Time–to-Build and Cycles

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Time–to–build and Cycles*

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Abstract

We analyze the dynamics of a simple growth model in which production occurs with a delay while new capital is installed (time–to–build). The time–to–build technology is shown to yield a system of functional (delay) differential equations with a unique steady state. We demonstrate that the steady state, though typically a saddle, may exhibit Hopf cycles on a measurable set of the parameter space. Furthermore, the optimal path to the steady state is oscillatory. A counter–example to the claim that "models with a time–to–build technology are not intrinsically oscillatory" is provided. We also provide a primer on the central technical apparatus — the mathematics of functional differential equations.
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1 Introduction

The preparation of this paper was prompted by a desire to clarify the precise theoretical relationship between time–to–build (investment gestation lags) and cycles in deterministic

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neoclassical growth models. Is the time–to–build feature essential to cyclical fluctuations— as argued by Kydland & Prescott (1982) — or is it some other feature of this class of models that delivers the oscillatory behavior — as argued by Ioannides & Taub (1992)? Of related interest is the optimality of oscillatory paths in this class of models.

How would one convince an economist of the relationship between time–to–build and cycles in this class of models? One obvious strategy is to write down a simple growth model, introduce a time–to–build technology through production lags, and then analyze the dynamics of the model using appropriate methods. Since it is well known that the standard deterministic growth model with one good never admits cycles; this strategy enables the precise theoretical role of the time–to–build technology to be clearly established.[1]

The above mentioned strategy is the one that we follow in this paper. We investigate the dynamics of a simple equilibrium growth model with an infinitely lived representative agent who trades a single consumption–investment good. Investment gestation lags are introduced by assuming production occurs with a delay while new capital is installed. We demonstrate that the optimality conditions of the model yield a system of functional (or delay) differential equations with a unique steady state. A constructive proof demonstrates that the steady state, though typically a saddle, may exhibit Hopf cycles on a measurable set of the parameter space. We show that the presence of cycles is due entirely to the time–to–build technology and that oscillating paths are optimal.

The idea that lags, represented by functional differential equations, can induce cyclical behavior is not new. Researchers at least since the time of Jevons have conjectured that production lags can induce cycles in output.[2] However, despite the long history of its use in economic analysis, delay differential equations have rarely been used in modern (equilibrium) neoclassical macroeconomics.[3] El–Hodiri *et al.* (1967) is to the best of our knowledge, the

[1]Benhabib & Nishimura (1979) show that the representative agent model can generate cycles only if there are three or more consumption goods.

[2]See also Frisch & Holme (1935) and James & Belz (1938) for early work along these lines.

[3]There are several examples of models using delay differential equations in microeconomics. Howroyd & Russell (1984) develop a Cournot oligopoly model in which each firm adjusts its output to counter competitors

first paper in modern macroeconomics to incorporate lags into an optimal growth model. However, the authors did not analyze the relationship between lags and cycles. Rustichini (1987, 1989) introduces a sophisticated lag structure in the optimal growth model and provides a rigorous proof of the existence of cycles.

The present analysis differs from these papers in that the production process is such that only the lagged state variable yields output and depreciates, not the current state. If production depends on both the lagged and current state, the optimality conditions yield a system of mixed functional differential equations with advanced and delayed time arguments, as in Rustichini (1989). In contrast, in the present analysis, we show that even a simple production lag structure will admit cycles.

The rest of the paper is organized as follows. In Section 2 we provide a primer on the mathematics of functional differential equations — the technical apparatus needed to understand the main ideas in the paper. In Section 3 we characterize the dynamics of a simple deterministic continuous–time growth model with a time–to–build technology. In Section 4 we demonstrate the optimality of Hopf cycles in this class of models. Section 5 concludes the paper.

2 FUNCTIONAL DIFFERENTIAL EQUATIONS WITHOUT TEARS

In this section we provide an accessible primer on the mathematics of functional differential equations (FDEs). A *retarded* functional differential equation is a differential equation in which the current behavior of the system depends on past history.[4] Henceforth, the abbreviation FDE will denote FDEs of retarded type. We limit our attention to FDEs of retarded type, because it is the class of equations relevant to the present analysis. Readers familiar

with a lag that signifies the time it takes for information to be received and processed. Béliar & Mackey (1989) and Mackey (1989) consider the dynamics of price adjustments in which prices are a function of past values which produces a system of delay differential equations.

[4]Functional differential equations that depend on future states are referred to as *advanced* functional differential equations.

with FDEs can proceed directly to the next section.

Functional differential equations generalize ordinary differential equations (ODEs) by allowing the state of the system at time t to depend on states other than at the current time, via a feedback mechanism. A typical representation is

$$\dot{x}(t) = F[t, x(t)g(x(t-r))], \tag{1}$$

where $\dot{x}(t)$ is the derivative of $x(t), g(\cdot)$ is the feedback function and $r > 0$ is the time delay.

One of the simplest FDEs is a delay differential equation (DDE) in which the feedback mechanism is the identity function. An example is the linear DDE

$$\dot{x}(t) = ax(t) + bx(t-r) + f(t),$$

where a and b are constants, and $f(t)$ is the forcing function. Other examples of DDEs are:

1. A two–delay differential equation

$$\dot{x}(t) - u(t-1) - u(t-2) = 0.$$

2. A second order delay differential equation

$$\ddot{u}(t) - \dot{u}(t-1) + u(t) = 0.$$

3. A *neutral* DDE (i.e. it has a time–delayed derivative)

$$\dot{x}(t) - c\dot{x}(t-r) - Ax(t) - Bx(t-r) - f(t) = 0.$$

4. A DDE often used to model population growth

$$\dot{N}(t) = k[1 - \frac{N(t-r)}{P}]N(t), \tag{2}$$

where the delay $r > 0$ represents periods before reproductive maturity is reached and P and k are positive constants.

2.1 *Solving functional differential equations*

DEFINITION 1 (HALE 1977) *Suppose $r \geq 0$ is a given real number, R^n is an n–dimensional vector space over the reals with norm $|\cdot|$ and $C([a,b],R^n)$ is the Banach space of continuous functions mapping the interval $[a,b]$ into R^n with the topology of uniform convergence. If $[a,b] = [-r,0]$, let $C = C([-r,0],R^n)$ and designate the norm of an element ϕ in C by $|\phi| = \sup_{-r \leq \theta \leq 0} |\phi(\theta)|$. If $\sigma \in R, A \geq 0$ and $x \in C([\sigma - r, \sigma + A], R^n)$, then for any $t \in [\sigma, \sigma + A]$ we let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta)$, for $-r \leq \theta \leq 0$. If D is a subset of $R \times C, f : D \to R$ is a given function and $\cdot$ represents the right hand derivative, we say that the relation*

$$\dot{x}(t) = f(t, x_t) \tag{3}$$

is a retarded functional differential equation on D. A function x is said to be a solution of (3) on $[\sigma - r, \sigma + A)$ if there are $\sigma \in R$ and $A > 0$ such that $x \in C([\sigma - r, \sigma + A], R^n), (t, x_t) \in D$ and $x(t)$ satisfies (3) with boundary condition ϕ at σ if there is an $A > 0$ such that $x(\sigma, \phi, f)$ is a solution on $C([\sigma - r, \sigma + A)$ and $x_\sigma(\sigma, \phi, f) = \phi$.

Definition 1 is sufficiently general to encompass ordinary differential equations, delay differential equations and integro–differential equations.

Next, we turn to the boundary conditions required for unique solutions to DDEs. Since DDEs rely on history to determine current behavior, a single datum such as $x(t_0) = x_0$, which would be used to pin down a solution for an ODE, doe not generally contain sufficient information to provide a solution to a DDE.[5] An initial function which identifies the relevant history of the equation before the system begins its motion must be specified. In (1) where the delay is $r > 0$, an initial function on $[t_0 - r, t_0]$ must be given when t_0 is the time at which the system starts.[6]

[5]Note that standard definitions used to describe ODEs equations such as "linear, " "homogeneous," "autonomous" and of "n^{th} order" all apply to DDEs.

[6]An example of an initial function would be the initial distribution of wealth. See Bélair & Mackey (1989) for an innovative use of this type of initial function.

The algebraic *methods of steps* can be used to prove the existence of a solution to a constant coefficient DDE. For example, consider a generic linear DDE with constant coefficients A and B and delay $r > 0$, given by

$$\dot{x}(t) = Ax(t) + Bx(t-r) + f(t). \tag{4}$$

The following theorem formalizes the notion of a solution to the generic DDE (4).

THEOREM 1 (HALE 1977) *If ϕ is a given continuous function on $[-r, 0]$, then there exists a unique function $x(\phi, f)$ defined on $[-r, \infty]$ which coincides with ϕ on $[-r, 0]$ and satisfies (4) for all $t > 0$.*

PROOF: Apply the method of steps repeated on intervals of length r beginning with $[-r, 0]$.

■

To illustrate the method of steps, consider the DDE used to model population growth given by (2). Observe that by making a change of variables $x(t) = N(rs)/P - 1$ and $c = kr$, (2) can be written as

$$\dot{x}(t) = -cx(t-1)[1 + x(t)]. \tag{5}$$

Now suppose one is given the initial function ϕ defined on $[-1, 0]$ and would like to solve the equation for a function x, where $x(t) = \phi(t)$ on the interval $[-1, 0]$ and where x satisfies (5) for $t > 0$. This can be done by partitioning $[0, \infty)$ into steps the size of the delay, which in this case is 1. Initially, we limit our attention to (5), defined on the interval $[0, 1]$, given the function $\phi(t)$ (which we assume is continuous in this example, though discontinuities can be handled as well). Writing the equation on $[0, 1]$ explicitly, we have

$$\dot{x}(t) + c\phi(t-1)x(t) = -c\phi(t-1),$$

which is an ordinary differential equation with initial condition $x(0) = \phi(0)$ because the function $\phi(t-1)$ is known. Using an integrating factor $\exp[\int_0^t c\phi(s-1)ds]$, one finds the unique solution on $[0, 1]$ is

$$x(t) = [\phi(0) + 1]e^{-\int_0^t c\phi(s-1)ds} - 1.$$

Once it is found that the solution is on the interval $[0, 1]$, the procedure is repeated on the interval $[1, 2]$ as the past history of $x(t)$ is now known for $[0, 1]$. It is readily apparent that the delay in a DDE provides a natural method by which the class of constant coefficient equations can be solved, even when the equations are nonlinear, as in this example. It is equally obvious that this method requires tedious computations and often yields cumbersome solutions.

An alternative method of solving DDEs—similar to methods used to solve ODEs — uses the characteristic equation associated with the DDE. The characteristic equation of a linear DDE with delays r_j

$$\dot{y}(t) = \sum_{j=1}^{n} A_j y(t - r_j), \tag{6}$$

is given by $\text{DET}[I\lambda - \sum_{j=1}^{n} A_j e^{-\lambda r_j}] = 0$. If $r_j = 0$ for all j, then (6) is an ordinary differential equation and the characteristic equation has the familiar polynomial form.

One solves a non–homogeneous FDE the same way as one would an ODE. First, find a general solution to the homogeneous problem; then, find a particular solution to the nonhomogeneous problem; finally, apply the principle of superposition. Recall that the principle of superposition states that a linear combination of solutions to a differential equation is also a solution. The eigenvalues which solve the characteristic polynomial and their associated eigenvectors are the elements from which a solution to a DDE is constructed.

To see this, consider the homogeneous part of (4). That is

$$\dot{x}(t) - Ax(t) - Bx(t - r) = 0.$$

The characteristic equation is

$$h(\lambda) = \lambda - A - Be^{-\lambda r} = 0. \tag{7}$$

The characteristic equation is found by substituting candidate solutions of the form $e^{-\lambda r}$ into the dynamical system. The exponential term associated with the delayed argument can be thought of as a "time consistency" construct.

The general solution to a non–homogeneous FDE is found by using the characteristic polynomial and a generalization of the method of steps—the Laplace transform. The particular Laplace transform we use is the inverse of the characteristic polynomial of the homogeneous problem, $h^{-1}(\lambda)$. An intermediate result is needed before we state the next theorem.

LEMMA 1 (EXISTENCE AND CONVOLUTION OF THE LAPLACE TRANSFORM) :
If $f : [0, \infty) \to R$ is measurable and satisfies $|f(t)| < ae^{bt}$ for $t \in [0, \infty)$ for some constants a and b, then the Laplace transformation $\mathcal{L}\ (f)$ given by $\mathcal{L}\ (f)(\lambda) = \int_0^\infty f(t)e^{-\lambda t}dt$ exists and is analytic for $Re(\lambda) > b$. If $f \circ g$ is defined by $f \circ g(t) = \int_0^t f(t-s)g(s)ds$, then $\mathcal{L}$ $(f \circ g) = \mathcal{L}(f)\mathcal{L}(g)$.

Lemma 1 establishes that a Laplace transform exists under weak restrictions on the function f. These are the conditions under which a solution to a general DDE exists.

THEOREM 2 (HALE 1977) *The solution to the generic DDE given by (4) with initial function $x(t) = 0$ for $t < 0$ and $x(t) = 1$ for $t = 0$ is the fundamental solution. That is, it satisfies $\mathcal{L}\ (x)(\lambda) = h^{-1}(\lambda)$; and, for any $c > b$, $x(t) = \int_{(c)} h^{-1}(\ \lambda)e^{\lambda t}d\lambda, t > 0$, where b is the bound on $x(t), |x(t)| \leq ae^{bt}, t > 0$ and the notation $\int_{(c)} \equiv \lim_{T\to\infty} 1/2 \int_{c-iT}^{c+it}$.*

Although Theorem 2 appears quite complicated, it states that solutions to FDEs will, in general, exist if the transition equation $F(t, x(t), x(t-r))$ in (1) is continuous and defined on a compact Banach space. Uniqueness follows if $F(\cdot)$is Lipschitzian in ϕ, the initial function, on a compact set. Solutions to FDEs are usually continuous functions of time and depend continuously on the parameters of the problem, including the delay, r.

Linear constant coefficient DDEs are often solvable for exact solutions using a modified variation of parameters procedure from ODEs. The homogeneous problem is solved first and then a particular solution is appended to form the general solution. However, it is nonlinear delay differential equations which generally arise in nontrivial economic applications and these almost never admit exact solutions.[7] As a result, after identifying several differences between DDEs and ODEs, the next section addresses the qualitative analysis of DDEs using characteristic equations.

Up to this point we have shown that the nature of the solutions and the methods used to solve DDEs are quite similar to ODEs. Nevertheless, DDEs exhibit more complicated behavior even in the simplest linear case. In particular, scalar linear first order homogeneous DDEs with real coefficients can have nontrivial oscillating solutions unlike ODEs. For example, Kalecki's (1935) model of business cycles has the investment equation $\dot{J}(t) = AJ(t) - BJ(t-r)$, which has cyclic solutions of the form $J(t) = e^{at}[c_1 \cos(t) + c_2 \sin(t)]$, for $a \gtreqless 0$. Secondly, solutions to DDEs may be discontinuous with backwards continuation of solutions being non–unique. This is markedly different from the existence and continuation solutions for ODEs. Winston & Yorke (1969) have shown that a solution may not exist at all if one defines the initial condition for $t \leq t_0$ as $x(t) = x_0$, as in ODEs.

2.2 Stability analysis

Consider the FDE $\dot{x}(t) = f(t, x(t))$ and suppose $f(t, 0) = 0$ for all t. Let $B(x(t), a)$ be an open ball of radius a. The solution $x(t) = 0$ is

- *stable* if for any $s \in R$ and $\mu > 0$, there exists a $\delta = \delta(\mu, s)$ such that the initial function $\phi \in B(0, \delta) \Rightarrow x(t)(s, \phi) \in B(0, \mu)$ for $t \geq s$.

- $x(t) = 0$ is *asymptotically stable* if it is stable and there exists $b = b(s) > 0$ such that $\phi \in B(0, b) \Rightarrow x(s, \phi(t)) \rightarrow 0$ as $t \rightarrow \infty$.

[7] As with ODEs, series solutions can be used to approximate exact solutions to nonlinear DDEs (Bellman and Cooke (1963 p.98ff) provide an elegant exposition). However, this method is generally tedious and yield complicated solutions which obscure many interesting analytical issues. As a result, we do not pursue it any further here.

In most applications, the stability of a fixed point of an FDE is established either by examining the roots of the characteristic equation or by constructing Lyapunov functionals. We focus on the former here (for the latter see Hale (1977 pp. 105ff)) and discuss stability issues for linear FDEs. Since the dynamics of nonlinear FDEs can be characterized by taking a first–order Taylor series approximation in a neighborhood of a fixed point, the techniques below are also used for local stability analysis of nonlinear systems.

A fixed point of an FDE can be shown to be stable if all roots of the characteristic equation have real parts less than zero. A fixed point is called *hyperbolic* if there are no eigenvalues λ such that $Re(\lambda) = 0$. It is worth noting that although stability analysis of FDE parallels that of ODEs, FDEs display some peculiar properties. For instance, the stability properties of constant coefficient non–autonomous DDEs may change as the initial time to varies.[8] A second peculiarity which at first appears untenable is that any hyperbolic fixed point of a linear FDE (subject to some technical restrictions on boundedness and continuity) is typically a saddle point.[9] The veracity of this claim will become clear after a short digression on the dimension of FDEs.

One generally determines the dimension of a dynamical system by examining the number of linearly independent roots of the characteristic polynomial. Applying the same logic to the homogeneous part of the linear DDE given by (4), which has characteristic polynomial (7), one arrives at the conclusion that (4) describes an infinite dimensional problem because (7) has an infinite number of (complex) roots.

To see this, suppose $\lambda = \mu + i\omega$ is a solution to (7), where μ and ω are real. Then, separating the real and imaginary parts and using Euler's Theorem, roots of the characteristic equation solve

$$\mu - A - Be^{\mu r}\cos(\omega r) = 0,$$

$$\omega - Be^{\mu r}\sin(\omega r) = 0.$$

[8]Driver (1974 p.362).
[9]Hale (1977, p.228).

It should be apparent that for almost every combination of parameter values, there is an infinite number of solutions to these equations. Note that in the example above, the value of the delay r is critical in determining the behavior of solutions to the characteristic equation, which is only well-defined if r is rational.[10]

The saddle-point property mentioned above should not seem so odd. With an infinite number of roots to the characteristic equation, it would be unlikely to find all roots having real parts strictly less than or strictly greater than zero. Thus, we expect fixed points of FDEs to have the saddle point property.

The stability of a one-delay differential equation has been solved by Hayes (1950). He derived conditions under which the characteristic polynomial

$$h(\lambda) = (\lambda + a)e^{\lambda} + b = 0, \tag{8}$$

for $a, b \in R$ indicates that a fixed point of the DDE is stable.[11]

THEOREM 3 (HAYES) *Equation (8) has all roots with negative real parts if and only if*

1. $a > -1$,

2. $a + b > 0$,

3. $b < p\sin(p) - a\sin(p)$,

where p is the root of $p = -\arctan(p), 0 < p < \infty$, if $a \neq 0$; $p = \pi/2$ if $a = 0$.

PROOF: See Bellman & Cooke, (1964) p.444. ∎

This important theorem is applicable to homogeneous scalar linear first-order one delay differential equations.

[10]Bellman & Cooke (1964) have an extensive chapter detailing the location of the roots of exponential polynomials. In many applications, one may apply the *Argument Principle* from complex variables to find zeros of analytic functions (see Churchill, Brown and Verhey, (1976 p296ff).

[11]The complexity of DDEs can be appreciated by noting that a general stability theorem does not exist for the two-delay differential equation (2DDE). Stability in a 2DDE varies radically depending on the particular values of the delays. See Mahaffy, Zak and Joiner (1995).

Next, we extend this analysis to nonlinear systems. Suppose the FDE in question is scalar, autonomous, and given by

$$a\dot{v}(t) = h(v(t), v(t-r)), \tag{9}$$

with an appropriately defined initial function. If $h(c, c) = 0$ for some constant c, then $w(t) = c$ is a constant solution for $t > r$. Differentiating (9) with respect to both arguments yields the *variational equation*

$$a\dot{u}(t) = h_1(c, c)u(t) + h_2(c, c)u(t-r), \tag{10}$$

where $u(t) = v(t) - w(t), t > r$, and h_i is the partial derivative of h with respect to the i^{th} term. By a generalization of the Poincaré–Lyapunov Theorem, solutions of (10) are isomorphic to those of (9) in some neighborhood of $w(t) = c$. More general methods of stability analysis for nonlinear systems, in particular the analysis of periodic solutions, may use higher order approximations as the analytical basis.[12]

Periodic solutions to FDEs typically arise in two ways, either in non–autonomous systems with forced periodic components, or via a Hopf bifurcation in autonomous systems.[13] The class of models considered below are all autonomous, and we restrict our discussion of cycles to this case. Hopf cycles appear when a fixed point loses or gains stability due to a change in a parameter value and, simultaneously, a cycle emerges from or collapses into the fixed point. There are two distinct cases of interest. One is a stable fixed point surrounded by an unstable cycle (a subcritical Hopf bifurcations). The other occurs when a stable fixed point loses stability and a stable cycle appears (a supercritical Hopf bifurcations) as some parameter a approaches a critical value $a^{\star}$. This critical value causes the fixed point to lose its hyperbolicity. That is, Hopf bifurcations appear about nonhyperbolic fixed points.

[12]Differentiation of current and lagged terms is not always possible when linearizing FDEs. For example, the famous Volterra equations have an infinite number of delays which precludes differentiating each term. See Hale (1977) and Gyori & Ladas (1991) for an analysis of DDEs with an infinite number of delays.

[13]There are other ways in which periodic orbits may arise, such as heteroclinic orbits. Since these are "rare" events and beyond the scope of the current paper, interested readers are referred to Walther (1989) for a more complete treatment.

To illustrate stability analysis we return to the DDE that describes population growth. The transformed DDE is

$$\dot{x}(t) = -ax(t-1)[1+x(t)], \tag{11}$$

where $a > 0$. The linear part of (11) is

$$\dot{y}(t) = -ay(t-1),$$

which has the characteristic equation

$$\lambda e^{\lambda} + a = 0. \tag{12}$$

Using Hayes' Theorem, it is straightforward to verify that all roots of (12) have negative real parts if $0 < a < \pi/2$. Extending this result, one can show that the roots are purely imaginary if $a = \pi/2$, and that (11) has a nontrivial periodic solution for all $a > \pi/2$.

The usual method of stability analysis is to linearize the equation about a fixed point and then apply theorems that provide criteria under which the steady state is stable or unstable. Hayes' Theorem is the natural first candidate one would use prove stability. However, for cases where this theorem does not apply, for instance, as in systems of DDEs, one must derive conditions for stability unique to the system. Determining critical parameter values at which hyperbolicity is lost identifies not only when cycles appear but also the boundary of the region of stability.[14]

FDEs often admit chaotic orbits with the behavior of orbits depending critically on the "smoothness" of the feedback mechanism $g(\cdot)$ in (1). In fact, there is no underlying vector field for (1), only a continuous *semiflow*. Recall that a semiflow only considers time in positive increments. Continuity of the semiflow can be shown piecewise using the method of steps (the solution is right–continuous at the initial point t_0).

The general method of analysis should now be clear and will be used in the next section to analyze the dynamics of a simple deterministic growth model. The steps may be summarized

[14]See Mahaffy, Zak and Joiner (1995) for an extensive discussion of this application.

as follows. First, linearize the nonlinear system. Second, decompose the characteristic equation into real and imaginary parts using $\lambda = \mu + i\omega$ along with Euler's Theorem. Third, use the criteria for stability and cycles along the lines of Hayes' Theorem. With this general method, many functional differential equation systems may be subjected to rigorous local analysis even when the original system is highly nonlinear.[15]

3 A SIMPLE TIME–TO–BUILD MODEL

We now apply the tools developed in the previous section to analyze a deterministic representative agent growth model with a simple time–to–build lag. Consider an economy that is inhabited by infinitely–lived households with unit aggregate measure. The representative individual's preferences are represented by a continuous, strictly increasing and concave utility function $U(c(t))$ and subjective discount rate $\rho > 0$. In this economy it takes $r > 0$ periods to install new capital equipment. The infinite horizon planning problem for this economy is given by

$$\max_{\{c(t)\}} \int_0^\infty U(c(t))e^{-\rho t}dt, \tag{13}$$

subject to

$$\dot{k}(t) = f(k(t-r)) - \delta k(t-r) - c(t),$$

$$k(t) = \phi(t) \text{ for all } t \in [-r, 0],$$

where $0 < c(t) \leq f(k(t-r)), \delta \in [0,1]$ is the rate at which capital depreciates, $f(.)$ is a neoclassical production function, $k(t-r)$ is the productive capital stock at time t, and $\phi(t)$ is the initial capital function. By assumption, the production function and utility function satisfy the standard Inada conditions.

[15]See Gyori & Ladas (1991) for a rigorous mathematical account as well as a generous spate of examples of delay differential equations.

Let $c(t)$ be the control variable and $k(t-r)$ the state variable. Setting up the Hamiltonian and solving for the Euler equations provides the pertinent system of equations. Note that the generalized Maximum Principle applies to control problems with time delays of the sort we study here (see Pontryagin *et al.* 1962). The first order conditions of this model do not yield an advanced time argument because the co–state variable has the same timing convention as the time the decision is made.

The Euler equations for the optimization problem are

$$\dot{c}(t) = \frac{U'(c(t))}{U''(c(t))}[\rho + \delta - f'(k(t-r))], \tag{14}$$

$$\dot{k}(t) = f(k(t-r)) - \delta k(t-r) - c(t). \tag{15}$$

These first order conditions are similar to those of the standard optimal growth model with the exception of the time delay in production. The steady state is exactly the same as that of the standard optimal growth model

$$c^\star = f(k^\star) - \delta k^\star,$$

$$f'(k^\star) = \rho + \delta,$$

and is unique given the Inada conditions.

Linearizing (14) and (15) about the steady state, and solving for the characteristic equation yields

$$h(\lambda) = \lambda^2 - \lambda B e^{-\lambda r} - C e^{-\lambda r} = 0, \tag{16}$$

where $B \equiv f'(k^\star) - \delta = \rho > 0$ and $C \equiv \frac{U'(c^\star)f''(k^\star)}{U''(c^\star)} > 0$. Since (16) has an infinite number of roots, the steady state is generally a saddle.

Decomposing the eigenvalue λ into real and imaginary parts, $\lambda = \mu + i\omega$, $\omega, \mu \in R$, yields a pair of transcendental equations which describe stability near the steady state

$$\mu^2 - \omega^2 - (\mu\rho + C)e^{-r\mu}\cos(r\omega) - \omega\rho e^{-r\mu}\sin(r\omega) = 0, \tag{17}$$

$$2\mu\omega + (\mu\rho + C)e^{-r\mu}\sin(r\omega) - \omega\rho e^{-r\mu}\cos(r\omega) = 0. \tag{18}$$

One can determine the region of stability for any production lag r, rate of time preference ρ, and steady state value of C defined above. The eigenvectors of the linearized system can be used to partition the (local) eigenspace into three distinct subspaces: the stable E^s, unstable E^u, and center subspaces E^c respectively.

The subspace E^s is spanned by the eigenvectors whose eigenvalues have positive real parts; E^u is associated with eigenvalues which have negative real parts; E_c is associated with eigenvalues which have real parts equal to zero. Under certain technical conditions, each of these subspaces has an invariant manifold that is tangent to it and on which the local dynamics are the same as that of the original system. Given the infinite number of roots of (16), we can restrict our investigation of the dynamics of the system to the dynamics on the finite dimensional center manifold, if it exists. The following theorem demonstrates that it does.

THEOREM 4 *Restrict the values of $\omega > 0$. Then, a center manifold for the system (14), (15) exists for some value of $r > 0$. In addition, the unstable subspace has dimension of at least 1.*

PROOF: To show that a well-defined solution $(r^\star, \omega^\star)$ exists for the decomposed characteristic equation system (17), (18), when $\lambda = i\omega$, it is sufficient to prove that a center manifold exists. Let $\lambda = i\omega$ and solve the imaginary equation (18) for

$$\tan(\omega r) = \frac{\omega\rho}{C}, \tag{19}$$

where ρ and C are strictly positive. Since equation (17) must also be satisfied, equation (19) can be decomposed into

$$\sin(\omega r) = \frac{-\omega\rho}{\sqrt{\omega^2\rho^2 + C^2}},$$

and

$$\cos(\omega r) = \frac{-C}{\sqrt{\omega^2\rho^2 + C^2}}.$$

Making this substitution into the first equation and solving for $\omega > 0$ yields,

$$\omega^\star = \frac{1}{\sqrt{2}}\sqrt{\rho^2 + \sqrt{\rho^4 + 4C^2}}.$$

¿From this equation the required $r^\star$ is

$$r^\star = \frac{1}{\omega^\star} \arctan\left(\frac{\omega^\star \rho}{C}\right) > 0.$$

Next, we demonstrate that the dimension of the unstable subspace is at least one. This is equivalent to proving that at least one root of (16) has positive real part. Observe that we can restrict all the complex roots of the characteristic equation to have negative real parts if either

$$r < \cot^{-1}\left(\frac{C + \rho\mu}{\rho\omega}\right), \tag{20}$$

for $r \in \left(\frac{2n\pi}{2\omega}, \frac{(2n+1)\pi}{2\omega}\right)$ or

$$r > \cot^{-1}\left(\frac{C + \rho\mu}{\rho\omega}\right), \tag{21}$$

for $r \in \left(\frac{(2n+1)\pi}{2\omega}, \frac{(2n+2)\pi}{2\omega}\right), n = 0, 1, 2, \ldots$.

We prove the second restriction on r, given by equation (21), because the proof of the first restriction (20) is identical except for a sign change. Using the decomposition of the characteristic equation into real and imaginary parts, note that $\mathrm{Im}[h(\lambda)] + \mathrm{Re}[h(\lambda)] = 0$. Thus, proving that $\mathrm{Im}[h(\lambda)] > 0$ implies that $\mathrm{Re}[h(\lambda)] < 0$.

$$\begin{aligned} \mathrm{Im}[h(\lambda)] &= 2\mu\omega + e^{-r\mu}[(C + \rho\mu)\sin(r\omega) - \rho\omega\cos(r\omega)] \\ &> e^{-r\mu}[(C + \rho\mu)\sin(r\omega) - \rho\omega\cos(r\omega)] \\ &> 0 \end{aligned}$$

This implies that

$$(C + \rho\mu)\sin(r\omega) > (\rho\omega)\cos(r\omega),$$

which in turn implies

$$\frac{C + \rho\mu}{\rho\omega} > \frac{\cos(r\omega)}{\sin(r\omega)} = \cot(r\omega),$$

or

$$\frac{\cot^{-1}\left(\frac{C+\rho\mu}{\rho\omega}\right)}{\omega} > r$$

for $r \in \left(\frac{(2n+1)\pi}{2\omega}, \frac{(2n+2)\pi}{2\omega}\right)$.

The production lag r is well defined because the cotangent function is positive on the specified interval. Thus, we can restrict all complex roots of the characteristic equation to have negative real parts.

Finally, consider the characteristic equation when $\lambda \in R$:

$$h(\lambda) = \lambda^2 e^{r\lambda} - \lambda\rho - C = 0.$$

Taking limits one finds that

$$\lim_{\lambda \to 0} h(\lambda) = -C,$$

and

$$\lim_{\lambda \to +\infty} h(\lambda) = \infty.$$

Therefore, a real positive root to the characteristic equation (16) always exists. ■

Remark 1 It is straightforward to show that the center manifold exists only if the rate of time preference, ρ, is strictly positive. In addition, just as in the standard optimal growth model where a particular initial condition "zeros–out" the positive (unstable) root of the local eigensystem yielding a stable one–dimensional manifold; an appropriate initial function in the time–to–build model nullifies all eigenvalues except a pair of purely imaginary ones. If the value of the delay r satisfies the conditions in Theorem 4, and we restrict attention to initial functions that begin the motion of the system on the center manifold, then the dynamics of the system as a whole are isomorphic to those on the center manifold.[16] □

The main theorem of this paper, which follows below, shows that the conditions for a generalized version of the Hopf bifurcation theorem to hold are satisfied on the center

[16] On this issue see Hale (1977) or Schmidt (1976).

manifold. That is, cyclic solutions to the model exist. We apply a theorem of Chaffee (1971) which covers the case when the "transverse crossing" condition fails in an FDE, as it does in this case. In the standard version of the Hopf bifurcation theorem, this condition says that roots to the characteristic equation must cross from the complex to the imaginary axis with positive speed for a Hopf bifurcation to occur. This condition is not met using (16) due to the periodicity of the sine and cosine functions. Chaffee's Theorem uses a speed of crossing condition on higher order derivatives which is satisfied for the production lag model considered here.

THEOREM 5 *The values $(r^\star, \omega^\star)$ defined in Theorem 4 induce a Hopf bifurcation.*

PROOF: (Sketch) We show that two conditions are met for values $(r^\star, \omega^\star)$. First, that no other eigenvalues, λ , have $\mathrm{Re}(\lambda) = 0$. Second, a transverse crossing condition must be satisfied.

¿From Theorem 4, note that the product $r^\star \omega^\star \in (0, \pi/2)$ and all complex eigenvalues have negative real parts if

$$\tan(r\omega) > \frac{\rho\omega}{C + \rho\mu},$$

for $r\omega \in (0, \pi/2)$. A center manifold exists if

$$\begin{aligned} \tan(r^\star \omega^\star) &= \frac{\rho\omega^\star}{C}, \\ &> \frac{\rho\omega^\star}{C + \rho\mu}, \end{aligned}$$

for any $\rho, \mu > 0$. Thus, at $(r^\star, w^\star)$ all other complex non-real eigenvalues λ that solve (16) have $\mathrm{Re}(\lambda) < 0$. In addition, we know that there is always one real positive root when $r = r^\star$ and $\omega = \omega^\star$. Hence, there are no other roots that have $\mathrm{Re}(\lambda) = 0$.

Finally, we show that at $r = r^\star$ and $\omega = \omega^\star$, the roots passing into the imaginary plane do so with nonzero speed. Direct calculation reveals that $d\frac{\mathrm{Re}[h(\lambda(r))]}{dr(r^\star \omega^\star)} = 0$. Differentiating a second time yields

$$d^2 \left. \frac{\mathrm{Re}[h(\lambda(r))]}{dr^2} \right|_{r=r^\star, \omega=\omega^\star} = C\omega^{\star 2} \cos(r^\star \omega^\star) + \omega^3 \rho \sin(r^\star \omega^\star). \tag{22}$$

Using $\tan(r^\star\omega^\star) = \frac{\omega^\star\rho}{C}$ to solve for $\sin(r^\star\omega^\star)$ and $\cos(r^\star\omega^\star)$ as in Theorem 5, equation (22) can be rewritten as

$$d^2 \left.\frac{\text{Re}[h(\lambda(r))]}{dr^2}\right|_{r=r^\star,\omega=\omega^\star} = \frac{-C^2\omega^{\star 2}\rho}{\sqrt{\omega^{\star 2}\rho^2 + C^2}} - \frac{\omega^{\star 4}\rho^2}{\sqrt{\omega^{\star 2}\rho^2 + C^2}} < 0.$$

Therefore, $d^2 \frac{Re[h(\lambda(r))]}{dr^2}|_{r=r^\star,\omega=\omega^\star} \neq 0$ and the transverse crossing condition of the Chaffee version of the Hopf bifurcation theorem is satisfied.[17] ■

Remark 2 Observe that Theorem 5 depends critically on the production lag, r, to generate cycles. If the lag is zero, there are no complex roots of the characteristic equation. Thus, cycles are not possible in the standard optimal growth model and their presence here is due entirely to the production lag. In addition, the restrictions on r and ω that generate cycles are defined over an interval. Hence, cyclic solutions occur for a measurable set of the parameter space. □

Remark 3 In a similar time–to–build model, Ioannides & Taub (1992) claim that such a model is "not intrinsically oscillatory." Using the method of proof above, we provide a counter–example to Ioannides & Taub's claim. In their model, a lag structure on investment produces a system of integro–differential equations. After a number of simplifying assumptions, a first–order approximation of the system has the characteristic equation

$$\lambda - (\Theta\lambda + \delta)\lambda e^{-\lambda J}, \tag{23}$$

where J is the time–to–build delay, $\Theta > 1$ is an amalgamation of production and preference parameters and $\delta \in [0, 1]$ is the rate of depreciation. It is straightforward to show that for the parameter values, $\Theta = 2$, $\delta = 1$, $J = 1.19$ this system exhibits a Hopf cycle without violating any primitive parameter restrictions.[18] □

[17] Several technical conditions are required to apply the Chaffee Theorem. Lengthy calculations show that these conditions are satisfied. The transverse crossing condition is nonstandard so it is verified directly while the calculations for the others are omitted. All omitted calculations are available on request from the authors.

[18] In their analysis, Ioannides & Taub (1992) correctly note that there is an infinite number of roots to the characteristic equation (23), but they fail to examine complex roots. As a result, they do not find cycles.

4 Optimality

The optimality of oscillatory paths in growth models is an important but often over–looked property of this class of models. In this section we show that a Hopf cycle is an optimal solution to the planning problem.[19] The presence of cyclic solutions requires that the standard transversality condition must be modified as a limiting boundary condition. The following transversality condition is valid for both cyclic and noncyclic solutions, including the standard model for which $r = 0$,

$$\limsup_{t \to \infty} e^{-\rho t} k(t - r) = 0 \tag{24}$$

for $\rho > 0$ and some fixed $r \geq 0$. We are now ready to demonstrate optimality.

Theorem 6 *If the conditions of Theorem 4 are satisfied and the initial function $\phi(t)$ for $t \in [r, 0]$ places the local dynamics of system (14), (15) on the center manifold, then a Hopf cycle is optimal.*

Proof: A solution to problem (13) is optimal if two conditions are satisfied, feasibility and maximization (Nishimura & Sorger (1996)). To establish feasibility, consider an initial function and the parameter restrictions of Theorem 4 that induce Hopf cycles. Given these conditions, the only dynamic paths of the system are a nondegenerate cycle or the degenerate dynamics of the fixed point. Consider the nondegenerate cycle. By construction, the cycle satisfies the resource constraint (13). In addition, since the solution is bounded for all t, it also satisfies the transversality condition (24). It is therefore feasible. Maximization follows directly since the first order conditions are necessary and sufficient for an optimum under the Inada conditions. Therefore the solution maximizes (13). Now, consider a degenerate cycle at the interior fixed point. Clearly, it is feasible and maximizing as it satisfies the resource constraint, transversality condition and optimality condition. Therefore, all Hopf cycles are optimal. ∎

[19]We are grateful to a referee for suggesting this line of inquiry and to Ken Cooke who motivated the theorem below with several insightful queries.

In Theorem 6 we have demonstrated that both nondegenerate and degenerate cycles are optimal. However, for all initial conditions that begin the dynamical system on the center manifold, the fixed point constitutes a set of measure zero. Therefore, we can restrict our attention to nondegenerate Hopf cycles. Moreover, the above Theorem shows that the particular Pareto optimal solution found for the planner's problem is, for a fixed set of parameter values, parameterized by the initial function ϕ. Since there are an infinite number of roots to the characteristic equation, most initial functions induce equilibrium paths that are high-dimensional saddles leading to the interior fixed point. Thus, depending on the initial function (for a given set of parameter values), the dynamic path of this economy either *i)* diverges and is therefore not an equilibrium path; *ii)* is a (high-dimensional) saddle path leading to the steady state which is optimal; or *iii)* is a Hopf cycle surrounding the steady state which is optimal. Figure 1 depicts the phase portrait of the system on the (local) center manifold showing an optimal cyclic solution to the model.

[PLACE FIGURE 1 ABOUT HERE.]

In an early contribution, Sutherland (1970) writes, "... there can exist several optimal programs which are stationary, so that the long-term behavior of an optimal program may depend on its initial state." (p. 588).[20] That is, the particular Pareto optimal solution one solves for depends upon initial conditions of the system. This is precisely what initiates cycles in the time-to-build model—parameter values and initial conditions such that the only solution is a cycle. Theorem 6 shows that such a cycle is optimal.

5 Conclusion

Business cycles are persistent and oscillatory. What is unclear is whether the oscillations are intrinsic—like the response of an undamped pendulum—or whether the oscillations are

[20]Surveys of neoclassical models with optimal cycles can be found in Reichlin (1997), Nishimura & Sorger (1996) and Boldrin & Woodford (1990).

like an overdamped pendulum in response to stochastic shocks. This paper revisits the debate over the source and nature of aggregate fluctuations. We analyze a version of the neoclassical growth model with a lag between investment and production (time–to–build) and demonstrate that the optimality conditions lead to a system of functional differential equations. The associated characteristic equation is the similar to that of the standard neoclassical growth model, except that there is an additional exponential factor due to the time–to–build feature that opens up the possibility of complex roots. Contrary to Ioannides & Taub (1992), we show that the dynamics are intrinsically oscillatory and that this is entirely due to the time–to–build technology. Moreover, the oscillating paths are shown to be optimal.

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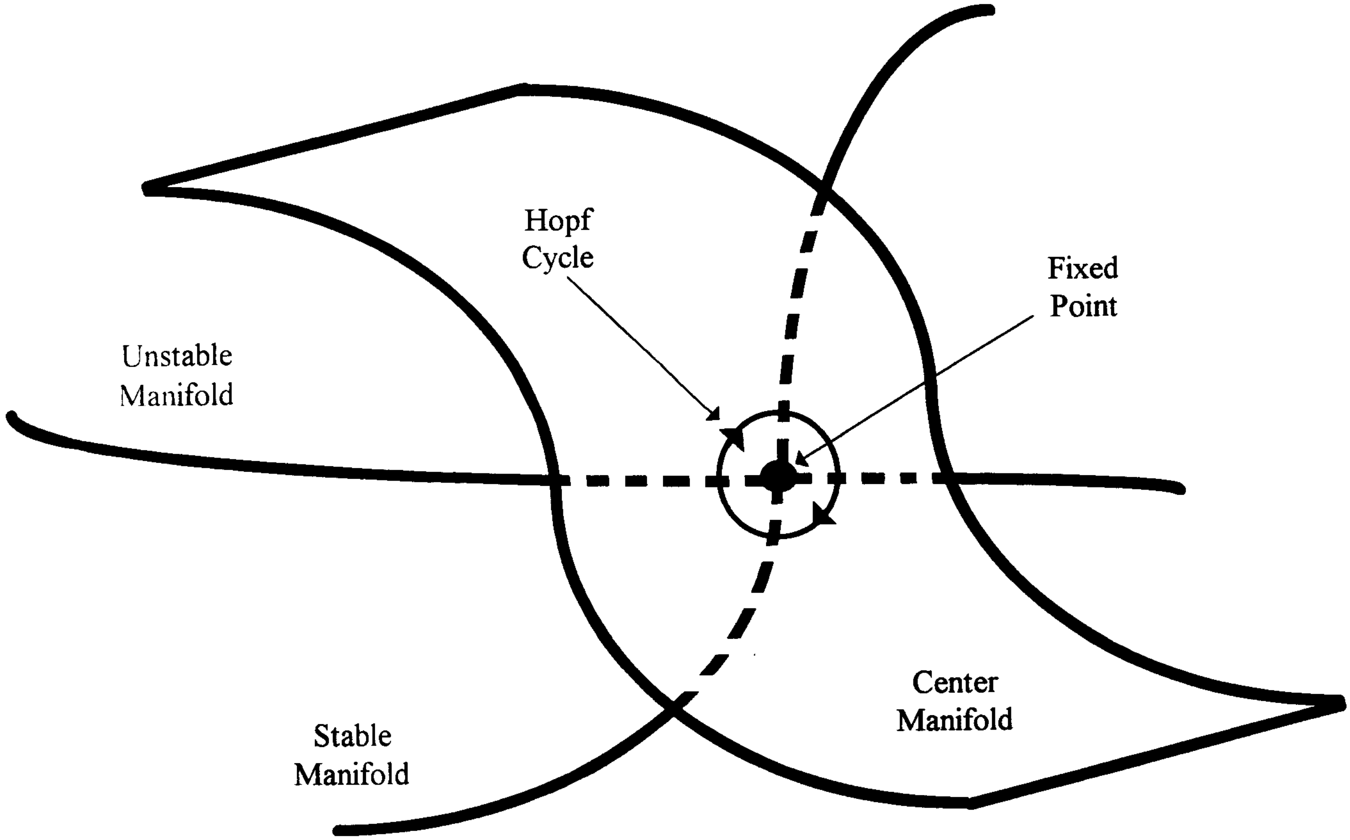

Figure 1: PHASE PORTRAIT OF THE SYSTEM ON THE CENTER MANIFOLD