delta_{(F,c_1)}) &= u_a(x, \delta_{(F,c_2)}) = 1 - e^{-x} \\ u_a(x, 2\delta_{(F,c_1)}) &= u_a(x, 2\delta_{(F,c_2)}) = 1 - e^{-x} \\ u_a(x, \delta_{(F,c_1)} + \delta_{(F,c_2)}) &= 2x \end{aligned}$$

Thus, both males and females hate belonging to a single club or two clubs of the same type, but love belonging to two clubs of different types. The core consists of a single point: all agents choose one club of each type and consume their endowments. This state is supported as an equilibrium by *any* private good prices and club membership prices such that $p > 0$ and

$$\begin{aligned} q(M, c_1) + q(F, c_1) &= 0 \\ q(M, c_2) + q(F, c_2) &= 0 \\ q(M, c_1) + q(M, c_2) &= 0 \\ q(F, c_1) + q(F, c_2) &= 0 \end{aligned}$$

(Because agents will never choose to belong to only one club or to two clubs of the same type, no matter how big the subsidy. ♣)

As these examples suggest, some of the indeterminacy would disappear if we regarded lists as the primary objects and priced them directly. However, doing so would lead to a less appealing notion of equilibrium, in which the price of a list might not be the sum of the prices of its component memberships. We shall therefore work directly with membership prices, keeping list prices in the background.

Finding upper and lower bounds for list prices is much simpler than finding a bounded domain for membership prices: If we normalize private good prices to sum to 1 and assume that individual endowments are uniformly bounded, then individual incomes also will be uniformly bounded. Hence if $q \in \mathbf{R}^M$ is a vector of membership prices, $\ell \in \mathbf{Lists}$ is a list, and $q \cdot \ell$ exceeds

the bound on individual incomes, then no agent will be able to afford the list ℓ , and the demand for ℓ will be 0. Thus the upper bound on individual incomes provides an upper bound for list prices. To construct a lower bound for list prices (keeping in mind that list prices might be negative), we show that, if some individuals are paying large negative list prices then others are paying large positive list prices, which is impossible. The construction we use is formalized in Lemma 8.1.

As does the proof of core/equilibrium equivalence (Theorem 6.1), the argument for Theorem 7.1 proceeds by constructing a pure transfer quasi-equilibrium. To deal with the difficulties identified earlier, we work with perturbations of the true economy, and then take limits as we make the perturbation disappear. The argument is divided into 8 steps:

- 1 For each k , construct a perturbed economy \mathcal{E}^k by adjoining to A a few agents of each external characteristic, with utility functions unbounded in private good consumption.
- 2 For the perturbed economy \mathcal{E}^k , identify a compact set of prices in which an equilibrium price will be found.
- 3 Construct an excess demand correspondence.
- 4 Find a fixed point of the correspondence that maximizes the value of excess demand.
- 5 Show that, at this fixed point, excess demand for private goods is equal to 0 and demand for club memberships is an element in \mathbf{Cons} .
- 6 Construct an equilibrium for \mathcal{E}^k .
- 7 Show that the equilibrium state can be supported by prices satisfying a uniform bound independent of k .
- 8 Take limits of these uniformly bounded equilibrium prices as $k \rightarrow \infty$ and apply Fatou's lemma to construct an equilibrium for \mathcal{E} .

8 Proofs

Here we collect proofs of most of the results in the text. We first show that if ν is “almost” consistent for B then it is exactly consistent for a large subset of B .

Proof of Lemma 6.3 If $\mathbf{Lists}_M \subset \mathbf{Cons}$ then $\text{dist}(\int_B \nu_b d\lambda(b), \mathbf{Cons}) = 0$, $\mathcal{D} = \emptyset$ and $D = \infty$, so we may take $B' = B$. Assume therefore that $\mathbf{Lists}_M \not\subset \mathbf{Cons}$.

For each $\ell \in \mathbf{Lists}_M$, write

$$B_\ell = \nu^{-1}(\ell) = \{b \in B : \nu_b = \ell\}$$

and let $L = \{\ell \in \mathbf{Lists}_M : \lambda(B_\ell) > 0\}$. Note that $\sum_{\ell \in L} \lambda(B_\ell) = \lambda(B)$ and that

$$\int_B \nu_b d\lambda(b) = \sum_{\ell \in L} \lambda(B_\ell) \ell = \frac{1}{\lambda(B)} \sum_{\ell \in L} \frac{\lambda(B_\ell)}{\lambda(B)} \ell$$

In particular

$$\int_B \nu_b d\lambda(b) \in \frac{1}{\lambda(B)} \text{conv}(L)$$

If $\text{conv}(L) \cap \mathbf{Cons} = \emptyset$ then the right hand side of (10) is non-positive, so we may take $B_\ell = \emptyset$ for each ℓ . We therefore assume $\text{conv}(L) \cap \mathbf{Cons} \neq \emptyset$.

Consider the linear programming problem:

$$\begin{aligned} & \text{maximize} && \sum_{\ell \in L} \beta_\ell \\ & \text{subject to} && 0 \leq \beta_\ell \leq \lambda(B_\ell) \\ & && \sum_{\ell \in L} \beta_\ell \ell \in \mathbf{Cons} \end{aligned}$$

The feasible set for this problem is non-empty (it contains the origin), so this problem has a solution; let $\{\beta_\ell : \ell \in L\}$ be any such solution.

For each ℓ , write $\alpha_\ell = \lambda(B_\ell) - \beta_\ell \geq 0$. Write $L' = \{\ell : \alpha_\ell > 0\}$. If $L' = \emptyset$ we are done, so assume not. Write $\alpha = \min\{\alpha_\ell : \ell \in L'\}$. If $\text{conv}(L') \cap \mathbf{Cons} \neq \emptyset$ there are non-negative real numbers ϵ_ℓ summing to

1 with $\sum_{L'} \epsilon_\ell \ell \in \mathbf{Cons}$. Set $\beta_\ell^* = \beta_\ell + \epsilon_\ell \alpha_\ell$ for $\ell \in L'$ and $\beta_\ell^* = \beta_\ell$ for $\ell \notin L'$. Then $\{\beta_\ell^* : \ell \in L\}$ satisfies the constraints in the linear programming problem and yields a larger value of the objective, contradicting the choice of $\{\beta_\ell\}$ as the solution. We conclude that $\text{conv}(L') \cap \mathbf{Cons} = \emptyset$.

For each $\ell \in \mathbf{Lists}_M$, non-atomicity of λ guarantees that we can choose $B'_\ell \subset B_\ell$ such that $\lambda(B'_\ell) = \beta_\ell$. Set

$$B' = \bigcup_{\ell \in L} B'_\ell$$

By construction,

$$\int_{B'} \nu_b d\lambda(b) = \sum_{\ell \in L} \lambda(B'_\ell) \ell \in \mathbf{Cons}$$

We need only estimate $\lambda(B')$. To this end, note first that, because \mathbf{Cons} is a linear subspace,

$$\text{dist}(x - y, \mathbf{Cons}) = \text{dist}(x, \mathbf{Cons})$$

and

$$\text{dist}(rx, \mathbf{Cons}) = r \text{dist}(x, \mathbf{Cons})$$

for every $x \in \mathbf{R}^M$, $y \in \mathbf{Cons}$, $r \in \mathbf{R}_+$. Hence

$$\begin{aligned} \text{dist}\left(\sum_{\ell \in L} \lambda(B_\ell) \ell, \mathbf{Cons}\right) &= \text{dist}\left(\left(\sum_{\ell \in L} \lambda(B_\ell) \ell - \sum_{\ell \in L} \beta_\ell \ell\right), \mathbf{Cons}\right) \\ &= \text{dist}\left(\sum_{\ell \in L'} (\lambda(B_\ell) - \beta_\ell) \ell, \mathbf{Cons}\right) \\ &= \text{dist}\left(\sum_{\ell \in L'} \alpha_\ell \ell, \mathbf{Cons}\right) \\ &= \text{dist}\left(\sum_{\ell \in L'} \alpha_\ell \sum_{\ell \in L'} \left[\frac{\alpha_\ell}{\sum_{\ell \in L'} \alpha_\ell}\right] \ell, \mathbf{Cons}\right) \\ &= \sum_{\ell \in L'} \alpha_\ell \text{dist}\left(\left[\sum_{\ell \in L'} \frac{\alpha_\ell}{\sum_{\ell \in L'} \alpha_\ell}\right] \ell, \mathbf{Cons}\right) \\ &\geq \sum_{\ell \in L'} \alpha_\ell \text{dist}(\text{conv}(L'), \mathbf{Cons}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\ell \in L'} (\lambda(B_\ell) - \beta_\ell) \text{dist}(\text{conv}(L'), \mathbf{Cons}) \\
&\geq D \sum_{\ell \in L'} (\lambda(B_\ell) - \beta_\ell) \\
&= D \sum_{\ell \in L} (\lambda(B_\ell) - \beta_\ell) \\
&= D[\lambda(B) - \lambda(B')]
\end{aligned}$$

Rearranging terms yields the desired inequality (10). ■

With this lemma in hand, we turn to the proof of core/equilibrium equivalence.

Proof of Theorem 6.1 Let $f = (x, \mu)$ be a core state. We show that f can be supported as a pure transfer quasi-equilibrium.

Step 1 For each agent a , consider the preferred set

$$\Phi(a) = \{(x, \ell) \in X_a : u_a(x, \ell) > u_a(x_a, \mu_a)\}$$

For each club $(\pi, \gamma) \in \mathbf{Clubs}$, consider the bundle $\frac{1}{|\pi|} \mathbf{inp}(\pi, \gamma)$; this is what each agent would be required to contribute to the club (π, γ) if inputs were imputed equally to all members. For $\ell \in \mathbf{Lists}(\omega_a)$, define

$$\tau(\ell) = \sum_{(\omega, \pi, \gamma) \in \mathcal{M}} \ell(\omega, \pi, \gamma) \frac{1}{|\pi|} \mathbf{inp}(\pi, \gamma)$$

This is the total an individual would be required to contribute to all clubs if ℓ is the chosen list of memberships and inputs were imputed equally to all members.

Define the net preferred set for agent a as:

$$\psi(a) = \{(x, \ell) \in \mathbf{R}^N \times \mathbf{R}^{\mathcal{M}} : (x + e_a - \tau(\ell), \ell) \in \Phi(a)\}$$

and set

$$\Psi(a) = \psi(a) \cup \{0\}$$

It is easily checked that Ψ is a measurable correspondence. Define the aggregate net preferred set to be the integral of the correspondence Ψ :

$$Z = \int_A \Psi(a) d\lambda(a)$$

(We refer the reader to Hildenbrand (1974) for discussion of the integral of a correspondence.)

Step 2 In view of the Lyapunov convexity theorem, Z is a non-empty convex subset of $\mathbf{R}^N \times \mathbf{R}^M$. (See Hildenbrand (1974).)

Step 3 Write $\mathbf{1} = (1, \dots, 1) \in \mathbf{R}_+^N$. By assumption, endowments are uniformly bounded; say $e_a \leq W\mathbf{1}$ for each $a \in A$. Set

$$C = \{(\bar{x}, \bar{\mu}) \in \mathbf{R}^N \times \mathbf{R}^M : \bar{x} \leq 0, \bar{\mu} \in \mathbf{Cons}\}$$

C is a convex cone in $\mathbf{R}^N \times \mathbf{R}^M$. The core property of f implies that $Z \cap C = \emptyset$ and hence that Z can be separated from C by a price pair (p, q) . Unfortunately, it might happen that the separating price has $p = 0$. (See Example 6.2.) In order to guarantee that $p \neq 0$, we separate Z from a “fatter” cone.

Define

$$C^* = \{(\bar{x}, \bar{\mu}) \in \mathbf{R}^L \times \mathbf{R}^M : \bar{x} < -\frac{W}{D} \text{dist}(\bar{\mu}, \mathbf{Cons})\mathbf{1}\}$$

We claim that $Z \cap C^* = \emptyset$.

To see this, suppose not; we construct a blocking coalition. Choose $z^* = (x^*, \mu^*) \in Z \cap C^*$. By definition, there is a measurable selection $a \mapsto (y_a, \nu_a)$ from the correspondence Ψ such that

$$z^* = \int_A (y_a, \nu_a) d\lambda(a)$$

Let $B = \{a \in A : (y_a, \nu_a) \in \psi(a)\}$; this is the set of agents for whom (y_a, ν_a) is in their net preferred set. Note that $\lambda(B) > 0$ and

$$z^* = \int_B (y_a, \nu_a) d\lambda(a)$$

so

$$x^* = \int_B y_a d\lambda(a), \quad \mu^* = \int_B \nu_a d\lambda(a) \tag{11}$$

We now apply Lemma 6.3 to choose $B' \subset B$ such that

$$\int_{B'} \nu_a d\lambda(a) \in \mathbf{Cons} \quad (12)$$

and

$$\lambda(B') \geq \lambda(B) - \frac{1}{D} \text{dist} \left(\int_B \nu_b d\lambda(b), \mathbf{Cons} \right) \quad (13)$$

We assert that B' is a blocking coalition. To see this, note first that, because endowments are bounded above by $W\mathbf{1}$, net preferred sets are bounded below by $-W\mathbf{1}$. Hence

$$\int_B y_a d\lambda(a) \geq -\lambda(B)W\mathbf{1}$$

Because $z^* = (x^*, \mu^*) \in C^*$, equation (11) entails that

$$\int_B y_a d\lambda(a) < -\frac{W}{D} \text{dist} \left(\int_B \nu_a d\lambda(a), \mathbf{Cons} \right)$$

Hence

$$\text{dist} \left(\int_B \nu_a d\lambda(a), \mathbf{Cons} \right) < \lambda(B)D$$

so $\lambda(B') > 0$. Define a state g by

$$g(a) = (y_a + e_a - \tau(\nu_a), \nu_a)$$

To see that the state g is feasible for B' note first that equation (11) and the definition of C^* entail that

$$x^* = \int_B y_a d\lambda \leq -\frac{W}{D} \text{dist}(\mu^*, \mathbf{Cons})\mathbf{1} = -\frac{W}{D} \text{dist} \left(\int_B \nu_a d\lambda(a), \mathbf{Cons} \right)\mathbf{1} \quad (14)$$

Additivity of integration entails that

$$\int_B y_a d\lambda = \int_{B'} y_a d\lambda + \int_{B \setminus B'} y_a d\lambda \quad (15)$$

Our bound on endowments and the definition of individual excess demand entails that

$$\int_{B \setminus B'} y_a d\lambda \geq -\lambda(B \setminus B')W\mathbf{1} \quad (16)$$

Combining equations (13), (14), (15) and (16) yields

$$\int_{B'} y_a d\lambda(a) \leq 0$$

and hence that

$$\int_{B'} [y_a + \tau(\ell_a)] d\lambda(a) \leq \int_{B'} e_a d\lambda$$

which is the material balance condition. Since equation (12) entails consistency for B' , we conclude that the state g is feasible for B' . By construction, g is preferred to $f = (x, \mu)$ by every member of B' , so this contradicts the assumption that f is a core state. We conclude that $Z \cap C^* = \emptyset$, as asserted.

Step 4 We now use the separation theorem to find prices $(p, q) \in \mathbf{R}_+^N \times \mathbf{R}^M$, $(p, q) \neq (0, 0)$ such that

$$\begin{aligned} (p, q) \cdot (\bar{x}, \bar{\mu}) &\leq 0 && \text{for each } (\bar{x}, \bar{\mu}) \in C^* \\ (p, q) \cdot z &\geq 0 && \text{for each } z \in Z \end{aligned}$$

Because C^* contains the cone $-\mathbf{R}_+^N \times \{0\}$, it follows that $p \geq 0$. Because C^* contains the subspace $\{0\} \times \mathbf{Cons}$, it follows that q vanishes on \mathbf{Cons} and hence that $q \in \mathbf{Trans}$. To see that $p \neq 0$, suppose to the contrary that $p = 0$. By construction, $(p, q^*) \neq (0, 0)$ so $q \neq 0$. Hence there is a $\bar{\mu} \in \mathbf{R}^M$ such that $q \cdot \bar{\mu} > 0$. For $\varepsilon > 0$ sufficiently small, $(-1, \varepsilon \bar{\mu}) \in C^*$, so that $(p, q) \cdot (-1, \varepsilon \bar{\mu}) \leq 0$. However

$$\begin{aligned} (p, q) \cdot (-1, \varepsilon \bar{\mu}) &= (0, q) \cdot (-1, \varepsilon \bar{\mu}) \\ &= \varepsilon q \cdot \bar{\mu} \end{aligned}$$

which, by our choice of $\bar{\mu}$, is positive. This is a contradiction, so we conclude that $p \neq 0$, as desired.

We claim that (f, p, q) is a pure transfer quasi-equilibrium. Feasibility of f is guaranteed by assumption; we need to check budget feasibility and quasi-optimization. To this end, let $E_1 \subset A$ be the set of agents for whom $f(a) = (x_a, \mu_a)$ is not in their budget set; that is, $a \in E_1$ if and only if expenditure strictly exceeds income:

$$\begin{aligned} \text{expenditure}(a) &= p \cdot [x_a + \tau(\mu_a)] + q \cdot \mu_a \\ &> p \cdot e_a = \text{income}(a) \end{aligned}$$

Write E_2 for the set of agents for whom income strictly exceeds expenditure. Measurability of the endowment mapping e implies that E_1, E_2 are measurable sets. Feasibility of f implies that the integral (over A) of expenditure must equal the integral of income. Hence, if $\lambda(E_1) > 0$ it must also be the case that $\lambda(E_2) > 0$. Strict monotonicity of preferences in private goods means that, for each $a \in A$ and each $\varepsilon > 0$, the choice vector $(x_a + \varepsilon \bar{e}, \mu_a)$ is strictly preferred to $f(a) = (x_a, \mu_a)$. Hence if $a \in E_2$ then there is an $\varepsilon_a > 0$ such that $(x_a + \varepsilon_a \bar{e}, \mu_a)$ costs strictly less than e_a and is strictly preferred to $f(a)$; we may choose ε_a to be a measurable function of a . Define $g : A \rightarrow \mathbf{R}_+^N \times \mathbf{R}^M$ by

$$g(a) = \begin{cases} (x_a + \varepsilon_a \bar{e} - e_a, \mu_a) & \text{if } a \in E_2 \\ (0, 0) & \text{otherwise} \end{cases}$$

By construction, g is a measurable selection from the correspondence Ψ , so $\int_A g(a) d\lambda(a) \in Z$. However, our construction guarantees that

$$(p, q) \cdot \int_A g(a) d\lambda(a) = \int_A (p, q) \cdot g(a) d\lambda(a) < 0$$

which contradicts the fact that (p, q) separates Z from C^* . We conclude that $\lambda(E_1) = 0$; that is, almost all agents are choosing in their budget set.

To check the quasi-optimization conditions, write E_3 for the set of agents who are not quasi-optimizing in their budget set; suppose that $\lambda(E_3) > 0$. Note that $a \in E_3$ if and only if there is a choice vector (y_a, ν_a) which is strictly preferred to (x_a, μ_a) and costs strictly less than a 's endowment; we may choose these choice vectors so that the mapping $a \mapsto (y_a, \nu_a)$ is measurable. Define $h : A \rightarrow \mathbf{R}_+^N \times \mathbf{R}^M$ by

$$h(a) = \begin{cases} (y_a - e_a, \nu_a) & \text{if } a \in E_3 \\ (0, 0) & \text{otherwise} \end{cases}$$

By construction, h is a measurable selection from Ψ so $\int_A h(a) d\lambda(a) \in Z$. However, our construction guarantees that

$$(p, q) \cdot \int_A h(a) d\lambda(a) = \int_A (p, q) \cdot h(a) d\lambda(a) < 0$$

which contradicts the fact that (p, q) separates Z from C^* . We conclude that $\lambda(E_3) = 0$; that is, almost all agents are quasi-optimizing in their budget set.

It follows that (f, p, q) is a pure transfer quasi-equilibrium. Setting

$$q_m^* = q_m + \frac{1}{|\pi|} p \cdot \text{inp}(\pi, \gamma)$$

for each $m \in \mathcal{M}$ yields a quasi-equilibrium f, p, q^* . If f is club linked, it follows from Proposition 3.4 that (f, p, q^*) is an equilibrium.

Finally, if \mathcal{E} is club irreducible, then every feasible state is club linked and hence every core state can be supported as an equilibrium. By Theorem 4.1, every equilibrium state belongs to the core. Hence the core coincides with the set of equilibrium states. ■

We now turn to the task of establishing existence of equilibrium. We begin by finding upper and lower bounds for list prices.

Write $\mathbf{Lists}_M = \{\ell \in \mathbf{Lists} : |\ell| \leq M\}$. By analogy with a notion from cooperative game theory, we say that a set $L \subset \mathbf{Lists}_M$ is *strictly balanced* if there are strictly positive real numbers $\{\epsilon_L(\ell) : \ell \in L\}$ (which we call *balancing weights*) such that

$$\sum_{\ell \in L} \epsilon_L(\ell) \ell \in \mathbf{Cons}$$

Lemma 8.1 *There is a constant R^* with the following property:*

If $L \subset \mathbf{Lists}_M$ is a strictly balanced collection and $q \in \mathbf{Trans}$ is a pure transfer then

$$\max_{\ell \in L} q \cdot \ell \geq -R^* \min_{\ell \in L} q \cdot \ell$$

Proof For each strictly balanced collection L , choose strictly positive balancing weights $\{\epsilon_L(\ell) : \ell \in L\}$ and set

$$R(L) = \frac{\max\{\epsilon_L(\ell) : \ell \in L\}}{\min\{\epsilon_L(\ell) : \ell \in L\}}$$

Define

$$R^* = \max_L (R(L)(|L| - 1))$$

where the maximum extends over the finite set of strictly balanced collections L .

To see that R^* has the desired property, let $q \in \mathbf{Trans}$ be a pure transfer and observe that

$$\sum_{\ell \in L} \epsilon_L(\ell)(q \cdot \ell) = q \cdot \sum_{\ell \in L} \epsilon_L(\ell)\ell = 0 \quad (17)$$

Choose a list $\ell^* \in L$ such that

$$q \cdot \ell^* = \max_{\ell \in L} q \cdot \ell$$

Rearranging terms in equation (17) and carefully keeping track of signs, we find:

$$\begin{aligned} q \cdot \ell^* &= -\frac{1}{\epsilon_L(\ell^*)} \sum_{\ell \in L, \ell \neq \ell^*} \epsilon_L(\ell)q \cdot \ell \\ &= -\sum_{\ell \in L, \ell \neq \ell^*} \frac{\epsilon_L(\ell)}{\epsilon_L(\ell^*)} q \cdot \ell \\ &\geq -(|L| - 1) \min_{\ell \in L} (R(L)(q \cdot \ell)) \\ &\geq -\left(\min_{\ell \in L} q \cdot \ell\right) R^* \end{aligned}$$

which is the desired inequality. ■

With this lemma in hand we establish the existence of equilibrium.

Proof of Theorem 7.1 By assumption, aggregate endowment \bar{e} is strictly positive and individual endowments are uniformly bounded above; say that $\bar{e} \geq w\mathbf{1} \gg 0$ and that $e_a \leq W_0\mathbf{1}$ for all $a \in A$. Write $W = \max\{W_0, 1\}$. We assume without loss that $\lambda(A) = 1$.

Step 1 Fix an integer $k > 0$. Choose a family $\{A_\omega^k : \omega \in \Omega\}$ of pairwise disjoint intervals in \mathbf{R} , each of length $1/k$. Write

$$A^* = \bigcup_{\omega \in \Omega} A_\omega^k$$

We define the agent space $(A^k, \mathcal{F}^k, \lambda^k)$ for the perturbed economy \mathcal{E}^k by setting $A^k = A \cup A^*$, defining \mathcal{F}^k to be the σ -algebra generated by \mathcal{F} and the Lebesgue measurable subsets of A^* , and defining λ^k to be λ on A and Lebesgue measure on A^* . Note that $\lambda^k(A^k) = 1 + \frac{|\Omega|}{k}$. External characteristics, consumption sets, endowments and utility functions of agents in A are just as in the original club economy \mathcal{E} . For agents $a \in A_\omega^k$, we define:

$$\begin{aligned}\omega_a &= \omega \\ X_a &= \mathbf{R}_+^N \times \{\ell \in \mathbf{Lists}_M : \ell(\omega', \pi, \gamma) = 0 \text{ if } \omega' \neq \omega\} \\ e_a &= W\mathbf{1} \\ u_a(x, \ell) &= |x|\end{aligned}$$

Step 2 The demand functions of these added agents is such that, for commodity prices near the boundary of the simplex and for membership prices that are large in absolute value, their commodity excess demand will be very large. This will lead to aggregate excess demands that are impossibly large. As a consequence, we can write down compact price sets that contain an equilibrium price for \mathcal{E}^k .

To define these sets, set

$$M^* = \max\{|\pi| : (\pi, \gamma) \in \mathbf{Clubs}\}$$

Choose a real number $\varepsilon > 0$ so small that

$$\left[1 - (N - 1)\varepsilon\right] \left[\frac{W}{kN\varepsilon} - W\left(1 + \frac{|\Omega|}{k}\right)\right] - \varepsilon(N - 1)W\left(1 + \frac{|\Omega|}{k}\right) > 0$$

Having chosen ε , choose a real number $R > 0$ so big that

$$\left[\frac{R}{kNM^*} - W\left(1 + \frac{|\Omega|}{k}\right)\right] \left[1 - (N - 1)\varepsilon\right] - \varepsilon(N - 1)W\left(1 + \frac{|\Omega|}{k}\right) > 0$$

Of course ε, R depend on k . Define a price simplex for private goods and a bounded price set for club memberships:

$$\begin{aligned}\Delta_\varepsilon &= \{p \in \mathbf{R}_+^N : p_n \geq \varepsilon \text{ for each } n\} \\ Q_R &= \{q \in \mathbf{Trans} : |q_m| \leq R \text{ for all } m \in \mathcal{M}\}\end{aligned}$$

Step 3 We define an excess demand correspondence. As in the proof of Theorem 6.1, define

$$\tau(\ell) = \sum_{(\omega, \pi, \gamma) \in \mathcal{M}} \ell(\omega, \pi, \gamma) \frac{1}{|\pi|} \mathbf{inp}(\pi, \gamma)$$

Let $p \in \Delta_\varepsilon, q \in Q_R$. For each agent $a \in A$, write

$$B(a, p, q) = \{(x, \ell) \in X_a : p \cdot x + q \cdot \ell + p \cdot \tau(\ell) \leq p \cdot e_a\}$$

As in the proof of Theorem 6.1, this is agent a 's budget set, assuming that he is required to pay his share of the inputs to club activities. Let

$$d(a, p, q) = \operatorname{argmax} \{u_a(x, \ell) : (x, \ell) \in B(a, p, q)\}$$

be the set of utility optimal choices in agent a 's budget set; that is, $d(a, p, q)$ is agent a 's demand set. Define agent a 's excess demand set to be

$$\zeta(a, p, q) = d(a, p, q) - (e_a, 0)$$

It is easily checked that excess demand sets are uniformly bounded (because endowments are bounded, private good prices are bounded away from 0 and club membership prices are bounded above and below). Moreover the correspondence $(a, p, q) \rightarrow \zeta(a, p, q)$ is measurable and, for each fixed a , is upper hemi-continuous (in p, q). Define the aggregate excess demand correspondence

$$Z : \Delta_\varepsilon \times Q_R \rightarrow \mathbf{R}_+^N \times \mathbf{R}^M$$

to be the integral of the individual excess demand correspondences:

$$Z(p, q) = \int_A \zeta(a, p, q) d\lambda(a)$$

As the integral of an upper hemi-continuous correspondence with respect to a non-atomic measure, Z is upper hemi-continuous, with compact, convex, non-empty values.

Step 4 We find a fixed point of the excess demand correspondence, in a slightly roundabout way. Note first that individual income comes from selling private good endowments and receiving subsidies for club memberships;

because private good endowments are bounded by W , private good prices are bounded below by ε and sum to 1, and club membership prices lie in the interval $[-R, +R]$, this means that individual demand for private goods is bounded above by $\frac{1}{\varepsilon}(W + RM)$. Hence individual (and aggregate) excess demands for private goods lie in the compact set

$$X = \{x \in \mathbf{R}^N : -W \leq x_n \leq \frac{1}{\varepsilon}(W + RM) \text{ for each } n\}$$

By assumption, agents can choose at most M memberships, so individual and aggregate demands for club memberships lie in the set

$$C = \{\bar{\mu} \in \mathbf{R}_+^M : \sum_{m \in \mathcal{M}} \bar{\mu}(m) \leq M\}$$

Define a correspondence

$$\Phi : \Delta_\varepsilon \times Q_R \times X \times C \rightarrow \Delta_\varepsilon \times Q_R \times X \times C$$

by

$$\Phi(p, q, x, \bar{\mu}) = \left[\operatorname{argmax} \{(p^*, q^*) \cdot (x, \bar{\mu}) : (p^*, q^*) \in \Delta_\varepsilon \times Q_R\} \right] \times Z(p, q)$$

It is easily checked that Φ is upper hemi-continuous with compact convex values. Hence Kakutani's fixed point theorem guarantees that Φ has a fixed point. Thus there is a price pair $(p^k, q^k) \in \Delta_\varepsilon \times Q_R$ and a consumption/club membership pair $(z^k, \bar{\mu}^k) \in Z(p^k, q^k)$ such that

$$(p^k, q^k) \cdot (z^k, \bar{\mu}^k) = \max \{(p^*, q^*) \cdot (z, \bar{\mu}) : (p^*, q^*) \in \Delta_\varepsilon \times Q_R, (z, \bar{\mu}) \in Z(p^k, q^k)\}$$

Walras's law implies that

$$(p^k, q^k) \cdot (z^k, \bar{\mu}^k) = 0$$

Step 5 We show that $z^k = 0$ and $\bar{\mu}^k \in \mathbf{Cons}$. The argument is in several parts.

Step 5.1 We show first that $q^k \cdot \bar{\mu}^k = 0$. Suppose that this is not so. We obtain a contradiction by looking at excess demands (at prices p^k, q^k) of agents in $A^k \setminus A$. Maximality and the definition of Φ entail that $q^k \cdot \bar{\mu}^k > 0$ (because $0 \cdot \bar{\mu}^k = 0$). Maximality entails that $q^k \in \text{bdy } Q_R$ so that $|q_m^k| = R$ for some $m \in \mathcal{M}$. The budget balance condition for clubs means that if some price has large magnitude and is positive then some other price must have large magnitude and be negative. Thus there is a membership m^* such that $q_{m^*}^k \leq -R/M^*$. The agents in $A_{\omega^*}^k$ (whom we have adjoined to the original set of agents, and whose external characteristic is ω^*), could obtain a subsidy of R/M^* by choosing the membership m^* (and no other). Because agents in $A_{\omega^*}^k$ don't care at all about club memberships and find all private goods to be perfect substitutes, they will choose to consume only the least expensive private good and to choose all club memberships whose prices are negative and no club memberships whose prices are positive. It follows that their excess demand for the least expensive private good — which we may as well suppose is good 1 — is at least

$$\zeta_1(b, p^k, q^k) \geq \frac{R}{NM^*}$$

Keeping in mind that $\lambda(A_{\omega^*}^k) = 1/k$ and that the excess demand of each agent is bounded below by $-W\mathbf{1}$, it follows that the aggregate excess demand for good 1 and for other private goods satisfy:

$$\begin{aligned} z_1^k &\geq \frac{1}{k} \frac{R}{NM^*} - W\left(1 + \frac{|\Omega|}{k}\right) \\ z_n^k &\geq -W\left(1 + \frac{|\Omega|}{k}\right) \end{aligned}$$

Define $p \in \Delta_\varepsilon$ by:

$$p_n = \begin{cases} 1 - (N-1)\varepsilon & \text{if } n = 1 \\ \varepsilon & \text{if } n > 1 \end{cases}$$

Calculation shows that

$$p \cdot z^k \geq [1 - (N-1)\varepsilon] \left[\frac{R}{kNM^*} - W\left(1 + \frac{|\Omega|}{k}\right) \right] - \varepsilon(N-1)W\left(1 + \frac{|\Omega|}{k}\right)$$

Our choices of R, ε guarantee that this is strictly positive, so that

$$(p, 0) \cdot (z^k, \bar{\mu}^k) > 0 = (p^k, q^k) \cdot (z^k, \bar{\mu}^k)$$

which contradicts maximality. We conclude that $q^k \cdot \bar{\mu}^k = 0$, as desired.

Step 5.2 We show next that $\bar{\mu}^k \in \text{Cons}$. If not, we could find a pure transfer $q^* \in \text{Trans}$ such that $q^* \cdot \bar{\mu}^k > 0$ and hence could find a $q^{**} \in Q_R$ such that $q^{**} \cdot \bar{\mu}^k > 0$, contradicting maximality.

Step 5.3 We claim that $p_n^k > \varepsilon$ for each n . Suppose not; we once again obtain a contradiction by considering the excess demand of agents in $A^* = A^k \setminus A$. Every agent in A^* finds all commodities to be perfect substitutes, and therefore demands only the least expensive commodities. Because agents in A^* have endowment $W1$ and hence wealth W , there is at least one commodity, say commodity 1, for which the excess demand of each agent in A^* is at least

$$\zeta_1(a, p^k, q^k) \geq \frac{W}{N\varepsilon}$$

Integrating over all agents and keeping in mind that individual excess demands are bounded below by $-W1$ and that $\lambda^k(A_\omega^K) = 1/k$, we conclude that

$$z_1^k \geq \frac{1}{k} \frac{W}{N\varepsilon} - W(1 + \frac{|\Omega|}{k})$$

$$z_n^k \geq -W(1 + \frac{|\Omega|}{k})$$

Define $p \in \Delta_\varepsilon$ by

$$p_n = \begin{cases} 1 - (N-1)\varepsilon & \text{if } n = 1 \\ \varepsilon & \text{if } n > 1 \end{cases}$$

Calculation gives

$$p \cdot z^k \geq [1 - (N-1)\varepsilon] \left[\frac{W}{kN\varepsilon} - W(1 + \frac{|\Omega|}{k}) \right] - \varepsilon(N-1)W(1 + \frac{|\Omega|}{k})$$

Our choice of ε guarantees that this is strictly positive and hence that

$$(p, 0) \cdot (z^k, \bar{\mu}^k) > 0 = (p^k, q^k) \cdot (z^k, \bar{\mu}^k)$$

which again contradicts maximality. We conclude that $p_n^k > \varepsilon$ for each n .

Step 5.4 We show that $z^k = 0$. If $z^k \neq 0$ there are indices i, j such that $z_i^k < 0$ and $z_j^k > 0$. Since $(p^k, q^k) \cdot (z^k, \bar{\mu}^k) = 0$ and $q^k \cdot \bar{\mu}^k = 0$ it follows that $p^k \cdot z^k = 0$. Since $p_i^k > \varepsilon$, we can construct a price $\hat{p} \in \Delta_\varepsilon$ by setting

$$\hat{p}_n = \begin{cases} p_i^k - \frac{1}{2}(p_i^k - \varepsilon) & \text{if } n = i \\ p_j^k + \frac{1}{2}(p_i^k - \varepsilon) & \text{if } n = j \\ p_n^k & \text{otherwise} \end{cases}$$

Since $p^k \cdot z^k = 0$, it follows that $\hat{p} \cdot z^k > 0$, a contradiction. We conclude that $z^k = 0$.

Step 6 By definition, there is a selection $g(a) = (y_a, \mu_a)$ from the individual excess demand sets which integrates to $(z^k, \bar{\mu}^k)$. Set

$$y_a^* = y_a + e_a - \tau(\mu_a)$$

Setting $f^k = (y^*, \mu)$ yields a state of the economy \mathcal{E}^k . Since we have just shown that commodity excess demand $z^k = 0$ and that $\bar{\mu}^k \in \mathbf{Cons}$, we conclude that (f^k, p^k, q^k) constitutes a pure transfer quasi-equilibrium for \mathcal{E}^k . Since \mathcal{E}^k is club irreducible, (f^k, p^k, q^k) in fact constitutes a pure transfer equilibrium for \mathcal{E}^k .

Step 7 Our price normalization entails that private good prices p^k are bounded by 1; our construction entails that club membership prices q^k are bounded by R , but R depends on k . We now replace the sequence of membership prices q^k by membership prices \bar{q}^k which lead to the same demands and are bounded independently of k .

Passing to a subsequence if necessary, we may assume that for each $\ell \in \mathbf{Lists}_M$ the sequence $(q^k \cdot \ell)$ converges to a limit G_ℓ , which may be finite or

infinite. Define:

$$\begin{aligned} L &= \{\ell \in \mathbf{Lists}_M : q^k \cdot \ell \rightarrow G_\ell \in \mathbf{R}\} \\ L_+ &= \{\ell \in \mathbf{Lists}_M : q^k \cdot \ell \rightarrow +\infty\} \\ L_- &= \{\ell \in \mathbf{Lists}_M : q^k \cdot \ell \rightarrow -\infty\} \end{aligned}$$

Choose $\bar{G} \in \mathbf{R}$ so large that $|q^k \cdot \ell| \leq \bar{G}$ for each k , each $\ell \in L$.

Define a linear transformation $T : \mathbf{Trans} \rightarrow \mathbf{R}^L$ by $T(q)_\ell = q \cdot \ell$. Write $\text{ran } T = T(\mathbf{Trans}) \subset \mathbf{R}^L$ for the range of T and $\ker T = T^{-1}(0) \subset \mathbf{Trans}$ for the kernel (null space) of T . The fundamental theorem of linear algebra implies that we can choose a subspace $H \subset \mathbf{Trans}$ so that $H \cap \ker T = \{0\}$ and $H + \ker T = \mathbf{Trans}$. Write $T|_H$ for the restriction of T to H . Note that $T|_H : H \rightarrow \text{ran } T$ is a one-to-one and onto linear transformation, so it has an inverse $S : \text{ran } T \rightarrow H$. Because S is a linear transformation, it is continuous, so there is a constant K such that $|S(x)| \leq K|x|$ for each $x \in \text{ran } T$.

Let R^* be the constant constructed in Lemma 8.1. Choose k_0 so large that $k \geq k_0$ implies

$$\begin{aligned} q^k \cdot \ell &> +2K\bar{G} + W && \text{if } \ell \in L_+ \\ q^k \cdot \ell &< -2K\bar{G} - R^*W && \text{if } \ell \in L_- \end{aligned}$$

Write ST for the composition of S with T . For each $k \geq k_0$ set

$$\bar{q}^k = ST(q^k) - ST(q^{k_0}) + q^{k_0} \in \mathbf{Trans}$$

Because $S, T|_H$ are inverses the composition TS is the identity, so

$$T(\bar{q}^k) = TST(q^k) - TST(q^{k_0}) + T(q^{k_0}) = T(q^k)$$

We claim that for $k > k_0$, the triple (f^k, p^k, \bar{q}^k) constitutes a pure transfer equilibrium. To see this, we first consider the prices of lists. For $\ell \in L$, $\bar{q}^k \cdot \ell = q^k \cdot \ell$ because $T(\bar{q}^k) = T(q^k)$. For $\ell \in L_+$, $\bar{q}^k \cdot \ell > W$ because $|ST(q^k)| \leq K\bar{G}$, $|ST(q^{k_0})| \leq K\bar{G}$ and $q^{k_0} \cdot \ell > W + 2K\bar{G}$. For $\ell \in L_-$, $\bar{q}^k \cdot \ell < -R^*W$ because $|ST(q^k)| \leq K\bar{G}$, $|ST(q^{k_0})| \leq K\bar{G}$ and $q^{k_0} \cdot \ell < -R^*W - 2K\bar{G}$.

To check the equilibrium conditions, keep in mind that individual demands for private goods and club memberships depend only on the prices of private goods and of lists, not directly on the prices of memberships. Because endowments are bounded above by W and private goods prices sum to 1, individual wealth is also bounded above by W . Hence no list whose price exceeds W is ever demanded; in particular, no list in L_+ is demanded at prices p^k, q^k or at prices p^k, \bar{q}^k . Moreover, because the set of lists demanded at an equilibrium is strictly balanced, it follows from Lemma 8.1 that no list in L_- is demanded at prices p^k, q^k . By construction, prices for lists in L_- are higher with respect to \bar{q}^k than with respect to q^k , so no lists in L_- are demanded at prices p^k, \bar{q}^k : if no one is willing to buy a list when a large subsidy is provided, no one will be willing to buy it when the subsidy is reduced. Since prices for lists in L are the same with respect to \bar{q}^k as with respect to q^k , it follows that demands are the same with respect to p^k, \bar{q}^k as they are with respect to p^k, q^k . (In words: When we replace membership prices q^k with membership prices \bar{q}^k we lower the prices of some unaffordable lists, but we keep them so high that they remain unaffordable. We also lower the subsidies of some lists, but lists that are not demanded when subsidies are large will not be demanded when subsidies are smaller. Hence we do not change demands.) It follows that (f^k, p^k, \bar{q}^k) is a pure transfer equilibrium for \mathcal{E}^k . By construction, $|\bar{q}^k \cdot \ell| \leq 2K\bar{G} + |q^{k_0} \cdot \ell|$ for k, ℓ ; because singleton memberships are themselves lists, it follows that (\bar{q}^k) is a bounded sequence in **Trans**.

Step 8 In view of this construction, we have bounded sequences (p^k) of private goods prices, (\bar{q}^k) of membership prices and (μ^k) of aggregate membership choices. Passing to a subsequence if necessary, we may assume that $p^k \rightarrow p^*$, $\bar{q}^k \rightarrow q^*$, $\bar{\mu}^k \rightarrow \bar{\mu}^*$. We may now employ Schmeidler's version of Fatou's lemma (see Hildenbrand (1974)) to conclude that there is a measurable mapping $f^* : A \rightarrow \mathbf{R}_+^N \times \mathbf{R}^{\mathcal{M}}$ such that

- for almost all $a \in A$, $f^*(a) \in B(a, p^*, q^*) \subset X_a$
- for almost all $a \in A$, $f^*(a)$ belongs to agent a 's quasi-demand set; that is, there does not exist $(x', \ell') \in X_a$ such that $u_a(x', \ell') > u_a(f^*(a))$

and $(p^*, q^*) \cdot (x', \ell') + p^* \cdot \tau(\ell') < p^* \cdot e_a$

- $\int_A f^*(a) d\lambda = (\bar{e}, \bar{\mu}^*)$

By definition, (f^*, p^*, q^*) is a pure transfer quasi-equilibrium for \mathcal{E} . Club irreducibility implies that (f^*, p^*, q^*) constitute a pure transfer equilibrium for \mathcal{E} , so the proof is complete. ■

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