Incentives, Contracts and Markets: A General Equilibrium Theory of Firms*

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Abstract

This paper takes steps toward integrating firm theory in the spirit of Alchian & Demsetz (1972) and Grossman & Hart (1986), contract theory in the spirit of Holmstrom (1979), and general equilibrium theory in the spirit of Arrow & Debreu (1954) and McKenzie (1959). In the model presented here, the set of firms that form and the contractual arrangements that appear, the assignments of agents to firms, the prices faced by firms for inputs and outputs, and the incentives to agents are all determined endogenously at equilibrium. Agents choose consumption — but they also choose which firms to join, which roles to occupy in those firms, and which actions to take in those roles. Agents interact anonymously with the (large) market, but strategically within the (small) firms they join. The model incorporates both moral hazard and adverse selection. Equilibria may not be Pareto optimal, and may even be Pareto ranked.

Keywords general equilibrium, incentives, contracts, firms, organizations

JEL Classification Numbers D2, D5, D71, D8, L2

1 Introduction

Incentives and contracts are at the heart of much of the modern theory of the firm. Alchian & Demsetz (1972) focuses on the implications of monitoring and incentives for the structure of the firm. Holmstrom (1979) formulates the incentive problem more broadly as a principal/agent problem, and explores the extent to which contracting on observable (or verifiable) events can or cannot overcome incentive problems. Grossman & Hart (1986) focuses on the incompleteness of contracts and the consequent role of control rights. These papers, and the enormous literature they have inspired, have provided a great deal of insight into the nature of the firm. However, this literature largely ignores the interaction between the firm and the market, taking as given the set of firms that form, the organizational structure of these firms, the assignments of agents to firms, the prices faced by firms for inputs and outputs, and the incentives for agents to take particular actions within a firm or even to participate in a firm at all.

The present paper begins an integration of firm theory in the spirit of Alchian & Demsetz (1972) and Grossman & Hart (1986), contract theory in the spirit of Holmstrom (1979), and general equilibrium theory in the spirit of Arrow & Debreu (1954) and McKenzie (1959). In the model presented here, the set of firms that form and the contractual arrangements that appear, the assignments of agents to firms, the prices faced by firms for inputs and outputs, and the incentives to agents are all determined endogenously at equilibrium. Agents choose consumption — but they also choose which firms to join, which roles to occupy in those firms, and which actions to take in those roles. Agents interact anonymously with the (large) market, but strategically within the (small) firms they join. Firms and the market are inextricably tied together: strategic choices within firms affect output, output affects market prices, market prices affect agents' budget sets and hence utilities, utilities affect incentives within firms, and incentives within firms affect strategic choices within firms. Competition determines the prices of goods and which agents consume those goods, and also the wages attached to jobs and which agents fill those jobs.

Because actions and skills of agents may be unobservable and uncontractible, the model incorporates both moral hazard and adverse selection. Because the model accommodates uninsurable idiosyncratic risk as well as moral hazard and adverse selection, it is no surprise that equilibria may not be Pareto optimal; equilibria may even be Pareto ranked and in particular need not be even constrained efficient.

The central notion of the model here is that of a type of firm, defined by a set of roles (jobs), a set of actions for each role, a set of skills that may be possessed by individuals who fill each role, a set of (observable and contractible) firm-specific production states, a stochastic production process, and a state-contingent profit-sharing plan. Wages (lump-sum transfers) are determined endogenously at equilibrium, as market prices for roles in firms. The possible types of firms are given exogenously (reflecting both technological possibilities and legal enforceability of contractual arrangements); many types of firms might be possible (although the number is required to be finite), but many fewer types of firms might actually form at equilibrium. Thus, the realized nature of firms is determined *endogenously* at equilibrium. (Because the set of firm types might encompass a fine grid approximation to all possible firms, the restriction to a finite set of possible firm types is not terribly restrictive. Allowing for an infinite set of firm types would lead to many technical and conceptual difficulties, including the necessity of an infinite dimensional space of wages.)

In the model, individual agents are exposed to two sources of uncertainty: the endogenously determined behavior of other agents with whom they join in various firms, and the exogenously specified stochastic nature of the production process. To make the model tractable, the matching of agents in firms is assumed to be random, the realization of individual uncertainty is assumed to be independent across firms, and the Law of Large Numbers is assumed to apply. The implications of these assumptions are that individuals face idiosyncratic risks — and therefore choose state-contingent consumption plans — but that aggregates (inputs, outputs, prices) are deterministic, not random. (The model accommodates aggregate risks through the familiar device of incorporating the state of nature into the description of commodities.)

A *common-beliefs equilibrium* of the model consists of commodity prices and wages, and consumption plans, firm-role-action choices and beliefs for each agent such that: commodity markets clear, job markets clear, agents optimize in their budget sets given beliefs, beliefs are correct and identical across agents. Beliefs must be specified as part of the equilibrium because optimization requires individual agents to consider firms that do not form in equilibrium as well as firms that do form. (Thus, this notion of equilibrium is analogous to perfect Bayesian equilibrium.) For firms that do form in equilibrium, correctness of beliefs implies commonality of beliefs across agents, but for firms that do not form in equilibrium, correctness of beliefs by itself would have no bite: requiring correctness of beliefs only for firms that form would leave open the possibility of equilibria in which some firms do not form only because agents hold contradictory beliefs about behavior of other members of the firm. Requiring common beliefs rules out such equilibria. Perhaps surprisingly, commonality of beliefs does not rule out miscoordination and self-fulfilling prophecies, which can occur at equilibrium. It also does not rule out absurd beliefs — for instance, that others use dominated strategies — but a refinement, *population perfect equilibrium* (an analog of trembling-hand perfect equilibrium), does rule out such absurd beliefs. Under natural assumptions, both common-beliefs equilibrium and population perfect equilibrium always exist.

In addition to the formal model, the definition of equilibrium, and proofs that equilibria exist, this paper offers several examples chosen to illustrate the main themes. (More realistic applications are intended for later papers.) The first, which is in the spirit of Holmstrom (1979), focuses on moral hazard and illustrates how the market determines incentives to join firms and to work or shirk within those firms. The second, which is in the spirit of Akerlof (1970), focuses on adverse selection and illustrates the possibility of self-fulfilling prophecies and Pareto ranked equilibria, and the implications of population perfection. The third, which is in the spirit of Bennardo & Chiappori (2003), illustrates how the market assigns agents to firms, how lottery-like outcomes can arise endogenously at equilibrium, and how the Harsanyi (1973) interpretation of agent types explains lotteries. The fourth illustrates the competitive choice of contracts.

There is by now a substantial literature that seeks to incorporate asymmetric information (including moral hazard and adverse selection) in a general equilibrium context. The seminal work in this literature is Prescott & Townsend (1984a, 1984b); more recent contributions include Bisin, Geanakoplos, Gottardi, Minelli & Polemarchakis (2001), Dubey & Geanakoplos (2002) and Rustichini & Siconolfi (2002). A different literature, which includes Ghosal & Polemarchakis (1997) and Minelli & Polemarchakis (2000), integrates economy-wide actions with economy-wide trade in commodities. In contrast with these literatures, the present paper is distinguished by the absence of pooling and of trade in lotteries, by the fact that agents are exposed to idiosyncratic risks — so that *ex ante* identical agents may make identical choices and obtain identical *ex ante* utilities but realize different *ex post* consumptions and different *ex post* utilities — and by the fact that agents *choose* the small firms to which they belong and in which they act strategically, and receive a wage (positive or negative) from each such firm. Closer to the present work is Prescott & Townsend (2001), in which agents choose small firms. However, the focus of that work is on on the role of monitoring and the possibility of supporting incentive efficient configurations as equilibria in an environment with one-sided moral hazard. McAfee (1993), which considers an environment in which the competitive market determines the choice of selling mechanisms, has some similarities in spirit — although not in breadth of purpose or execution — with the present paper.

The framework of the present paper builds on Makowski (1976) and Ellickson, Grodal, Scotchmer and Zame (1999, 2001, 2003). As does the present paper, these papers build frameworks in which agents interact with the market and with each other in small groups, determined endogenously in equilibrium. (Cole & Prescott (1997) takes a rather different approach to the same problem, and is somewhat further from the present work.) However, these papers do not allow for hidden actions (moral hazard) or unobserved skills (adverse selection) or stochastic production. The focus in those papers, and in Makowski & Ostroy (2004), which uses a similar framework, is on competition and efficiency (or constrained efficiency) in the presence of clubs (in the sense of Buchanan (1965)) or teams (in the sense of Marshack & Radner (1972)). Rahman (2005) builds a framework in which actions and trades of each group (firm) are coordinated by an entrepreneur, and focuses on achieving incentive efficiency with (personalized) Lindahl prices.

Following this Introduction, Section 2 presents the moral hazard example (in a slightly informal way), in order to motivate the model and illustrate some of the ideas and implications. Section 3 presents the bones of the formal model. Section 4 addresses common beliefs equilibrium and Section 5 addresses the refinement I call population perfect equilibrium. Section 6 presents the remaining examples and Section 7 concludes. Proofs are collected in the Appendix.

2 A Motivating Example

To introduce and motivate the model, this Section presents an informal description and analysis of a simple example, much in the spirit of Holmstrom (1979), that illustrates how trade with the market affects incentives within a firm.

Example 1: Moral Hazard Consider a world with two goods and a single kind of productive firm. Production requires the participation of two agents. The input to the production process is 2 units of the first good; the output of the production process depends on the realized state: 28 units of the second good in the Good state, 2 units of the second good in the Bad state. The realization of the state depends stochastically on whether the agents Work (exert high effort) or Shirk (exert low effort); the probability that the Good state is realized is given in Figure 1 below.

	W	S
W	1	$\frac{1}{2}$
S	$\frac{1}{2}$	0

Figure 1

All agents are identical. Endowments are $e = (2, 2)$. Utility for certain consumption is Cobb-Douglas $u(c_1, c_2) = (c_1 c_2)^{1/2}$; faced with risky prospects, agents maximize expected utility. Working is unpleasant: an agent who chooses to Work incurs a disutility of δ. For the purpose intended, consider the range of disutilities $2 - \frac{1}{2}\sqrt{3} < \delta < 3$.

Finally: output is observable and contractible but effort is not, so each member of the firm provides half the input and receives half the output.

As a benchmark, consider first the setting in which there are only two agents. Combining the assumption that ROW and COL each provide half the input and receive half the output with the assumptions about endowments, utility functions and risk attitudes leads to Figure 2, which expresses the expected utility of ROW as a function of the choices of ROW and COL. (Expected utility for COL is obtained by interchanging roles.)

	W	S
W	$4-\delta$	$2+\frac{1}{2}\sqrt{3}-\delta$
S	$2+\frac{1}{2}\sqrt{3}$	$\sqrt{3}$

Figure 2

Because $2-\frac{1}{2}\sqrt{3}<\delta$, Shirk is a dominant strategy. If participation in the firm is viewed as a choice, then each agent prefers to consume his/her endowment of $(2,2)$ rather than enter the firm, so *equilibrium is autarkic*: the firm does not form and each agent consumes his/her endowment $(2,2)$.

Now suppose there are *many* agents, each of whom is free to enter or not enter a firm and *also* to trade with other agents in an anonymous competitive market. In this case, *autarky cannot be an equilibrium.* To see this, note that if autarky were an equilibrium, so that each agent consumed endowment $(2,2)$, then prices would be $(\frac{1}{2},\frac{1}{2})$. In that case, an agent who entered a firm would be able to sell his/her share of the output at the prevailing market prices and buy the optimal bundle. Because utility for certain consumption is Cobb-Douglas, imputed utilities after trade would be as shown in Figure 3.

	W	S
W	$\frac{17}{2}-\delta$	$\frac{11}{2}-\delta$
S	$\frac{17}{2}$	2

Figure 3

Because $\delta<3$, W would now be a dominant strategy. In particular, each agent would prefer to enter a firm and Work rather than to consume his/her endowment. Hence autarky cannot be an equilibrium.

To solve for equilibrium it is convenient to work backwards: rather than fixing δ and solving for the form of the equilibrium, fix the form of equilibrium and find the range of δ in which such an equilibrium is possible. As noted, autarky cannot be an equilibrium; equivalently, there is no equilibrium in which all agents choose not to enter a firm. Similarly, there is no equilibrium in which all agents enter a firm and Work and there is no equilibrium in which all agents enter a firm and Shirk. (If all agents enter the firm and Work, prices would be $(1, \frac{1}{16})$; at those prices, agents would prefer not to enter the firm but rather to trade their endowment with the market. If all agents enter the firm and Shirk, prices would be $(1, \frac{1}{3})$, and again, agents would prefer not to enter the firm but rather to trade their endowment with the market.) Thus, there are no (pure strategy) symmetric equilibria. Put differently, the only equilibria have the property that *ex ante* identical agents make different choices. Thus there are three possible equilibrium configurations to look for:

- Look first for an equilibrium in which some agents enter a firm and Work and the remaining agents do not enter a firm. Write α for the fraction of agents who enter a firm and Work and normalize so commodity prices are $(1, q)$. To determine α and q, equate demand and supply for good 1 and then equate utility for agents who enter a firm and agents who do not enter a firm:

$$\begin{aligned} \alpha\left(\frac{1+16q}{2}\right) + (1-\alpha)(1+q) &= \alpha(1) + (1-\alpha)(2) \\ \frac{1+16q}{2}q^{-1/2} - \delta &= (1+q)q^{-1/2} \end{aligned}$$

  Solve these equations to obtain

$$\alpha = \frac{2-2q}{1+14q} \quad (1)$$

$$q = \frac{2-\alpha}{2+14\alpha} \quad (2)$$

$$\delta = (7q - \frac{1}{2})q^{-1/2} \quad (3)$$

  (It is messy to solve (3) for q in terms of δ.) It remains to determine when Work (rather than Shirk) is an optimal choice for agents who join a firm. To do this, observe first that utility conditional on high consumption and utility conditional on low consumption are:

$$u_H = \left(\frac{1+16q}{2}\right) q^{-1/2}\ ,\ u_L = \left(\frac{1+3q}{2}\right) q^{-1/2}$$

Work is an optimal choice if $u_H - \delta \geq \frac{1}{2}(u_H + u_L)$. It follows from (1) – (3) that this true when $\delta \leq \frac{13\sqrt{30}}{60}$. Summarizing: if $2 - \frac{1}{2}\sqrt{3} < \delta \leq \frac{13\sqrt{30}}{60}$, there is an equilibrium in which some agents enter a firm and Work and the remaining agents do not enter a firm. Because agents are *ex ante* identical, all agents obtain the same expected utility, but agents who enter a firm and Work consume more, as compensation for bearing the unpleasantness of work.

- Straightforward calculations (along the lines above) show that there is no equilibrium in which some agents enter a firm and Work and the remaining agents enter a firm and Shirk.

- Finally, look for an equilibrium in which some agents enter a firm and Work, some agents enter a firm and Shirk, and some agents do not enter a firm. Straightforward (but tedious) calculations show that such an equilibrium exists if $\frac{13\sqrt{30}}{60} \leq \delta < 3$. Because agents are *ex ante* identical, all agents obtain the same expected utility, but agents who enter a firm are exposed to risk (some will be matched with partners who Work and some with partners who Shirk, and some will be exposed to production risk as well) and must be paid a premium (in the form of higher expected consumption) to bear this risk. Agents who enter a firm and are lucky (obtaining High output) consume more than agents who enter the firm and are unlucky (obtaining Low output), so these agents do not enjoy identical *ex post* utilities.

Equilibrium is unique except when $\delta = \frac{13\sqrt{30}}{60}$. ◇

3 The Formal Model

Example 1 embodies many of the features of the general model formalized below, but two important features are missing. Most obviously, the general model allows for *many* potential types of firms, embodying many different production technologies and contractual structures. This feature allows for endogenous determination of contracts and firm structures. More subtly, the general model allows for the possibility that production depends stochastically on the actions of agents *and* on the (endogenously acquired) characteristics (skills) of these agents. Because skills may not be observable, this feature allows for adverse selection.

The data of the model consists of a finite set of perfectly divisible commodities (private goods), a finite set of firm types, and a space of agents.

3.1 Commodities

There are $L \geq 1$ perfectly divisible commodities (private goods), traded on competitive markets, so the commodity space is $\mathbb{R}^L_+$. Commodity prices are required to be strictly positive. It is convenient to normalize commodity prices to sum to 1 so the price simplex is

$$\Delta = \{p \in \mathbb{R}^L_{++} : \sum p_\ell = 1\}$$

3.2 Firm Types

A firm type is described by a finite set of roles (jobs), a finite set of actions and a compact metric space of skills for each role, a finite set of observable and contractible states, a specification of inputs and outputs in each state, a state-dependent distribution of income, and a stochastic mapping from actions and skills to outcomes.

Formally, a *firm type* is described by the following data:

- a finite set R of *roles* (or *jobs*)

- for each role $r \in R$: a compact metric space S_r of *skills* (or *individual characteristics*) and a finite set A_r of *actions*
- a finite set Ω of (observable and contractible) *states*
- an *input/output mapping*

$$y : \Omega \to \mathbb{R}^L$$

- a bounded, continuous *profit sharing plan*

$$D : R \times \Omega \times \Delta \to \mathbb{R}$$

  such that

$$\sum_r D(r, \omega, p) = p \cdot y(\omega)$$

  for each $\omega \in \Omega$ and each $p \in \Delta$

- continuous *conditional probabilities*

$$\pi : (S_1 \times A_1) \ldots (S_R \times A_R) \to \mathbf{P}(\Omega)$$

  where $\mathbf{P}(\Omega)$ is the space of probability measures on Ω

The interpretation intended is that the state ω summarizes what is observable or verifiable to an outside agency, and hence what is contractible; $\pi(\omega|\mathbf{s}, \mathbf{a})$ is the conditional probability that state ω occurs if the various roles are filled by individuals whose skill profile is $\mathbf{s} = (s_1, \ldots, s_R)$ and who choose action profile $\mathbf{a} = (a_1, \ldots, a_R)$; $D(r, \omega, p)$ is the share of the net income accruing to the firm from its realized production that is distributed to the agent filling role r if the realized state is ω and market prices for commodities are p. Note that some shares $D(r, \omega, p)$ may be negative, in which case such agents subsidize others. The requirement that distributions sum to net income (i.e., that the budget balances) is just an accounting identity.

A firm type delineates an array of possibilities for a set of agents; a *firm* is particular instance of a firm type, arising from the choices of individual agents. There are $J \geq 0$ exogenously specified firm types; I use superscripts for the parameters in firm type j. It is frequently convenient to abuse notation and view J as either the number of firms or as the set of firms, to view as R^j as the either the set of roles in firm type j or the number of roles in firm type j, and so forth.[1]

3.3 Agents

An agent is described as a tuple consisting of a choice set, an endowment, a utility function, and a skill mapping.

3.3.1 Choices

Each element of the choice set specifies a state-contingent consumption plan, the firms to which the agent chooses to belong, the roles chosen in each firm, and the action chosen in each role.

Although agents choose consumption contingent on the realization of the uncertainty they face, it is convenient to adopt notation in which consumption is contingent on the realization of all uncertainty and later constrain the choice to be independent of the realization of uncertainty that is not faced by the agent. Hence, define a *consumption plan* to be a random variable

$$\tilde{x} : \Omega \to \mathbb{R}^L_+$$

(In parallel with familiar probabilistic notation, I use tildes to denote random

[1]By definition, each role in a firm is filled by a single agent and each individual agent may choose to belong to at most one firm. This modeling choice involves no loss of generality, because otherwise identical roles or identical firm types could always be distinguished by giving them different names. In addition to being a bit simpler than the alternatives, this modeling choice has two additional advantages. The first is that it preserves the possibility that memberships distinguished only by name pay different wages, and are chosen by agents who are different or who choose different actions, leading to an asymmetric equilibrium. The second is that it preserves the possibility that firms distinguished only by name attract agents who coordinate on different actions. Put differently: names can serve as a correlating device.

— i.e., state-dependent — consumptions and plain letters to denote non-random consumptions.) Write $\mathbb{R}_+^{L\Omega}$ for the space of all consumption plans.

To formalize firm-role-action choices, it is convenient to define a dummy role-action choice in each firm, so write:

$$\begin{aligned}
\mathbb{F}_0^j &= \{(r,a) : 1 \leq r \leq R^j, a \in A_r^j\} \\
\mathbb{F}^j &= \mathbb{F}_0^j \cup \{0\} \\
\mathbb{F}_0 &= \mathbb{F}_0^1 \times \ldots \times \mathbb{F}_0^J \\
\mathbb{F} &= \mathbb{F}^1 \times \ldots \times \mathbb{F}^J
\end{aligned}$$

An element $\phi = (\phi^1, \ldots, \phi^J) \in \mathbb{F}$ is a choice of role and action in each firm type; by convention, $\phi^j = 0$ represents the choice *not to belong* to firm type j. For each j, define

$$\begin{aligned}
\rho^j &: \mathbb{F}^j \to R^1 \cup \ldots \cup R^J \cup \{0\} \\
\alpha^j &: \mathbb{F}^j \to A_1^j \cup \ldots \cup A_{R^j}^j \cup \{0\}
\end{aligned}$$

by $\phi^j = (\rho^j(\phi), \alpha^j(\phi))$. (I abuse notation in the obvious way so that $\rho^j(\phi) = \alpha^j(\phi) = 0$ if $\phi^j = 0$.)

For $\Phi \subset \mathbb{F}$ a non-empty set of firm-role-action choices, define the corresponding *choice set* to be:

$$X(\Phi) = \{(\tilde{x}, \phi) \in \mathbb{R}_+^{L\Omega} \times \Phi : \phi^j = 0 \Rightarrow \tilde{x} \text{ is independent of } \omega^j\}$$

Defining consumption sets in this way carries out the earlier promise that individual consumption choices depend formally on the realization of all uncertainty, but are independent of the realization of uncertainty not faced.

3.3.2 Endowments

An *endowment* $(e, \epsilon) \in X(\Phi)$ specifies an initial claim to consumption and to firm-role-action choices. By assumption, e is *not random*. In the leading case, $\epsilon \equiv 0$, so the agent does not initially belong to any firm. However it is both natural and convenient to allow for the more general possibility.[2]

Because profit shares may be negative, a non-zero initial endowment of firm-role-action choices implies an initial liability as well; if this liability exceeds the value of the initial endowment of private goods, an agent with those characteristics will not be able to survive without trade, and the budget set may be empty. Hence I make the following assumption:

Survival If the initial endowment is (e, ϵ) with $\epsilon \neq 0$, then there is some $c > 0$ such that

$$p \cdot e + \sum_{\epsilon^j \neq 0} D^j(\rho^j(\varepsilon), \omega^j, p) \geq c > 0$$

for every $\omega \in \Omega$, $p \in \Delta$.

3.3.3 Utilities

Agents care about consumption, about their own choices, about the actions and skills of others in the firm to which they belong, and about the realized state. For simplicity, agents are assumed to be expected utility maximizers, so it suffices to specificy utility for *non-random* consumption. As above, it is convenient to view utility as defined over all tuples but require that it be independent of irrelevant components.

To formalize this, fix a consumption set $X(\Phi)$. Write

$$\begin{aligned} \mathbf{S}^j &= S_1^j \times \ldots \times S_{R^j}^j \\ \mathbf{S} &= \mathbf{S}^1 \times \ldots \times \mathbf{S}^J \\ \mathbf{A}^j &= A_1^j \times \ldots \times A_{R^j}^j \\ \mathbf{A} &= \mathbf{A}^1 \times \ldots \times \mathbf{A}^J \end{aligned}$$

Utility is a mapping

$$u : \mathbb{R}_+^L \times \mathbb{F} \times \mathbf{S} \times \mathbf{A} \times \Omega \to \mathbb{R}$$

[2] For instance, if there is a single good, no production and no profit shares, and if each agent is endowed with and must choose, a particular role in a particular firm, the framework reduces to the structure of a game with random matchings.

so that $u(x, \phi, \mathbf{s}, \mathbf{a}, \omega)$ is the utility obtained if the agent consumes x, chooses the firm-role-action profile ϕ, faces firm members with skill-action profile $\mathbf{s}, \mathbf{a}$, and the state ω occurs. I require that

$$\phi^j = 0 \;\Rightarrow\; u \text{ is independent of } \omega^j$$
$$\phi^j = (r, a) \;\Rightarrow\; u \text{ is independent of } \mathbf{s}^j_r, \mathbf{a}^j_r$$

That is, utility is independent of skills and actions in firms to which the agent does not belong, and utility is independent of skills and actions of others in roles which the given agent fills. I assume utility is continuous in its arguments and strictly increasing in each commodity. In addition, I require the following three additional assumptions.[3]

A1 $u(0, \phi, \mathbf{s}, \mathbf{a}, \omega) < u(e, \epsilon, \mathbf{s}, \mathbf{a}, \omega)$ for all $\phi, \mathbf{s}, \mathbf{a}, \omega$

A2 $\lim_{|x| \to \infty} u(x, \epsilon, \mathbf{s}, \mathbf{a}, \omega) = \infty$ for every $\mathbf{s}, \mathbf{a}, \omega$

A3 there is a $k > 0$ such that $u(x, \epsilon, \mathbf{s}, \mathbf{a}, \omega) \leq k(1 + |x|)$ for every $\mathbf{s}, \mathbf{a}, \omega$

3.3.4 Skills

Formally, all skills are acquired, according to the full array of firm-role-action choices. The *skill mapping* $\sigma : \Phi \to \mathbf{S}$ specifies the skill the agent would have in each role in each firm as a function of the agent's firm-action-role choices.

3.3.5 The Space of Agent Characteristics

Write $\mathcal{C}$ for the space of agent characteristics $(\Phi, u, (e, \epsilon), \sigma)$. If $\xi \in \mathcal{C}$ is a typical characteristic, I sometimes abuse notation to write $\xi = (X_\xi, u_\xi, (e_\xi, \epsilon_\xi), \sigma_\xi)$.

Let $\mathcal{V}$ be the space of continuous functions $v : \mathbb{R}^L_+ \times \mathbb{F} \times \mathbf{S} \times \mathbf{A} \times \Omega \to \mathbb{R}$, equipped with the topology of uniform convergence on compact sets, give $2^{\mathbb{F}} \times \mathbb{R}^L_+ \times \mathbb{F} \times \mathcal{V} \times \mathbf{S}^{\mathbb{F}}$ the product topology, and give $\mathcal{C}$ the subspace topology.

[3]Because the choice to belong to a firm is indivisible, some assumption such as **A1** is necessary to guarantee that optimal choices are upper-hemi-continuous in prices, and some assumptions such as **A2**, **A3** are necessary to guarantee that expected demand blows up as commodity prices approach the boundary of the price simplex.

3.4 The Economy

An *economy* consists of a finite set of commodities, a finite set of firms, and a Borel probability measure λ on $\mathcal{C}$.

4 Common Beliefs Equilibrium

An equilibrium of the model will be defined as a set of market prices and individual choices with the property that markets clear and individuals optimize. However, it does not seem obvious which objects should be priced, nor which markets should clear, nor the sense in which individuals should optimize. All of all of these issues depend on the interpretation of the model and in particular on the interpretation of uncertainty in the model. Because these issues are central, and the formalization is complicated, an informal discussion may help.

In the model, individuals face two sources of uncertainty. One is the exogenously specified stochastic nature of the production process. The other arises from the behavior and skills of other agents with whom they might be matched in a particular firm. (Recall that a firm is a particular instance of a firm type, formed by matching individual agents.) I assume here that matching of agents is random and uniform, that the realizations of uncertainty within firms (i.e., the realized states) are drawn independently from the induced distribution — in the language of Prescott & Townsend (1984a), *shocks are private* — and that the Law of Large Numbers applies.[4] The implication of these assumptions is that, although each individual agent faces risk, and accommodates these risks by choosing state-contingent consumption plans, these risks are purely idiosyncratic and wash out in the aggregate. Thus, *there is no aggregate risk*; in particular, aggregate input, output and consumption are deterministic (given the choices of agents).[5] I therefore

[4]The applicability of the Law of Large Numbers can be justified as in Sun (1998).

[5]For instance, consider Example 1 of Section 2. Suppose $1/3$ of the agents enter a firm in the role of ROW and Work, $2/9$ of the agents enter a firm in the role of COL and Work, $1/9$ of the agents enter a firm in the role of COL and Shirk, and the remaining $1/3$ of the agents do not enter the firm. From the point of view of an agent in the role of ROW, this means that the probability of being matched with an agent (in the role of COL) who Works is $2/3$ and the probability of being matched with an agent who Shirks is $1/3$. In the aggregate, the probability that the High state will obtain is $2/3 + (1/2)(1/3) = 5/6$ and the probability that the Low state will obtain is $(1/2)(1/3) = 1/6$. The aggregate production of good 2 is therefore $(1/3)[(5/6)(44) + (1/6)(2)] = 222/18$ units, and the total supply of good 2 is $222/18 + (1/3)(1) + (1/3)(1) + (1/3)(3) = 252/18$ units.

require prices to be deterministic as well.

In the model, agents choose (state-dependent) consumption of private goods and also choose firms, roles and actions. The point of view taken here is that consumption of private goods and choices of firm and role are observable, verifiable and contractible, and hence are priced, but that action choices are not observable — or at least not verifiable - and hence are not contractible and are not priced. Similarly, skills are not contractible and are not priced. (The non-contractibility of actions seems the essence of moral hazard, and the non-contractibility of skills seems the essence of adverse selection.) Prices paid for choices of roles in a given firm, which it is natural to term *wages*, represent transfers within the firm, and so sum to 0. Requiring wages to be specified as part of equilibrium seems very natural and probably needs little justification, but it is also necessary to guarantee that equilibrium exists.[6]

At equilibrium, the market for private goods must clear, so aggregate consumption must equal aggregate endowment plus aggregate net production. (As noted above, these quantities are not random, so private goods markets clear in the usual deterministic sense.) At equilibrium, I also require that the market for jobs clears; that is, for each firm, an equal mass of agents (perhaps 0) choose each role.

A notion of equilibrium in the spirit of Nash equilibrium would view market prices as constraining feasible choices for each agent, and require that actual choices be optimal, among feasible choices, given the actions of others. However, this notion of equilibrium would seem too weak. In Example 1, for instance, autarky would be an equilibrium. (Planning to join a firm would not be optimal if no one else planned to join a firm.) Indeed, such a notion would always admit an equilibrium in which *no* multi-agent firms form. Such a notion of equilibrium would also seem to violate the tenet of general equilibrium theory that individual demand should not take account of aggregate supply. (In context: an individual's demand for a particular

[6] In Example 1, the symmetric division of input and output guarantees that equilibrium wages are 0, but if ROW were entitled to the entire output then equilibrium wages could not be 0, else all agents would prefer the role of ROW to the role of COL.

role in a particular firm should not take account of others' demand for the complementary roles in that firm.) A stronger notion of equilibrium, in the spirit of perfect Bayesian equilibrium, would require that each agent form beliefs about the behavior of others and optimize given those beliefs, and that beliefs about firms that form should be correct. However, this notion of equilibrium would also seem too weak, because it would admit equilibria in which agents hold contradictory beliefs about behavior in firms that do not form. I therefore require that all agents hold the same beliefs.

Thus, a *common-beliefs equilibrium* consists of commodity prices p, wages w, beliefs β and a probability distribution on $\mathcal{C} \times \mathbb{R}_+^{L\Omega} \times \mathbb{F}$ such that: the equilibrium distribution is consistent with the population distribution, commodity markets clear, job markets clear, agents optimize in their budget sets given beliefs, and beliefs are correct. Formal definitions are given below.

4.1 Wages

In addition to private goods, roles in firms are priced. It is natural to interpret a role in a firm as a job and its price as a *wage*, and to adopt the convention that wages are paid *to* the members of the firm.[7] Because profits of the firm are apportioned according to the specified profit shares, wages are transfers among the various members of a firm. To formalize this, write

$$\mathbb{M} = \{(j, r) : j \in J, r \in R^j\}$$

for the set of *memberships*. A *wage structure* (or *wage* for short) is a function $w : \mathbb{M} \to \mathbb{R}$ for which $\sum_{r \in R^j} w(j, r) = 0$ for each j. (The requirement that wages in a firm sum to 0 is just an accounting identity.) Write $\mathbb{W} \subset \mathbb{R}^{\mathbb{M}}$ for the space of wages. I emphasize that, in keeping with the interpretation that skills and actions are not observable and/or not contractible, skills and actions are *not* priced.

[7] This is the opposite sign convention from that adopted in Ellickson, Grodal, Scotchmer and Zame (1999, 2001, 2003).

4.2 Budget Sets

If w is a wage structure and $\phi \in \mathbf{F}$ is a profile of role-action choices, it is convenient to abuse notation and write

$$w \cdot \phi = \sum_{\phi^j \neq 0} w(j, \rho^j(\phi))$$

for total wage income, given the wage structure w and the choices ϕ. (Of course agents do not receive wages in firms to which they do not belong.)

Given private good prices p and wages w, the choice $(\tilde{x}, \phi)$ is *budget feasible* for an agent with endowment (e, ϵ) if consumption choices are budget feasible in each state $\omega = (\omega^1, \ldots, \omega^J) \in \Omega$ given income in that state:

$$p \cdot \tilde{x}(\omega) \leq p \cdot e + (w \cdot \phi - w \cdot \epsilon) + \sum_{\phi^j \neq 0} D(\rho^j(\phi), \omega^j, p)$$

The left side is expenditure on private goods in state ω; the right side is the value of the (state-independent) endowment of private goods plus net wages plus the share of profits in firms to which the agent belongs. Write $B(e, \epsilon, p, w)$ for the set of budget feasible choices. (Keep in mind that, because agents are exposed to risks, they receive state-dependent profit shares and choose state-dependent consumption; however, because risks within each firm are idiosyncratic to that firm, prices are *not* state-dependent.)

4.3 Beliefs and Expected Utility

A system of *beliefs* is a probability measure β on $\mathbf{S} \times \mathbf{A}$, the space of all skill-action profiles for all firm types. Given beliefs β, write β^j for the marginal of β on $\mathbf{S}^j \times \mathbf{A}^j$ and β^j_{-r} for the marginal of β on $\mathbf{S}^j_{-r} \times \mathbf{A}^j_{-r}$.[8]

Consider an agent with characteristics $(\Phi, u, (e, \epsilon), \sigma)$ who holds beliefs β and chooses $(\tilde{x}, \phi) \in X(\Phi)$. Fix $\omega = (\omega^1, \ldots, \omega^J) \in \Omega$ and a skill-action profile $(\mathbf{s}, \mathbf{a}) \in \mathbf{S} \times \mathbf{A}$. If $\phi^j \neq 0$, the probability that this agent observes

[8] As usual, I write $\mathbf{S}^j_{-r}$ for the profiles of skills in all roles *except* role r, etc.

state ω^j in firm type j when complementary agents have skill-action profile $(\mathbf{s}, \mathbf{a})$ is

$$\pi^j(\omega^j|\phi, \mathbf{s}, \mathbf{a}) = \pi(\omega^j|\sigma^j_{\rho^j(\phi)}, \alpha^j(\phi), \mathbf{s}^j_{-r}, \mathbf{a}^j_{-r})$$

If $\phi^j = 0$, define

$$\pi^j(\omega^j|\phi, \mathbf{s}, \mathbf{a}) = \pi^j(\omega^j|\mathbf{s}^j, \mathbf{a}^j)$$

The probability that this agent observes ω when complementary agents have skill-action profile $(\mathbf{s}, \mathbf{a})$ is

$$\pi(\omega|\phi, \mathbf{s}, \mathbf{a}) = \prod_{j \in J} \pi^j(\omega^j|\phi, \mathbf{s}, \mathbf{a})$$

Thus the agent's expected utility is:

$$Eu(\tilde{x}, \phi|\beta) = \int_{\mathbf{S}\times\mathbf{A}} u(\tilde{x}(\omega), \phi, \mathbf{s}, \mathbf{a}, \omega)\pi(\omega|\phi, \mathbf{s}, \mathbf{a})\, d\beta(\mathbf{s}, \mathbf{a})$$

4.4 Job Market Clearing

For each j, r, let

$$T^j_r = \{(\xi, \tilde{x}, \varphi) \in \mathcal{C} \times \mathbb{R}^{L\Omega}_+ \times \mathbb{F} : \rho^j(\phi) = r\}$$

This is the set of characteristics of agents who choose role r in firm type j. Say that the probability measure μ on $\mathcal{C} \times \mathbb{R}^{L\Omega}_+ \times \mathbb{F}$ is *consistent*, or that *the job market clears* if for each j and each $r, r' \in R^j$

$$\mu(T^j_r) = \mu(T^j_{r'})$$

That is, the same number of agents choose each of the roles in a given firm.

4.5 Aggregate Output

A choice of firms, roles and actions for each agent in the economy induces a distribution of skills and behaviors within each firm. From this distribution, the assumption of random matching and Law of Large Numbers determines aggregate output. Formally, let μ be a consistent probability measure on $\mathcal{C} \times \mathbb{R}_+^{L\Omega} \times \mathbb{F}$. For each j, r for which $\mu(T_r^j) \neq 0$, define $h : T_r^j \to S_r^j \times A_r^j$ by

$$h(\xi, \tilde{x}, \phi) = (\sigma_r^j(\phi), \alpha^j(\phi))$$

Let $\gamma_r^j = h_* \mu$ be the direct image measure and set $\gamma_r^j = \gamma_r^j / \|\gamma_r^j\|$; γ_r^j is the distribution of skills and actions in role r in firm type j, given the choices μ. Because matching is random, the distribution of skills and actions over all roles in firm type j is $\gamma^j(\cdot|\mu) = \gamma_1^j \times \ldots \times \gamma_{R^j}^j$ and the distribution of outcomes across firms of type j is given by

$$\Gamma^j(\omega^j|\mu) = \int_{\mathbf{S}^j \times \mathbf{A}^j} \pi^j(\omega^j|\mathbf{s}, \mathbf{a})\, d\gamma^j(\cdot|\mu)$$

The number of firms of type j that form is $\mu(T_1^j)$ (which, by consistency, coincides with $\mu(T_r^j)$ for each $r \in R^j$), so the Law of Large Numbers implies that the total output of all firms of type j is

$$Y^j(\mu) = \mu(T_1^j) \sum_{\omega^j \in \Omega^j} y^j(\omega^j) \Gamma^j(\omega^j)$$

Hence aggregate output is

$$Y(\mu) = \sum_{j \in J} Y^j(\mu)$$

4.6 Aggregate Consumption

Because individual consumption is random (state-dependent), aggregate consumption cannot be defined simply as the integral of individual consumption with respect to the population measure. However, it follows from the Law of Large Numbers that aggregate consumption can be defined as the integral of individual expected consumption with respect to the population measure. Because the risks individuals face are correlated with their consumption decisions, some care must be taken in formalizing this definition.

Fix a consistent probability measure μ on $\mathcal{C} \times \mathbb{R}_+^{L\Omega} \times \mathbb{F}$. For $\phi \in \mathbb{F}$, write

$$\begin{aligned} J(\phi) &= \{j \in J : \phi^j \neq 0\} \\ T(\phi) &= \{(\xi, \tilde{x}, \psi) \in \mathcal{C} \times \mathbb{R}_+^{L\Omega} \times \mathbb{F} : \psi = \phi\} \end{aligned}$$

For $j \in J(\phi)$, define

$$
\begin{aligned}
S^j(\phi) &= \begin{cases} S^j & \text{if} \quad j \notin J(\phi) \\ S^j_{-\rho^j(\phi)} & \text{if} \quad j \in J(\phi) \end{cases} \\
A^j(\phi) &= \begin{cases} A^j & \text{if} \quad j \notin J(\phi) \\ A^j_{-\rho^j(\phi)} & \text{if} \quad j \in J(\phi) \end{cases} \\
\mathbf{S}(\phi) &= S^1(\phi) \times \ldots \times S^J(\phi) \\
\mathbf{A}(\phi) &= A^1(\phi) \times \ldots \times A^J(\phi)
\end{aligned}
$$

For $\mathbf{s} \in \mathbf{S}(\phi), \mathbf{a} \in \mathbf{A}(\phi)$, $j \in J$ and $\omega = (\omega^1, \ldots, \omega^J) \in \Omega$, define $\pi^j(\omega^j|\phi, \mathbf{s}, \mathbf{a})$ and $\pi(\omega|\phi, \mathbf{s}, \mathbf{a})$ as in Subsection 4.3 and $\gamma^j(\cdot|\mu)$ as in Subsection 4.5. Set $\gamma(\cdot|\mu) = \gamma^1(\cdot|\mu) \times \ldots \times \gamma^J(\cdot|\mu)$ and let $\gamma_\phi(\cdot|\mu)$ be the marginal of $\gamma(\cdot|\mu)$ on $\mathbf{S}(\phi) \times \mathbf{A}(\phi)$; $\gamma_\phi(\cdot|\mu)$ is the distribution of skills and actions in all roles in all firms except those in ϕ. Expected consumption of $(\Phi, u, (e, \epsilon), \sigma, \bar{x}, \phi)$ is

$$E(\bar{x}|\mu) = \sum_{\omega \in \Omega} \int_{\mathbf{S}(\phi) \times \mathbf{A}(\phi)} \bar{x}(\omega) \pi(\omega|\phi, \mathbf{s}, \mathbf{a}) \, d\gamma_\phi(\mathbf{s}, \mathbf{a}|\mu)$$

The total expected consumption of agents in $T(\phi)$ is $\int_{T(\phi)} E(\bar{x}|\mu) \, d\mu$, so the total expected consumption of all agents, which (using the Law of Large Numbers) I identify with aggregate consumption, is

$$
\begin{aligned}
X(\mu) &= \sum_{\phi \in \mathbb{F}} \int_{T(\phi)} E(\bar{x}) \, d\mu \\
&= \sum_{\phi \in \mathbb{F}} \int_{T(\phi)} \sum_{\omega \in \Omega} \int_{\mathbf{S}(\phi) \times \mathbf{A}(\phi)} \bar{x}(\omega) \pi(\omega|\phi, \mathbf{s}, \mathbf{a}) \, d\gamma_\phi(\mathbf{s}, \mathbf{a}|\mu) \, d\mu
\end{aligned}
$$

(Of course, some of the sets $T(\phi)$ may have measure 0, so the corresponding contributions to aggregate consumption are 0 as well.)

4.7 Equilibrium

An *equilibrium* consists of prices $p \in \Delta$, wages $w \in \mathbb{R}^{\mathbb{M}}$, a system of beliefs β, and a probability measure μ on $\mathcal{C} \times \mathbb{R}_+^{L\Omega} \times \mathbb{F}$ such that

- the marginal of μ on $\mathcal{C}$ is λ
- almost all choices are physically and budget feasible:

$$\mu\{(\Phi, u, (e, \epsilon), \sigma, \tilde{x}, \phi) : (\tilde{x}, \phi) \notin X(\Phi) \cap B(e, \epsilon, p, w)\} = 0$$

- almost all choices are optimal given prices and beliefs:

$$\mu\{(\Phi, u, (e, \epsilon), \sigma, \tilde{x}, \phi) : \\ \exists(\tilde{x}', \phi') \in X(\Phi) \cap B(e, \epsilon, p, w), Eu(\tilde{x}', \phi'|\beta) > Eu(\tilde{x}, \phi|\beta)\} = 0$$

- the job market clears: $\mu(T^j_r) = \mu(T^j_{r'})$ for each $j \in J$ and each $r, r' \in R^j$
- commodity markets clear: $X(\mu) = Y(\mu) + \int e d\lambda$
- beliefs are correct for firms that form: $\mu(T^j_r) \neq 0 \Rightarrow \beta^j = \gamma^j(\cdot|\mu)$

Common beliefs equilibrium exists; the proof is deferred to the Appendix.

Theorem 1 *If*

a) individual endowments are uniformly bounded

b) λ is consistent in the sense that, for each $j \in J$, $r, r' \in R^j$

$$\lambda\{\xi \in \mathcal{C} : \rho^j(\epsilon^j_\xi) = r\} = \lambda\{\xi \in \mathcal{C} : \rho^j(\epsilon^j_\xi) = r'\}$$

c) all goods are represented in the aggregate; that is, $\int_{\mathcal{C}} e\, d\lambda \gg 0$

then a common beliefs equilibrium exists.

Because the model admits unobservable actions and skills and uninsurable idiosyncratic risk, equilibrium need not be Pareto optimal. If skills and actions do not matter and there is no idiosyncratic risk, then the model reduces to that of Ellickson, Grodal, Scotchmer & Zame (2003) (except that the present model is described in distributional form), and in that case, equilibrium outcomes are Pareto optimal.

4.8 Endogenous Contracting

The approach taken here describes a firm type as a set of roles, a set of actions and a set of skills in each role, a set of states, a stochastic production process, and a state-contingent profit-sharing plan. Equilibrium determines wages and the set of firms that actually form. This approach is much less restrictive than it might appear, because the set of firm types could be very large, allowing for a set of firm types that incorporate many different production technologies and many different profit sharing plans, only a few of which are actually formed at equilibrium. Put differently, the production technologies and profit sharing plans that are actually observed — as opposed to those which are scientifically or legally feasible — are determined endogenously at equilibrium. (See Example 4 in Section 6.) In particular, contractual arrangements may be determined endogenously by market forces. The one restriction made is that the number of firm types is finite; in particular, the number of different profit sharing plans is finite. However, because the possible profit sharing plans might constitute a very fine — albeit finite — grid, this seems a minor restriction.

An alternative approach would have been to describe a firm as a set of roles, a set of actions and a set of skills in each role, a set of states, and a stochastic production process, but to take the entire contractual arrangement — both wages and profit-sharing plans — as part of the definition of equilibrium. This approach would certainly lead to a consistent model, but it would be much less satisfactory than it might appear, because equilibrium would *always* be highly indeterminate (at least if there is at least one firm type with at least two states).

To make the point, consider the world of Example 1 with the sole difference that the firm type is not a partnership; rather, the entire contractual structure is to be determined endogenously at equilibrium. As has already been shown, there is an equilibrium in which the contractual structure is a partnership. However, Theorem 1 guarantees that *any* profit-sharing plan can be supported at equilibrium by some lump-sum wage. In particular, there is an equilibrium in which ROW provides all the input and owns all the output and pays COL for his participation, and there is an equilibrium in which COL provides all the input and owns all the output and pays ROW for her participation, and there is an equilibrium in which ROW owns the output in the Good state and COL owns the output in the bad state, and so forth. More precisely, every profit-sharing plan, supported by an appropriate wage, can be part of an equilibrium. If we restrict attention to profit-sharing plans that are independent of prices, this yields a one-dimensional space of equilibria, each with a distinct contractual arrangement; if we allow profit-sharing plans that depend on prices, the space of equilibria is infinite dimensional.

Similar indeterminacy is *not* a characteristic of the approach taken here. Consider Example 1 again, but suppose there are two firm types, which differ only in the profit-sharing plan: in the first firm type, ROW and COL are partners, in the second firm type, ROW owns all the output. Suppose the disutility parameter δ is in the range $2 - \frac{1}{2}\sqrt{3} < \delta < \frac{13\sqrt{30}}{60}$. Straightforward calculations, which I leave to the reader, show that there is a unique equilibrium, and in that equilibrium, *only the partnership forms.*[9]

4.9 Observable Skills and Actions

The framework described here focuses on unobservable and unpriced skills and actions. It is important to note, however, that it also permits *observable and priced* skills and actions, in exactly the same way as Ellickson, Grodal, Scotchmer & Zame (2003), by appropriately coding skills into consumption sets and actions into the description of firms. For instance, consider the firm type of Example 1 (Section 2). To model ROW's actions as observable, posit two firm types, rather than one: in the first, ROW will be constrained to Work; in the second, ROW will be constrained to Shirk. To model that output depends on the skill of COL, and that skill is observable, posit as many firm types as there are possible skills of COL, and restrict consumption sets so that only agents who have a particular skill can choose the role of

[9]In the second firm type, ROW cannot induce COL to Work, but still must pay COL a wage equal to what COL could earn in the partnership. Such a wage is so high that ROW would prefer to be in the partnership, not pay a wage, and Work — even if COL Shirked.

COL in the corresponding firm type. Observable skills that are acquired can be modeled in the same way; see especially Example 1 of Ellickson, Grodal, Scotchmer & Zame (2003).

5 Population Perfect Equilibrium

The definition of common beliefs equilibrium rules out contradictory beliefs, but not absurd beliefs — for example that others use dominated strategies. Here I describe a refinement that rules out such absurd beliefs.

The refinement I offer is suggested by trembling hand perfection. Recall that a Nash equilibrium of a normal form game is trembling hand perfect if it is the limit of equilibria of perturbed games in which each agent is required to play each of his/her strategies with small but positive probability. Informally, each agent optimizes given that others behavior is subject to small trembles. A literal translation of trembling hand perfection would not make good sense in the present framework. One obvious reason is that I require agents to choose pure strategies. A less obvious, but more important, reason is that the set of feasible strategies depends on prices, so that some trembles may not be budget feasible.

To avoid these difficulties, I consider trembles *in the population*, rather than trembles in the behavior of individual agents. The effect of these trembles is to guarantee that in the equilibria of the perturbed economies, each action in each role in each firm is chosen by at least a small fraction of agents. From the point of view of an individual, trembles in the population play the same role as trembles in the behavior of others: they guarantee that the agent's potential partners in a given firm choose each given action with strictly positive probability. Population perfect equilibria always exist, and population perfection rules out absurd beliefs, including the belief that others would use dominated strategies. Example 2 in Section 6 provides an example and further discussion.

5.1 Trembles and Population Perfect Equilibrium

Fix private goods, firm types, and the distribution of agent characteristics λ. For each $j \in J$, $r \in R^j$ and $a \in A^j_r$, define $\phi_{jra} \in \mathbb{F}$ by

$$\phi^k_{jra} = \begin{cases} (r, a) & \text{if} \quad k = j \\ 0 & \text{if} \quad k \neq j \end{cases}$$

$$e_{jra} = \left(\max\left\{1, -\inf_{\omega\in\Omega, p\in\Delta} \sum_{j\in J} D^j(r,\omega,p)\right\}\right)(1,\ldots,1) \in \mathbb{R}^L_+$$

Define $h_{jra} : \mathcal{C} \to \mathcal{C}$ by

$$h_{jra}(\Phi,(e,\epsilon),u,\sigma) = ((\{\phi_{jra}\},(e_{jra},\phi_{jra}),u,\sigma)$$

and let λ_{jra} be the direct image of λ under h_{jra}. Characteristics in the support of λ_{jra} represent agents who are endowed with and forced to choose the firm jm the role r, and the action a, and who are given enough endowment of private goods to in the role r in the firm type j (so that they meet the survival requirement) but who otherwise have the same characteristics as agents in the support of λ). Set

$$\widehat{\lambda} = \frac{1}{|\mathbb{F}_0^1| + \ldots |\mathbb{F}_0^J|} \sum_{j\in J}\sum_{r\in R^j}\sum_{a\in A_r^j} \lambda_{jra}$$

Note that $\widehat{\lambda}$ is a probability measure, and so defines an economy.

Given an initial distribution λ, the common beliefs equilibrium (p,w,β,μ) is *population perfect* if there exists sequences $\{\varepsilon_n\}$ of positive real numbers and $\{(p_{\varepsilon_n}, w_{\varepsilon_n}, \beta_{\varepsilon_n}, \mu_{\varepsilon_n})\}$ of equilibria of the economies $\widehat{\lambda}_{\varepsilon_n} = (1-\varepsilon_n)\lambda + \varepsilon_n\widehat{\lambda}$ such that

- $\varepsilon_n \to 0$ as $n \to \infty$
- $(p_{\varepsilon_n}, w_{\varepsilon_n}, \beta_{\varepsilon_n}, \mu_{\varepsilon_n}) \to (p,w,\beta,\mu)$ as $n \to \infty$

5.2 Existence of Population Perfect Equilibrium

THEOREM 2 *If*

a) individual endowments are uniformly bounded

b) λ is consistent, in the sense that for each $j \in J$, $r, r' \in R^j$

$$\lambda\{\xi \in \mathcal{C} : \rho^j(\epsilon^j_\xi) = r\} = \lambda\{\xi \in \mathcal{C} : \rho^j(\epsilon^j_\xi) = r'\}$$

c) all goods are represented in the aggregate; that is, $\int e \, d\lambda \gg 0$

then a population perfect equilibrium exists.

6 Examples

Example 1 focused on pure moral hazard to illustrated how trade with the market can affect incentives within a firm. The following example (suggested by the familiar "lemons" example of Akerlof (1970)), focuses on adverse selection to illustrate the possibility of self-fulfilling prophecies and Pareto ranked equilibria, and the implications of population perfection.

Example 2: Adverse Selection Consider a world with two goods and one firm type, with two roles. Output of the firm depends on the skill of the worker, but skills are hidden, so there is adverse selection.

Formally, the single firm type has two roles, Supervisor and Worker; each role has a single action (so actions are suppressed). The space of Supervisor skills is a singleton (so Supervisor skills are suppressed); the space of Worker skills is $[0,1]$. There are two states: G (good), B (bad); state-dependent output is $y(G) = (0,1)$, $y(B) = (0,0)$. Conditional probabilities over states depend only on the skill of the Worker: $\pi(G|s) = s$. The Supervisor owns all the output of the firm: $D(S,\omega,p) = p \cdot y(\omega)$.

There are two types of agents. Type I agents can choose to be a Supervisor or not to enter a firm (but cannot choose to be a Worker). Type I agents have endowment $(1, a; 0)$ (where $1/3 < a < 1/2$); utility depends only on consumption (so I suppress other variables):

$$u_I(x_1, x_2) = \sqrt{x_1 x_2}$$

Type II agents can choose to be a Worker or not to enter a firm at all (but cannot choose to be a Supervisor). Type II agents have endowment $(1,0;0)$; utility of type II agents depend on own skill s, on consumption, and on firm-role choice:

$$\begin{aligned} u_s(x_1, x_2; 0) &= x_1 \\ u_s(x_1, x_2; W) &= x_1 - s \end{aligned}$$

Note that more skilled agents have higher disutility for work.[10]

[10]Nothing would change if type II agents' marginal utility for the second good were

As a benchmark, consider first the setting in which there is a single agent of each type, and the skill of the type II agent is drawn from the uniform distribution on $[0,1]$ (the realization known to the type II agent but not to the type I agent). To solve, take the first good as numeraire. If a wage $w > 0$ is offered to a Worker, then the type II agent having skill s will be willing to be a Worker if and only if $s \leq w$. A type I agent who pays the wage w and becomes a Supervisor will collect output equal to 1 unit of the second good with probability $w/2$, and enjoy expected utility:

$$\overline{u} = \frac{w}{2}\left[(1-w)(a+1)\right]^{1/2} + \left(1-\frac{w}{2}\right)\left[(1-w)(a)\right]^{1/2}$$

On the other hand, a type I agent can also consume her endowment $(1,a)$ and enjoy utility $\sqrt{a}$. A little algebra shows that $\overline{u} < \sqrt{a}$ for every $w > 0$, so choosing to be a Supervisor is an inferior strategy if $w > 0$. Hence the unique equilibrium is autarkic, supported by the wage $w = 0$ and by the (correct) belief that all type II agents who choose to be Workers have skill 0. Of course, this (deliberately) parallels Akerlof (1970).

Now consider a setting with many agents: the total mass of type I agents is $\alpha > 1/2$, the total mass of type II agents is $1-\alpha < 1/2$, and the distribution of skills of type II agents is uniform on $[0,1]$. Note first that there is an autarkic equilibrium, supported by the wage $w = 0$ and by the (correct) belief that all type II agents who choose to become Workers have skill 0.

However, *there is also a non-autarkic equilibrium*, driven by the possibility that a Supervisor can *trade* good 2 with the market, rather than consume it herself. To see this, normalize so commodity prices are $(1,q)$ and posit a wage w, $0 \leq w \leq 1$. Assign the type II agents who have skill $s \leq w$ to be Workers and an equal number of type I agents to be Supervisors. Given stochastic production, calculate the price q for good 2 at which demand equals supply. Keeping in mind the fraction of Supervisors who obtain output and the fraction who do not, as well as the fraction of type II agents who do not enter a firm, it may be seen that q is the unique solution to

$$\frac{w^2}{2}\left(\frac{1+qa-w+q}{2q}\right)+\frac{w(1-w)}{2}\left(\frac{1+qa-w}{2q}\right)+(1-w)\left(\frac{1+qa}{2q}\right) = \frac{w^2}{2}+a$$

strictly positive, but small; the given specification is merely simpler.

This is a first degree equation in q, so has a unique solution. Having solved for q, compute the imputed (indirect) expected utility $V_S(w)$ for type I agents who become Supervisors and $V_0(w)$ for type I agents who do not:

$$\begin{aligned} V_S(w) &= \left(\frac{1}{2}\right)\left[\left(\frac{w^2}{2}\right)(1+qa-w+q)+(1-\frac{w^2}{2})(1+qa-w)\right]q^{-1/2} \\ V_0(w) &= \left(\frac{1}{2}\right)(1+qa)q^{-1/2} \end{aligned}$$

Comparison of $V_S(w)$ with $V_0(w)$ leads to a cubic equation in w which is hard to solve in closed form. However, it is easy to show that $V_S(0) = V_0(0)$, $V_S(\varepsilon) > V_0(\varepsilon)$ for $\varepsilon > 0$ small, and $V_S(1) < V_0(1)$. Hence there is some w^*, $0 < w^* < 1$, with $V_S(w^*) = V_0(w^*)$. (A little curve sketching shows that w^* is unique.) For this wage w^* and the corresponding commodity prices $(1, q^*)$, all agents optimize and markets clear, so this yields a common beliefs equilibrium with trade. Because they are *ex ante* identical, all type I agents obtain the same *ex ante* expected utility. However, some type I agents become Supervisors and are lucky (obtain output), some type I agents become Supervisors and are unlucky (do not obtain output), and the remaining type I agents do not enter a firm; these various type I agents obtain different realized consumption and different *ex post* realized utility.

The autarkic equilibrium might appear intuitively implausible: Given the autarkic commodity prices $(1, 1/a)$, a Type I agent could foresee that offering a wage $w > 0$ would attract type II agents whose quality is uniformly distributed on the interval $[0, w]$, and would thus yield, with probability $w/2$, one unit of good 2, which could then be traded in the market at the prevailing market prices $(1/a)$. Straightforward computation (which I leave to the reader) shows that for w positive but small, this alternative would yield higher expected utility than choosing not to be a Supervisor.

This implausibility is reflected in the fact that the autarkic equilibrium is not population perfect. To see that it is not, suppose $(p_\varepsilon, w_\varepsilon, \beta_\varepsilon, \mu_\varepsilon)$ is an equilibrium for the economy $\widehat{\lambda}_\varepsilon$ (see Subsection 5.1), where $\varepsilon > 0$ is small, p_ε is close to the autarkic commodity prices $(1, 1/a)$ and w_ε is close to the autarkic wage $w = 0$. If $w_\varepsilon < 0$, then all type I agents demand to be a Supervisor (because they receive a subsidy), so the job market cannot clear.

If $w_\varepsilon = 0$ then again all type I agents demand to be a Supervisor (because some type II agents of quality greater than 0 choose to become Workers), and again the job market cannot clear. Finally, if $w_\varepsilon > 0$, all type II agents of quality less than w_ε and a uniform fraction of all other type II agents choose to become Workers. Direct computation shows that if $\varepsilon, w_\varepsilon$ are small and positive and p_ε is close to the autarkic market prices $(1, a)$, then becoming a Supervisor is strictly preferred to not becoming a Supervisor, so again all type I agents demand to be Supervisor, and again the job market cannot clear. Hence $\widehat{\lambda}_\varepsilon$ does not admit an equilibrium $(p_\varepsilon, w_\varepsilon, \beta_\varepsilon, \mu_\varepsilon)$ with p_ε close to the autarkic market prices and w_ε close to autarkic wage $w = 0$, and the autarkic equilibrium is not population perfect. $\diamond$

The next example, which is suggested by Bennardo & Chiappori (2003), illustrates the effects of market competition for workers.

Example 3: Bertrand Competition and Zero Profits Consider a world with one good and one firm type. Output of a firm depends on the effort of the Worker, but effort is unobservable so there is moral hazard. Agents compete for Workers, but high wages induce Workers to exert Low effort, so there is a tension between finding a Worker and inducing the Worker to exert High effort.

Formally, there is one type of firm, with two roles: Investor and Worker. The Investor has a single action (which I suppress); the worker chooses High effort H or Low effort L. There are two states, Good G and Bad B; output is $y(G) = 4$, $y(B) = 0$. The space of skills is a singleton, so I suppress skill in what follows. Conditional probabilities are $\pi(G|H) = 1/2$, $\pi(G|L) = 0$. The Investor is entitled to 3/4 of the output: $D(I, \omega) = \frac{3}{4}y(\omega)$.

There are two types of agents in the economy. Type I agents can be an Investor or not enter a firm (but cannot be a Worker), and have endowment $(2; 0)$. Type I agents care only about consumption, and are risk neutral; suppressing irrelevant variables their utility function is:

$$u_I(x) = x$$

Type II agents can be a Worker or not enter a firm (but cannot be an Investor), and have endowment $(0;0)$[11]. Type II agents have utility functions that depend on consumption, on firm-role choice, on effort, and on a preference parameter $s \in [0,1]$:

$$\begin{aligned} u_s(x;0) = u_s(x;L) &= x \\ u_s(x;H) &= sx \end{aligned}$$

Consider first a simple case: there is a positive mass of agents of each type, but more type I agents than type II agents; all type II agents have identical skill $s = 1/2$. To solve for common beliefs equilibrium, focus on the wage w paid to Workers and the resulting effort choice. If $w \leq 0$ then Investors would all make positive profit, whence all type I agents would demand to become Investors; since there are more type I agents than type II agents, the job market could not clear so this cannot be an equilibrium. If $0 < w < 1/2$, then all type II agents would demand to be Workers and choose to exert High effort; if $w > 1/2$ then type II agents would demand to be Workers and choose to exert Low effort. (Notice that higher wages induce lower effort.) In the former case, all Investors would make positive profit, so all type I agents would demand to be Investors, and the job market could not clear; in the latter case, all Investors would make a loss, so no agents of type I would choose to become Investors, and again the job market could not clear. The only remaining possibility is that the equilibrium wage is $w = 1/2$. At this wage, type II agents demand to be Workers, and are indifferent between exerting High effort and Low effort. Because all type I agents obtain the same *ex ante* utility, and there are more type I agents than type II agents, at equilibrium Investors must make 0 expected profit. This determines the fractions of Workers who exert High and Low effort: 1/3 of Workers exert High effort and 2/3 exert Low effort.

Bennardo & Chiappori (2003) model such environments as two-stage games: in the first stage, type I agents propose a wage; in the second stage, type II agents choose contracts and effort. Bennardo & Chiappori focus on the

[11] Nothing qualitative would change if type II agents were endowed with a small amount of the consumption good; the given specification just makes computation easier.

subgame perfect equilibrium in which all type I agents propose the wage $w = 1/2$, and all type II agents choose to become Workers and exert High effort. In this equilibrium, those type I agents who attract a Worker make positive profit, and those type I agents who do not attract a Worker make zero profit. The point of Bennardo & Chiappori (2003) is that the usual Bertrand undercutting story does not work here: type I agents who do not attract a Worker can only attract one by offering a higher wage, but a higher wage induces Low effort and thus leads to a loss, so offering a higher wage is not an improving deviation.

Two points should be noted. The first point is that the common beliefs equilibrium requires Workers to behave in a very particular way ($1/3$ choosing High effort, $2/3$ choosing Low effort) when indifferent. Moreover, if Workers do not behave in this way, it is not clear how equilibrium might be restored. For instance, if more than $1/3$ of Workers were to exert High effort, then Investors would make positive profits. Positive profits for Investors would suggest that more Type I agents should seek to become Investors, and so wages should rise. However, higher wages destroy the incentive for Workers to exert High effort, and hence lead to losses for all Investors. Hence the common beliefs equilibrium is in some sense unstable. (But keep in mind that there are otherwise well-behaved exchange economies with a unique equilibrium that is unstable for obvious dynamics.)

The second point is that the analysis of Bennardo & Chiappori (2003) follows a common convention in assuming that, when indifferent, agents of type II choose the action that is most favorable to agents of type I. However, there are many other (pure strategy) equilibria as well, including an equilibrium in which all type I agents offer the wage $w = 1/2$, all type II agents choose to become Workers, but only $1/3$ of Workers choose to exert High effort. At this equilibrium, which coincides with the common beliefs equilibrium, all Investors make 0 expected profit.

However, both the instability of the common beliefs equilibrium and the existence of a subgame-perfect equilibrium in which Investors make positive profits are artifacts of the assumption that all type II agents are identical. Following Harsanyi (1973), consider a perturbation in which the type II agents are not precisely identical, but are distinguished by values of the parameter s that are uniformly distributed on some small interval $[a, b]$, where $a < 1/2 < b$ and $|b - a|$ is small.

First look for a common beliefs equilibrium. If the wage $w > 0$ is offered to Workers, type II agents who have preference parameter s will choose to become a Worker and exert High effort if $s > \frac{2w}{1+2w}$ and will choose to become a Worker and exert Low effort if $s < \frac{2w}{1+2w}$. Because there are more type I agents than type II agents, equilibrium profits of Investors must be 0, so the equilibrium wage w^* solves the equation:

$$0 = \left(\frac{3}{2}\right)\left(\frac{b - \frac{2w}{1+2w}}{b - a}\right) - w = 0 \tag{4}$$

($3/2$ is the expected return to the Investor, conditional on obtaining a Worker who exerts High effort; $\frac{b-\frac{2w}{1+2w}}{b-a}$ is the fraction of Workers who exert High effort; w is the wage paid.) A little algebra shows that

$$\frac{a}{2(1-a)} < w^* < \frac{b}{2(1-b)}$$

(In particular, $w^* \to 1/2$ as $a, b \to \frac{1}{2}$.) At equilibrium, only agents of type II who have skill level exactly

$$s^* = \frac{2w^*}{1 + 2w^*}$$

are indifferent between exerting High effort and Low effort; of course this is a set of measure 0.

Now look for a subgame perfect equilibrium (in pure strategies) for the two-stage game. Say the equilibrium wage proposal is $\overline{w}$. If $\overline{w} > w^*$ then Investors make losses, which cannot occur at equilibrium. On the other hand, if $\overline{w} < w^*$ so Investors make profits, the usual Bertrand undercutting argument applies: type I agents who do not become Investors could offer a wage w above $\overline{w}$ but below w^*; such a wage would attract a Worker who would exert High effort, and hence would make positive profit. It follows that the equilibrium wage must be $\overline{w} = w^*$. As noted above, type II agents with preference parameter $s < s^* = \frac{2w^*}{1+2w^*}$ prefer to exert Low effort and type II agents with preference parameter $s > s^*$ prefer to exert High effort; only type II agents who have preference parameter exactly $s*$ — a set of measure 0 — are indifferent. Hence all Investors make 0 expected profit, and the equilibrium of the two-stage game coincides with the common beliefs equilibrium. As $a, b \to \frac{1}{2}$, this equilibrium converges to the zero-profit equilibrium, which is the unique subgame perfect equilibrium that is "stable" with respect to small perturbations in preferences. ◇

The final example illustrates how market forces determine the choices of contracts.

Example 4: The Endogenous Choice of Contracts Consider a world with one good and one production process, but two contractual arrangements. Competition determines which contracts are observed at equilibrium and which agents choose which contracts.

Formally, there are two types of firms:

- Firm type $\mathcal{P}$ is a *partnership*. There are two roles 1, 2, a single action for each role, two states G, B. State probabilities depend on skills of agents:
  $$\pi^{\mathcal{P}}(G|s_1, s_2) = \min\{1, s_1 + s_2\}$$
  Output is $y(G) = 1$, $y(B) = 0$. Profits are shared equally:
  $$\begin{aligned} D^{\mathcal{P}}(1,G) &= D^{\mathcal{P}}(2,G) = \tfrac{1}{2} \\ D^{\mathcal{P}}(1,B) &= D^{\mathcal{P}}(2,B) = 0 \end{aligned}$$
- Firm type $\mathcal{O}$ is an *ownership*. There are two roles Owner O and Employee E, a single action for each role, two states G, B. State probabilities depend on skills of agents:
  $$\pi^{\mathcal{P}}(G|s_O, s_E) = \min\{1, s_O + s_E\}$$
  Output is $y(G) = 1$, $y(B) = 0$. Profits belong to the owner:
  $$\begin{aligned} & & D^{\mathcal{O}}(O,G) &= 1 \\ D^{\mathcal{O}}(E,G) = D^{\mathcal{O}}(1,B) &= & D^{\mathcal{O}}(2,B) &= 0 \end{aligned}$$

All agents have endowment $e = (1; 0)$; agents care only about consumption, and are risk neutral:

$$u(x) = x$$

Agent skills are fixed, and uniformly distributed on $[0, 1]$.

Because there are two different types of firm, market forces may lead agents to sort themselves more or less efficiently. There are three kinds of equilibria.

1) **Only Partnerships form** For Partnerships, agents are matched randomly, wages are $w(\mathcal{P}, 1) = w(\mathcal{P}, 2) = 0$ (else only one role would be demanded), and the common beliefs are correct. (That is: for each role the distribution of skills is uniform on $[0, 1]$.) For Ownerships (which do not form), the shadow wage is $w(\mathcal{O}, E) = 1/4$, and the common beliefs are that both roles are filled by agents with skill 0. (Other shadow wages and beliefs are possible as well.)

   Checking that this is an equilibrium is straightforward. The expected utility of any agent in either role of a Partnership is at least $1/4$. Given the shadow wage and beliefs for Ownerships, the expected utility of the same agent in the Employee role is $1/4$ and in the Owner role is no more than $1/4$, so choosing a Partnership is optimal.

2) **Only Ownerships form** For Ownerships, agents with skills in $[0, 1/2]$ choose to be Employees, agents with skills in $[1/2, 1]$ choose to become Owners, Employees and Owners are matched randomly, the wage of the Employee is $w(\mathcal{O}, E) = 3/8$, and the common beliefs are correct. For Partnerships the shadow wages are $w(\mathcal{P}, 1) = w(\mathcal{P}, 2) = 0$ and the common beliefs are that both roles in the Partnership are filled by agents with skill 0. (Other shadow wages and beliefs are possible as well.)

   Checking that this is an equilibrium is again straightforward. Agents with skill $s \in [0, 1/2]$ choose to become Employees and enjoy wages $3/8$; in a Partnership with an agent of skill 0 they would expect payoff of $s/2 \leq 1/4$. Agents with skills $s \in [1/2, 1]$ choose to become Owners, and expect profits of $s + 1/4$, from which they must pay wages of $3/8$, hence net $s - 1/8$; in a Partnership with an agent of skill 0 they expect $s/2$ which is less than $s - 1/8$ (because $s \geq 1/2$).

3) **Both Ownerships and Partnerships form** Agents in $[0, c]$ become Employees, agents in $[c, d]$ enter partnerships, agents in $[d, 1]$ become Owners. The wages in the partnership are $w(\mathcal{P}, 1) = w(\mathcal{P}, 2) = 0$ (else only one role would be demanded), common beliefs are correct. The employee wage $w(\mathcal{O}, E)$ and the variables c, d are then determined by the requirements that the job market must clear, agents with skill c must be indifferent between being an employee and being in a partnership, and agents with skill d must be indifferent between being in a partnership and being an owner. The simple calculations imply $c = 1/4, d = 3/4$ and $w(\mathcal{O}, E) = 3/8$.

I leave it to the reader to check that the "Only Partnerships" and "Only Ownerships" equilibria are *not* population perfect, but the "Both Partnerships and Ownerships" equilibrium is.

It is enlightening to compare the social gains for the various equilibria. For the "Only Partnerships" equilibrium, the "Only Ownerships" equilibrium and the "Both Partnerships and Ownerships" equilibria the social gains are $80/192, 88/192, 91/192$ respectively. By comparison, for the social optimum (in which agents of skill s are matched with agents of skill $1 - s$) the social gain is $96/192$. Of course the social optimum cannot be achieved if skills are private information.

7 Conclusion

This paper offers a model in which agents trade anonymously with the market but interact strategically in small productive groups (firms). The model allows for both moral hazard and adverse selection. Equilibrium exists, but may not be Pareto optimal. Several examples are offered to illustrate the basic principles; specific applications are intended for succeeding papers.

Several extensions of the basic model seem natural and worthwhile. The most obvious would be to relax the finiteness requirements to allow allow for a continuum of types of firm, for a continuum of possible production states in each firm, and for a continuum of actions in each role. Such extensions seems conceptually straightforward, although the modeling and technical details involved (especially wages in a continuum of firms) seem daunting. A more important extension involves time. This paper follows a familiar general equilibrium strategy of incorporating time as a characteristic of a commodity, but it seems important to understand a properly dynamic version of the model in which skills and actions in a given firm depend on the realization of previous uncertainty.

This paper assumes throughout that agents take prices as given, but there does not seem any obvious way to justify price-taking in this context.

This paper assumes a continuum of agents, and relies on the Law of Large Numbers for the definition of equilibrium and for the demonstration that equilibrium exists. It is a challenge to construct satisfactory versions of the model and of the notion of equilibrium (or of approximate equilibrium) with a finite number of agents.

Appendix: Proofs

Before beginning the proofs of the Theorems, I collect some technical material. The proofs of the following two lemmas are straightforward and omitted.

LEMMA 1 *$\mathcal{C}$ is a Borel subset of $2^{\mathbb{F}} \times \mathbb{R}_+^L \times \mathbb{F} \times \mathcal{V} \times \mathbf{S}^{\mathbb{F}}$, which is a complete separable metric space.*

LEMMA 2 *Let $K \subset \mathcal{C}$ be a compact set of agent characteristics.*

(i) For every consumption level $c^* \in \mathbb{R}_+$ there is a utility level $u^* < \infty$ such that if $\|\bar{x}\| = \sup_\omega |\bar{x}(\omega)| \leq c^*$ then

$$u_\xi(\bar{x}, \phi, \mathbf{s}, \mathbf{a}, \omega) \leq u^*$$

for every $\xi \in K$, $\phi \in \Phi_\xi$, $\mathbf{s} \in \mathbf{S}$, $\mathbf{a} \in \mathbf{A}$, $\omega \in \Omega\}$

(ii) For every utility level $u^* < \infty$ there is a consumption level set $c^* \in \mathbb{R}_+$ such that if $|x| \geq c^*$ then:

$$u_\xi(x, \varepsilon_\xi, \mathbf{s}, \mathbf{a}, \omega) \geq u^*$$

for every $\xi \in K$, $\mathbf{s} \in \mathbf{S}$, $\mathbf{a} \in \mathbf{A}$, $\omega \in \Omega$.[12]

Now fix a set of firms. Define $\rho^* : \mathbb{F} \to \mathbb{R}^{\mathbf{M}}$ by

$$\rho^*(\phi)(j, r) = \begin{cases} 1 \text{ if } r = \rho^j(\phi) \\ 0 \text{ if } r \neq \rho^j(\phi) \end{cases}$$

Set

$$\nabla = \{\eta \in \mathbb{R}^{\mathbf{M}} : \eta^j_r = \eta^j_{r'} \text{ for all } j \in J, r, r' \in R^j\}$$

The following characterization of consistent distributions is straightforward and left to the reader.

[12]Note that the conclusion is only asserted for the *initial* membership.

LEMMA 3 *The distribution* μ *on* $\mathcal{C} \times \mathbb{R}_+^{L\Omega} \times \mathbb{F}$ *is consistent if and only if* $\int \rho^*(\phi)\, d\mu \in \nabla$.

The next two lemmas are derived from Ellickson, Grodal, Scotchmer & Zame (1999): the first translates Lemma 7.2 to the present language (so I omit the proof); the second improves on Step 7 in the proof of Theorem 6.1.

LEMMA 4 *There is a constant* $b > 0$ *such that if* $w \in W$, $\phi_1, \ldots, \phi_I \in \mathbb{F}$ *and there are* $\alpha_1, \ldots, \alpha_I > 0$ *for which* $\sum_i \alpha_i \rho^*(\phi_i) \in \nabla$ *then*

$$\min_i w \cdot \phi_i \leq -b \max_i w \cdot \phi_i$$

LEMMA 5 *For every* $E > 0$ *there is a constant* b_E *such that if* ν *is an economy in which individual endowments are bounded by* E *and* (p, w, β, μ) *is an equilibrium for* ν *then there is a wage* $w^* \in \mathbb{W}$ *such that* (p, w^*, β, μ) *is also an equilibrium for* ν *and* $\|w^*\| = \sup_{j,r} |w^*(j,r)| \leq b_E$.

Proof The proof is by contradiction. Suppose the assertion is not true; then for every n there is an economy λ_n in which individual endowments are bounded by E and an equilibrium $(p_n, w_n, \beta_n, \mu_n)$ for λ_n with the property that there does not exist a wage structure w_n^* such that $(p_n, w_n^*, \beta_n, \mu_n)$ is an equilibrium for λ_n and $\|w_n^*\| \leq n$. Passing to a subsequence if necessary, assume that for each $\phi \in \mathbb{F}$ the sequence $\{w_n \cdot \phi\}$ has a limit — perhaps infinite. Set

$$\begin{aligned}
\mathbb{F}^0 &= \{\phi \in \mathbb{F} : w_n \cdot \phi \to v(\phi) \text{ for some } v(\phi) \in \mathbb{R}\} \\
\mathbb{F}^+ &= \{\phi \in \mathbb{F} : w_n \cdot \phi \to +\infty\} \\
\mathbb{F}^- &= \{\phi \in \mathbb{F} : w_n \cdot \phi \to -\infty\}
\end{aligned}$$

Define a linear map $F : \mathbb{W} \to \mathbb{R}^{\mathbb{F}^0}$ by $F(w)(\phi) = w \cdot \phi$; let $\operatorname{ran} F$ be the range of F. Use the fundamental theorem of linear algebra to find a linear map $G : \operatorname{ran} F \to \mathbb{W}$ such that the composition FG is the identity on $\operatorname{ran} F$. Because G is continuous, there is a constant $\|G\|$ so that $|G(f)| \leq \|G\|\,|f|$ for every $f \in \mathbb{R}^{\mathbb{F}^0}$.

Let b be the constant given by Lemma 4. Set

$$\overline{D} = \sup\{D^j(r,\omega^j,p) : j \in J, r \in R^j, \omega \in \Omega^j, p \in \Delta\}$$

Choose n_0 sufficiently large so that if $n \geq n_0$ then

$$\begin{array}{rcll} |w_n \cdot \phi - v(\phi)| & \leq & +1 & \text{for every} \quad \phi \in \mathbb{F}^0 \\ w_n \cdot \phi & \geq & +(1+b^{-1})(1+\|G\|)(E+J\overline{D}) & \text{for every} \quad \phi \in \mathbb{F}^+ \\ w_n \cdot \phi & \leq & -(1+b^{-1})(1+\|G\|)(E+J\overline{D}) & \text{for every} \quad \phi \in \mathbb{F}^- \end{array}$$

For each $n \geq n_0$ set

$$w_n^* = GF(w_n) - GF(w_{n_0}) + w_{n_0}$$

I claim that $(p_n, w_n^*, \beta_n, \mu_n)$ is an equilibrium for λ_n if $n \geq n_0$. To see this, it is only necessary to show that optimal choices given p_n, w_n, β_n are optimal choices when wages are w_n^*.

The first task is to show that (almost all) optimal choices given p_n, w_n, β_n are feasible when wages are w_n^*. To this end, for each index $n \geq n_0$ and each $\phi \in \mathbb{F}$, set

$$\begin{array}{rcl} T(\phi) & = & \{(\xi, \tilde{x}, \psi) \in \mathcal{C} \times \mathbb{R}_+^{L\Omega} \times \mathbb{F} : \psi = \phi\} \\ \mathbb{F}(\mu_n) & = & \{\phi \in \mathbb{F} : \mu_n(T(\phi)) > 0\} \end{array}$$

I assert that $\mathbb{F}(\mu_n) \subset \mathbb{F}^0$. To see this, let $\phi \in \mathbb{F}(\mu_n)$. If $\phi \in \mathbb{F}^-$ then $w_n \cdot \phi < -E - J\overline{D}$. However, in each of the economies λ_n, individual endowments are bounded by E and profit shares in each firm are bounded by $\overline{D}$, so no agent can have income from endowment and profit distributions greater than $E + J\overline{D}$ (at any prices). Hence, no agent can afford to pay wages greater than $E + J\overline{D}$. Put differently: $w_n \cdot \phi \geq -E - J\overline{D}$, which means $\phi \notin \mathbb{F}^-$. On the other hand, if $\phi \in \mathbb{F}^+$ then Lemma 3 and consistency of μ_n imply

$$\sum_{\phi \in \mathbb{F}(\mu_n)} \mu_n(T(\phi,\mu_n)) = \sum_{\phi \in \mathbb{F}} \mu_n(T(\phi,\mu_n)) = \int \rho^*(\phi)\, d\mu_n \in \nabla$$

and Lemma 4 implies that there is some $\phi' \in \mathbb{F}(\mu_n)$ such that $w_n \cdot \phi' < -E - J\overline{D}$, which again is a contradiction. It follows that $\mathbb{F}(\mu_n) \subset \mathbb{F}^0$, as asserted. However, because FG is the identity on ran F,

$$F(w_n^*) = FGF(w_n) - FGF(w_{n_0}) + F(w_{n_0}) = F(w_n)$$

Equivalently, $w_n^* \cdot \phi = w_n \cdot \phi$ for every $\phi \in \mathbb{F}^0$. Hence, choices that are optimal given p_n, w_n, β_n have the same cost when wages are w_n^*, and in particular are feasible when wages are w_n^*.

The second task is to show that optimal choices given p_n, w_n^*, β_n are feasible when wages are w_n. To see this, let $(\tilde{x}, \phi)$ be optimal (for some characteristics) given p_n, w_n^*, β_n. Observe that ϕ cannot belong to $\mathbb{F}^-$, because if it were then the construction would guarantee that $w_n^* \cdot \phi < E - J\overline{D}$, and again ϕ could not be part of a feasible choice at at prices p_n and wages w_n^*. If $\phi \in \mathbb{F}^0$ then $w_n \cdot \phi = w_n^* \cdot \phi$, so budget feasibility of $(\tilde{x}, \phi)$ at p_n, w_n^* implies budget feasibility at p_n, w_n. Finally, if $\phi \in \mathbb{F}^+$ then the construction guarantees that $w_n^* \cdot \phi \leq w_n \cdot \phi$, so if $(\tilde{x}, \phi)$ is budget feasible given p, w_n^* it is certainly budget feasible given p, w_n, as desired.

Thus, $(p_n, w_n^*, \beta_n, \mu_n)$ is an equilibrium for λ_n. On the other hand, the definition of $\mathbb{F}^0$ implies that $\|F(w_n)\| \leq \max_\phi v(\phi)$, so

$$\|w_n^*\| \leq \|G\| \max_\phi v(\phi) + \|w_{n_0}\| + \|G\| \, \|F(w_{n_0})\|$$

Because the right hand side is fixed, it is less than n for n sufficiently large, which contradicts the supposition that there does not exist a wage structure w_n^* such that $(p_n, w_n^*, \beta_n, \mu_n)$ is an equilibrium for λ_n and $\|w_n^*\| \leq n$. This contradiction completes the proof. ■

With these preliminaries in hand I turn to the proof of Theorem 1.

Proof of Theorem 1 The proof first constructs artificial economies for which prices can be bounded away from 0 and wages can be bounded above and below, constructs common beliefs equilibria of these artificial economies, and then constructs a common beliefs equilibrium for the given economy as a limit of common beliefs equilibria for the artificial economies. The proof is in a number of steps. For each $\varepsilon > 0$, Step 1 constructs an artificial economy; Step 2 constructs a compact convex space of prices, wages, beliefs and choice distributions; Step 3 constructs a correspondence from this space to itself; Step 4 constructs a fixed point for this correspondence, and shows that this fixed point is an equilibrium for the artificial economy. To take a limit, it is necessary that equilibrium wages for these artificial economies remain bounded as $\varepsilon \to 0$. The equilibrium wages constructed in Step 4 may not have this property, but Step 5 shows that it is possible to modify the equilibria so that modified wages do remain bounded as $\varepsilon \to 0$. Step 6 shows that equilibrium prices in the artificial economies stay away from the boundary of the price simplex, and Step 7 shows that the limit of a subsequence of the modified equilibria for the artificial economies is an equilibrium for the given economy.

Step 1 The artificial economies are constructed to contain a few agents whose demands are easy to estimate and whose commodity demands are unsatisfiable when any prices are sufficiently close to 0 or any wages are sufficiently high or low. To accomplish this, write $\overline{e} = \int e\, d\lambda$ for the aggregate endowment; by assumption, $\overline{e} \gg 0$. For each $j \in J$, $r \in R^j$, choose and fix an arbitrary skill $s_{jr} \in S^j_r$ and action $a_{jr} \in A^j_r$. Define $\delta_{jr} \in \mathbb{F}$ by

$$\delta^{j'}_{jr} = \begin{cases} (r, a_{jr}) & \text{if} \quad j' = j \\ 0 & \text{if} \quad j' \neq j \end{cases}$$

For each j, r, define a characteristic $\xi_{jr} = (\Phi, e, \epsilon, u, \sigma) \in \mathcal{C}$ by

$$\Phi = \{0, \delta_{jr}\}\ ,\ e = \overline{e}\ ,\ \varepsilon = 0$$
$$u(x, \phi, \mathbf{s}, \mathbf{a}, \omega) = |x|\ ,\ \sigma^{j'}_{r'} \equiv s_{j'r'}$$

Now fix $\varepsilon > 0$. Write $\overline{R} = R^1 + \ldots + R^J$; this is the total number of roles in all firms. Set:

$$\lambda_\varepsilon = (1 - \varepsilon)\lambda + \varepsilon \left(\frac{1}{\overline{\overline{R}}}\right) \sum_{j,r} \delta_{jr}$$

Note that λ_ε is a probability measure, so defines an economy.

Step 2 The spaces of prices and wages are constructed so that the commodity demands of the artificial agents are impossibly large when prices or wages are on the boundary. To accomplish this, recall that profit shares are bounded, and set

$$\overline{D} = \sup\{|D^j(r, \omega^j, p)| : j \in J, r \in R^j, \omega \in \Omega^j, p \in \Delta\}$$

Write $M = \overline{e} + L\overline{D}\,\overline{R} + J\max_{j,\omega^j}|y^j(\omega^j)|$ (this will serve as an upper bound for the norm of supply). By assumption, $\overline{e} \gg 0$ so $\overline{e} \geq e_0(1,\ldots,1) = e_o\mathbf{1}$ for some $e_0 > 0$. Set

$$t = \frac{\varepsilon e_0}{2LM + \overline{R}(M + \overline{D})}$$

Define price and wage spaces by

$$\begin{aligned}\Delta_\varepsilon &= \{p \in \Delta : p_\ell \geq t \text{ for all } \ell\}\\ \mathbb{W}_\varepsilon &= \{w \in \mathbb{W} : |w(j,r)| \leq \frac{1}{t} \text{ for all } j,r\}\end{aligned}$$

Note that $\Delta_\varepsilon, \mathbb{W}_\varepsilon$ are compact convex sets.

The compact space of beliefs and choice distributions is constructed so that choice distributions are consistent with the population measure λ_ε and with the bounds on prices and wages. To this end, let E be an upper bound for individual endowments, set

$$\overline{C} = \frac{1}{t}(E + J\overline{D}) + \frac{J}{t^2}$$

and let $\mathbb{M}_\varepsilon$ be the space of probability measures on $\mathcal{C} \times \mathbb{R}_+^{L\Omega} \times \mathbb{F}$ that are supported on $\mathcal{C} \times [0,\overline{C}]_+^{L\Omega} \times \mathbb{F}$ and have marginal on $\mathcal{C}$ equal to λ_ε. Write B for the space of beliefs; that is, probability measures on $\mathbf{S} \times \mathbf{A}$. Say that beliefs $\beta \in B$ and a choice distribution $\mu \in \mathbb{M}_\varepsilon$ are *compatible* if $\beta_r^j = \gamma_r^j(\cdot|\mu)$ whenever $\mu(T_r^j) \neq 0$. (See Subsection 4.5 and 4.6.) Finally, write

$$H_\varepsilon = \{(\beta,\mu) \in B \times \mathbb{M}_\varepsilon : \beta \text{ is compatible with } \mu\}$$

By construction, λ_ε is a Borel measure on $\mathcal{C}$, which (by Lemma 1) is a Borel set in $2^{\mathbb{F}} \times \mathbb{R}_+^L \times \mathbb{F} \times \mathcal{V} \times \mathbf{S}^{\mathbb{F}}$. Hence λ_ε may be regarded as a probability measure on $2^{\mathbb{F}} \times \mathbb{R}_+^L \times \mathbb{F} \times \mathcal{V} \times \mathbf{S}^{\mathbb{F}}$ for which $\lambda_\varepsilon(\mathcal{C}) = 1$. Because $2^{\mathbb{F}} \times \mathbb{R}_+^L \times \mathbb{F} \times \mathcal{V} \times \mathbf{S}^{\mathbb{F}}$ is a complete separable metric space, λ_ε is tight and regular; it follows that H_ε is tight and hence relatively compact in the topology of weak convergence of measures. (See Billingsley (1968).). It is evident that H_ε is closed, hence compact, and it is convex and non-empty.

Step 3 The correspondence

$$\Psi : \Delta_\varepsilon \times \mathbb{M}_\varepsilon \times H_\varepsilon \to \Delta_\varepsilon \times \mathbb{M}_\varepsilon \times H_\varepsilon$$

is constructed so that, given (p, w, β, μ), the image set $\Psi(p, w, \beta, \mu)$ consists of all (p', w', β', μ') for which μ' is an optimal choice distribution given p, w, β, β' is compatible with μ', and p', w' maximize the value of aggregate excess demand at β, μ.

The first part of the construction is easy. Given (p, w, β, μ), the construction in Steps 1 and 2 guarantees that all budget feasible commodity choices for every characteristic ξ in the support of λ_ε belong to $[0, \overline{C}]^{L\Omega}$. Hence I may define $\Psi_1(p, w, \beta, \mu)$ as the set of pairs $(\beta', \mu') \in H_\varepsilon$ such that the marginal of μ' on C is λ_ε and μ' is supported on tuples $(\xi, \bar{x}, \phi)$ for which $(\bar{x}, \phi)$ is optimal in ξ's budget set, given prices p, wages w, and beliefs β.

Aggregate demand is defined exactly as in Subsection 4.6:

$$X(\mu) \quad = \quad \sum_{\phi \in \mathbb{F}} \int_{T(\phi)} \sum_{\omega \in \Omega} \int_{\mathbf{S}(\phi) \times \mathbf{A}(\phi)} \bar{x}(\omega) \pi(\omega | \phi, \mathbf{s}, \mathbf{a}) \, d\gamma_\phi(\mathbf{s}, \mathbf{a} | \mu) \, d\mu \qquad (5)$$

Because μ may not be consistent, the definition of aggregrate supply is more roundabout. Write $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^L_+ \subset \mathbb{R}^{L\Omega}_+$. For each firm j, role r, price p define

$$y^j(\omega, r, p) = \begin{cases} y^j(\omega) + [D^j(1, \omega, p) - p \cdot y^j(\omega)] \, \mathbf{1} & \text{if} \quad r = 1 \\ D^j(r, \omega, p) \mathbf{1} & \text{if} \quad r \neq 1 \end{cases} \qquad (6)$$

Notice that

$$\begin{aligned} p \cdot y^j(\omega, r, p) &= D^j(r, \omega, p) \quad \text{for each } r \\ \sum_r y^j(\omega, r, p) &= y^j(\omega) \end{aligned} \qquad (7)$$

Define aggregate output by

$$Y(\mu, p) = \sum_{\phi \in \mathbb{F}} \int_{T(\phi)} \sum_{\omega \in \Omega} \int_{\mathbf{S}(\phi) \times \mathbf{A}(\phi)} y^j(\omega, r, p) \pi(\omega | \phi, \mathbf{s}, \mathbf{a}) \, d\gamma_\phi(\mathbf{s}, \mathbf{a} | \mu) \, d\mu \qquad (8)$$

and aggregate supply as $\overline{e} + Y(\mu, p)$. (This definition of aggregate output agrees with the definition in Subsection 4.5 when μ is consistent.)

Define firm-role supply and demand $f, g(\mu) \in \mathbb{R}^{\mathrm{M}}$ by

$$\begin{aligned} f_r^j &= \lambda_\varepsilon\{\xi \in \mathcal{C} : \rho^j(e_\xi) = r^j\} \\ g_r^j(\mu) &= \mu(T_r^j) = \mu\{(\xi, \bar{x}, \phi : \rho^j(\phi) = r^j\} \end{aligned}$$

and set

$$\begin{aligned} z(p, w, \beta, \mu) &= X(\mu) - \overline{e} - Y(\mu, p) \\ \zeta(p, w, \beta, \mu) &= g(\mu) - f \end{aligned} \tag{9}$$

The quantity z plays the role of excess commodity demand, and the quantity ζ plays the role of excess firm-role demand.

Define $\Psi_2(p, w, \beta, \mu)$ to be the set of price-wage pairs that maximize the value of excess demand:

$$\Psi_2(p, w, \beta, \mu) = \arg\max \{p' \cdot z(p, w, \beta, \mu) - w' \cdot \zeta(p, w, \beta, \mu) : p' \in \Delta_\varepsilon, w' \in \mathbb{W}_\varepsilon\}$$

(By construction, both wages and excess firm-role demand belong to $\mathbb{R}^{\mathrm{M}}$; $w' \cdot \zeta(p, \beta, \mu)$ is the ordinary inner product. The quantity $w' \cdot \zeta(p, \beta, \mu)$ is subtracted, in order to be consistent with the sign convention for wages.)

Finally, the correspondence Ψ is defined as the product of Ψ_2 with Ψ_1:

$$\Psi(p, w, \beta, \mu) = \Psi_2(p, w, \beta, \mu) \times \Psi_1(p, w, \beta, \mu)$$

Step 4 It is straightforward to check that Ψ is an upper-hemi-continuous correspondence, and has compact, convex, non-empty values. (Continuity of Ψ relies on upper-hemi-continuity of individual demand, which is a consequence of Assumption A1.) Kakutani's fixed point theorem implies that Ψ has a fixed point $(p_\varepsilon, w_\varepsilon, \beta_\varepsilon, \mu_\varepsilon)$; I claim that $(p_\varepsilon, w_\varepsilon, \beta_\varepsilon, \mu_\varepsilon)$ is an equilibrium for λ_ε.

To see this, write $z_\varepsilon = z(p_\varepsilon, w_\varepsilon, \beta_\varepsilon, \mu_\varepsilon)$, $\zeta_\varepsilon = \zeta(p_\varepsilon, w_\varepsilon, \beta_\varepsilon, \mu_\varepsilon)$. It is immediate from the definitions of budget sets, aggregate consumption, the quantities $y^j(\omega, r, p)$, aggregate output and excess demands (equations (5), (6), (8), (9)) that Walras's Law holds in the aggregate:

$$p_\varepsilon \cdot z_\varepsilon - w_\varepsilon \cdot \zeta_\varepsilon = 0 \tag{10}$$

Next I show that p_ε does not belong to the boundary of the price simplex Δ_ε. To see this, note first that because $(p_\varepsilon, w_\varepsilon, \beta_\varepsilon, \mu_\varepsilon)$ is a fixed point, Walras's Law, equation (10), implies that the maximum value of excess demand is 0. Because we can always take $w' = 0$ it follows that the maximum value of commodity excess demand is at most 0. If p_ε is on the boundary of Δ_ε then some commodity ℓ has price $(p_\varepsilon)_\ell = t$. Because the artificial agents have endowment $\overline{e}$, they have wealth at least $p_\varepsilon \cdot \overline{e} \geq e_0 > 0$, and hence can purchase at least e_0/t units of good ℓ, which would yield utility at least e_0/t. In view of the nature of their utility functions, their commodity demand must be at least e_0/t in expectation. Hence the norm of total demand of the artificial agents is at least $\varepsilon e_0/t$. Because every agent's demand is non-negative and the norm of aggregate supply is bounded by M, it follows that excess demand for at least one good is at least $\varepsilon e_0/tL - M$ and that excess commodity demand for other goods is at least $-M$. Hence the value of excess demand at the price $(1/L, \ldots, 1/L)$ is at least

$$\frac{1}{L}\left(\frac{\varepsilon e_0}{tL} - LM\right) > 0$$

This contradicts the fact that p_ε maximizes the value of commodity excess demand, so we conclude that p_ε does not belong to the boundary of the price simplex Δ_ε, as desired.

Because the value $p \cdot z_\varepsilon$ of commodity excess demand is linear in prices, and the maximum is attained for a price in the interior of the price simplex Δ_ε, the value of commodity excess demand must be independent of prices. Hence commodity excess demand must be $-c\mathbf{1}$ for some $c \geq 0$.

I now show that w_ε does not belong to the boundary of the wage space $\mathbb{W}_\varepsilon$. If w_ε is on the boundary of $\mathbb{W}_\varepsilon$, then some wage has absolute value $1/t$; because wages in each firm type sum to 0, it follows that there is some j, r so that $w(j, r) \geq 1/t\overline{R}$. Each artificial agent with characteristics ϕ_{jr} could choose role r in firm j, obtain income at least $1/tL - \overline{D}$, and spend this income on the cheapest private good, obtaining utility at least $1/tL - \overline{D}$. The expected total commodity demand of each such artificial agent must be at least $1/tL - \overline{D}$. Because the total mass of such agents is $\varepsilon/\overline{R}$, total demand of these artificial agents is at least

$$\frac{\varepsilon}{\overline{\overline{R}}}\left(\frac{1}{tL}-\overline{D}\right)$$

Because the total commodity demand of all other agents is no less than $-M$, it follows that excess demand for at least one commodity is strictly positive, which contradicts the fact that commodity excess demand is $-c\mathbf{1}$. It follows that w_ε does not belong to the boundary of the price simplex $\mathbb{W}_\varepsilon$.

Write

$$\nabla = \{\eta \in \mathbb{R}^{\mathbb{M}} : \eta^j_r = \eta^j_{r'} \text{ for all } j \in J, r, r' \in R^j\}$$

Because the value $-w' \cdot \zeta_\varepsilon$ of firm-role excess demand is linear in w', and the maximum is attained for a wage in the interior of W_ε, the value of firm-role excess demand must be independent of wages. Because W_ε is closed under multiplication by -1, firm-role excess demand must actually be 0. Because wages are are arbitrary, subject to the bound $1/t$ and the constraint that wages in each firm type sums to 0, firm-role excess demand ζ_ε must belong to ∇. Because λ and hence λ_ε are consistent, $f \in \nabla$, so $g(p_\varepsilon, w_\varepsilon, \beta_\varepsilon, \mu_\varepsilon) \in \nabla$ as well. This implies that μ_ε is consistent.

Because firm-role excess demand ζ_ε belongs to ∇, the value of firm-role excess demand is 0 at every wage $w \in \mathbb{W}$. Because the maximum value of excess demand is 0, the maximum value of commodity excess demand z_ε must also be 0. Because commodity excess demand is $-c\mathbf{1}$, this implies that commodity excess demand $z_\varepsilon = 0$.

Straightforward algebraic manipulations, using the consistency of μ_ε together with the definitions and properties of $y^j(\omega, r, p)$ (equations (6) and (7)) show that $Y(\mu_\varepsilon, p) = Y(\mu_\varepsilon)$, and hence that $(p_\varepsilon, w_\varepsilon, \beta_\varepsilon, \mu_\varepsilon)$ is an equilibrium for λ_ε.

Step 5 Individual endowments for the economy λ_ε are bounded by E. Apply Lemma 5 to find wages w^*_ε such that $(p_\varepsilon, w^*_\varepsilon, \beta_\varepsilon, \mu_\varepsilon)$ is an equilibrium for λ_ε and $\|w^*_\varepsilon\| \leq b_E$.

Step 6 The next step is to show that prices $\{p_\varepsilon\}$ are bounded away from the boundary of the price simplex Δ. Suppose this were not so; then there is a subsequence $\{p_{\varepsilon_n}\}$ converging to some price $\overline{p} \in \text{bdy}\,\Delta$. I find a set of agents whose total demand at p_{ε_n} blows up as $n \to \infty$; this will provide a contradiction.

It is convenient to consider two cases. Write $\mathcal{C}^* = \{\xi \in AGENTS : \varepsilon_x i \neq 0\}$, and suppose first that $\lambda(\mathcal{C}^*) > 0$. For each index k let $\mathcal{C}_k^*$ be the set of $\xi \in \mathcal{C}^*$ such that, for every $x, \phi, \mathbf{s}, \mathbf{a}, \omega, p$:

$$\begin{aligned} u(x, \phi, \mathbf{s}, \mathbf{a}, \omega) &\leq k(1 + |x|) \\ p \cdot e_\xi + \sum_{\varepsilon^j \neq 0} D^j(\rho^j(\varepsilon_\xi), \omega^j, p) &\geq \frac{1}{k} \end{aligned}$$

Each $\mathcal{C}_k^*$ is a Borel set and $\mathcal{C}^* = \bigcup \mathcal{C}_k^*$, so there is an index k^* such that $\lambda(\mathcal{C}_{k^*}^*) > 0$. Use regularity and tightness of λ to find a compact subset $K \subset \mathcal{C}_{k^*}^*$ such that $\lambda(K) > 0$.

By assumption, $\overline{p} \in \text{bdy}\,\Delta$; say $\overline{p}_\ell = 0$. Write $\delta_\ell \in \mathbb{R}_+^L$ for the bundle that is 1 unit of commodity ℓ and 0 units of all other commodities. For each $p \in \Delta$, the budget set of an agent with characteristics $\xi \in K$ includes $((1/k\overline{p}_\ell)\delta_\ell, \epsilon)$. Because $p_{\varepsilon_n} \to \overline{p}$, it follows that agents with characteristics $\xi \in K$ can choose a bundle of the form $(\alpha\delta_\ell, \epsilon_\xi)$, where $\alpha \to \infty$ (as $n \to \infty$) uniformly on K. Lemma 2 guarantees that $u_\xi(\alpha\delta_\ell, \varepsilon_\xi, \mathbf{s}, \mathbf{a}, \omega) \to \infty$ (as $n \to \infty$) uniformly on K. Hence the utility of the equilibrium consumption $(x_x i(p_{\varepsilon_n}), \phi_x i(p_{\varepsilon_n}))$ also tends to ∞ (as $n \to \infty$), uniformly on K. The definition of $\mathcal{C}_k^*$ implies that the expectation of the equilibrium consumption must also tend to ∞, uniformly on K. However, since $\lambda(K) > 0$ this implies that the total demand of agents with characteristics in K tends to ∞ (as $n \to \infty$). Because individual endowments and aggregate supply are uniformly bounded, independently of n, this is impossible. This disposes of the first case.

Now suppose $\lambda(\mathcal{C}^*) = 0$. Because $\overline{p} \neq 0$, there is some index m with $\overline{p}_m > 0$. Write $\mathcal{C}^m = \{\xi : (e_\xi)_m > 0\}$ and note that $\lambda(\mathcal{C}_m) > 0$. For each index k, let $\mathcal{C}_k$ be the set of characteristics $\xi \in \mathcal{C}^m$ such that for every $x, \phi, \mathbf{s}, \mathbf{a}, \omega, p$:

$$u(x, \phi, \mathbf{s}, \mathbf{a}, \omega) \leq k(1 + |x|)$$

Because $\mathcal{C}^m = \bigcup \mathcal{C}_k$, there is some index k' for which $\lambda(\mathcal{C}_{k'}) > 0$. Use regularity and tightness of λ to find a compact subset $K \subset \mathcal{C}^*_{k'}$ such that $\lambda(K) > 0$. For each $p \in \Delta$, the budget set of an agent with characteristics $\xi \in K$ includes $((p_m(e_\xi)_m/\overline{p}_\ell)\delta_\ell; 0)$. Arguing exactly as in the previous case we see that, as $n \to \infty$, and uniformly on K, agents with characteristics in K can afford arbitrarily large consumption of good ℓ, hence obtain arbitrarily high utility, hence demand arbitrarily high expected consumption. And again as in the previous case, because $\lambda(K) > 0$ this implies that the total demand of agents with characteristics in K tends to ∞ (as $n \to \infty$). Because individual endowments and aggregate supply are uniformly bounded, independently of n, this is impossible. This disposes of the second case, and hence proves that the family $\{p_\varepsilon\}$ is bounded away from the boundary of Δ.

Step 7 Because prices $\{p_\varepsilon\}$ are bounded away from the boundary of the simplex Δ, the family $\{\mu_\varepsilon\}$ of choice distributions is tight. Because prices $\{p_\varepsilon\}$ and wages w^*_ε lie in bounded sets and beliefs β_ε lie in the compact set B, the family $\{(p_\varepsilon, w^*_\varepsilon, \beta_\varepsilon, \mu_\varepsilon)\}$ lies in a compact set, so some subsequence converges: say $(p_{\varepsilon_n}, w^*_{\varepsilon_n}, \beta_{\varepsilon_n}, \mu_{\varepsilon_n}) \to (p, w, \beta, \mu)$, with $p \in \Delta$. It is easily checked that (p, w, β, μ) is an equilibrium for the economy λ, so the proof is complete. ■

Proof of Theorem 2 For each $\varepsilon > 0$, Theorem 1 guarantees that the economy $\widehat{\lambda}_\varepsilon = (1-\varepsilon)\lambda + \varepsilon\widehat{\lambda}$ has an equilibrium $(p_\varepsilon, w_\varepsilon, \beta_\varepsilon, \mu_\varepsilon)$. If individual endowments in the economy λ are bounded by E then individual endowments in the economy $\widehat{\lambda}_\varepsilon$ are certainly bounded by $E^* = \max\{E, J\overline{D}\}$, where, as in the proof of Lemma 5 and Step 2 in the proof of Theorem 1, $\overline{D}$ is the supremum of the absolute value of all profit shares. In view of Lemma 5, there is no loss in assuming that $\|w_\varepsilon\| \le b_{E^*}$ for each ε. Arguing exactly as in Steps 6 and 7 of the proof of Theorem 1 produces a subsequence $(p_{\varepsilon_n}, w_{\varepsilon_n}, \beta_{\varepsilon_n}, \mu_{\varepsilon_n})$ which converges to some (p, w, β, μ), with $p \in \Delta$; (p, w, β, μ) is an equilibrium of the economy λ, and hence is population perfect. ■

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