

Liquidity and Spending Dynamics*

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Abstract

How do financial frictions affect the response of an economy to aggregate shocks? In this paper, we address this question, focusing on liquidity constraints and uninsurable idiosyncratic risk. We consider a search model where agents use a liquid asset to smooth individual income shocks. We show that the response of this economy to aggregate shocks depends on the rate of return on liquid assets. When liquid assets pay a low return, agents hold smaller liquidity reserves and the response of the economy tends to be larger. In this case, agents expect to be liquidity constrained and, due to a precautionary motive, their consumption decisions are more sensitive to changes in expected income. On the other hand, when liquid assets pay a large return, agents hold larger reserves and their consumption decisions are more insulated from income uncertainty. Therefore, aggregate shocks tend to have larger effects in economies where liquid assets pay a lower rate of return.

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1 Introduction

In times of economic distress, the demand for liquid assets typically increases. Facing the prospect of temporary shortfalls in revenue, agents tend to increase their precautionary reserves of cash, government bonds, gold or other safe assets. A symptom of this behavior is the counter-cyclical pattern of various liquidity premia, measured by the spread between the yield of assets with different liquidity, e.g., between commercial paper and treasury bills of the same maturity. In recent emerging market crises, this phenomenon has been dubbed a “flight to liquidity.” What are the aggregate implications of this behavior? When agents scramble to build reserves of liquid assets in a recession, this might affect negatively their spending decisions. Can this amplify the initial shock which triggered the recession? In this paper, we explore these questions in a general equilibrium model with a single liquid asset, money, and decentralized production and exchange. We find that the answers to the questions above depend crucially on the total supply of liquidity in the economy. When this supply is abundant, a negative aggregate shock leads to a reduction in activity, but there is no amplification due to the agents’ precautionary behavior. When, instead, the real value of liquid balances is relatively low, an aggregate shock has a magnified effect on the economy, as agents reduce their consumption in an attempt to protect their reserves. In a simple quantitative exercise, we show that this effect can be sizeable, leading to an increase in aggregate volatility by up to 50%, in economies with severe shortages of liquid assets.

We consider a model of decentralized production and exchange in the tradition of search models with money. Agents are anonymous and, thus, credit arrangements are not feasible and transactions are financed using a government-supplied asset. There is a large number of households made of a consumer and a producer. We introduce idiosyncratic uncertainty by assuming that producers are exposed to heterogeneous productivity shocks. Consumers have to make their consumption decisions before knowing the realized income of the producer. Therefore, consumption decisions are determined both by initial real money balances and by income expectations.

The government issues a fixed supply of interest-bearing notes, money, and pays a constant interest rate on them. The interest payments are financed by lump-sum taxation. The equilibrium real value of the money supply is an increasing function of the interest rate chosen by the government. Therefore, a regime with a high rate of return, is identified as a regime of

“abundant liquidity.” To derive our main analytical results, we focus on two extreme regimes. In the first case, the rate of return is equal to the inverse of the agents’ discount factor. This is a “Friedman rule” regime and, in this case, the economy achieves the first-best allocation. In the second case, the rate of return is so low that agents expect to be liquidity constrained after any realization of their income shock. We refer to this case as a “fully constrained economy.” We then compare the effect of an aggregate shock in the two regimes described. Under the Friedman rule, real money balances are large and are sufficient to completely buffer agents’ consumption against temporary income losses. Therefore, a change in the probability of low income realizations has no effect on consumption decisions. If there is a negative aggregate shock, there will be more producers with low productivity. These producers will charge higher prices and the consumers that buy from them will consume less. However, *for given prices*, the consumers’ behavior is unchanged. In a fully constrained economy, instead, future consumption will be affected by the realization of the income shock. In this case, consumption decisions today are affected by the probability attached to different income realizations. A bad aggregate shock, by increasing the probability of low income realizations, induces consumers to reduce spending and increase their precautionary reserves. This pushes down the demand curve facing each producer, and leads to a larger reduction in aggregate activity, compared to the Friedman rule economy.

The approach in this paper is closely related to the large literature on money in models with search, going back to Diamond (1984) and Kiyotaki and Wright (1989). The search model in Diamond (1981) has a built-in amplification mechanism, due to the assumption of increasing returns in the matching function. Our model shares his focus on coordination motives in decentralized trading, but we look at a different mechanism, which works through risk aversion and the precautionary behavior of agents. From a methodological point of view, our model uses quasi-linear preferences as in Lagos and Wright (2005) to simplify the analysis of the cross-sectional distribution of money balances.

The paper is also related to the large literature exploring the relation between financial frictions and aggregate volatility, including Bernanke and Gertler (1989), Acemoglu and Zilibotti (1997), Greenwood and Jovanovic (1990), Bencivenga and Smith (1991), Kiyotaki and Moore (1997). To the best of our knowledge, this is the first paper to address this issue from the point of view of limited liquidity supply.

The rest of the paper is organized as follows. In Section 2 we introduce our environment

and solve for the first-best allocation of resources. In Section 3 we define and characterize the competitive equilibrium. In particular, we analyze separately the Friedman rule economy and the fully constrained economy. Section 4 addresses the main question of the paper, that is, how the two economies react to an aggregate shock. In that section, we derive our main analytical results and present a simple quantitative exercise. Section 5 concludes. The appendix contains all the proofs that are not presented in the text.

2 The Model

Consider an economy with a continuum of infinitely-lived households, composed of two agents, a consumer and a producer. Time is discrete and, each period, agents produce and consume a single, perishable consumption good. The economy has a simple periodic structure, each time period t is divided in three sub-periods, $s = 1, 2, 3$. To simplify the exposition, we will call them “periods,” whenever there is no risk of confusion. There is an exogenous supply of money, which is the only asset available in the economy.

In periods 1 and 2, the consumer and the producer from each household travel to spatially separated markets, or islands, where they interact with consumers and producers from other households. On each island there is a competitive market, as in Lucas and Prescott (1974). Trading on the islands is characterized by anonymity, therefore the only type of trades that are feasible are spot exchanges of money for goods. There is a continuum of islands and each island receives the same mass of consumers and producers in both periods 1 and 2. The assignment of agents to islands is random and satisfies a law of large numbers, so that each island receives a representative sample of consumers and producers. Specifically, the distribution of money holdings among producers and consumers located in each given island, in each period, is identical to the economy-wide distribution. The consumer and the producer from the same household do not communicate during periods 1 and 2. However, at the end of both periods, they meet and share money holdings and information. In period 3 consumers and producers trade in a centralized market.

In period 1 at time t the producer has access to the linear technology

$$y_{1,t} = \theta_t n_t$$

where $y_{1,t}$ is output, n_t is labor effort by the producer, and θ_t is a random productivity parameter, which is the same for all the producers in a given island. The productivity shock

is realized after producers and consumers have been assigned to the islands, and is revealed to all the agents in the island. The distribution of productivity shocks across islands is given by a distribution with support $\Theta = [0, \bar{\theta}]$ and cumulative distribution function $F(\theta)$. It will be useful to assume that the distribution F has a positive atom at 0, i.e., $F(0) > 0$, and is continuous on $(0, \bar{\theta}]$. In periods 2 and 3, at each time t , the producers have fixed endowments of consumption goods, e_2 and e_3 . We will assume that the value of e_3 is large, so as to ensure that the consumption c_3 is non-negative in all the equilibria we study.

The shock θ is the source of idiosyncratic income volatility in our economy. The consumer in period 1 makes his spending decisions before knowing the shock faced by his producer. This introduces a precautionary motive in his behavior. The strength of this precautionary motive will depend on the ability of households to self-insure against income shocks. This, in turns, will depend on the monetary regime, which we will describe below.

The preferences of the household are represented by

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t (u(c_{1,t}) - v(n_t) + U(c_{2,t}) + c_{3,t}),$$

where $c_{s,t}$ is consumption in period (s, t) , n_t is labor effort, and $\beta \in (0, 1)$. The functions u and U are increasing and strictly concave, and the function v , representing the disutility of effort, is increasing and convex. A number of technical assumptions will be useful in the analysis. First, both u and U satisfy standard Inada conditions and v satisfies the condition $\lim_{n \rightarrow \bar{n}} v'(n) = \infty$. Second, both u and U display a coefficient of risk aversion smaller or equal than 1, i.e., $-u''(c)c/u'(c) \leq 1$ and $-U''(c)c/U'(c) \leq 1$ for all $c \geq 0$. Finally, the elasticity of v is bounded below, $v''(n)n/v'(n) \geq \underline{\eta} > 0$. We will discuss the role of these assumptions below.

The fact that producers and consumers are located in the same island and have linear utility in period 3 is essential for tractability. In particular, it allows us to derive an equilibrium with a degenerate distribution of money balances in period $(1, t)$, as in Lagos and Wright (2005).¹

Finally, we need to specify the monetary regime. Let M denote the fixed stock of money. At the end of period 3, the government levies a lump sum tax T and pays a gross interest R on the residual money balances held by each household. The tax is set so as to keep the money

¹The extension of the Lagos and Wright (2005) environment to a 3-period setup is also pursued in Berentsen et al. (2005), which uses it to study the distributional effects of monetary policy.

stock constant and, hence, satisfies

$$R(M - T) = M.$$

Monetary policy is fixed and is characterized by the two parameters R and M . Since we want to focus on equilibria with stationary nominal prices, we restrict R to be in the interval $[0, 1/\beta]$.

Notice that we allow for $R \stackrel{\leq}{\geq} 1$. The assumption of interest-paying money balances is a general way of introducing a government-supplied liquid asset. In the case $R > 1$ the asset resembles a nominal government bond, while in the case $R < 1$ it looks more like money subject to a positive inflation tax. The model can be rewritten with a zero interest rate on money balances, and a constant growth rate of the money stock. In the appendix, we show that all the stationary equilibria derived below correspond to stationary equilibria of an economy with constant money growth and constant inflation.

2.1 First-best allocation

In this section we describe the first-best allocation, as a benchmark for our economy. Consider a social planner who allocates consumption to households and decides the labor effort of the producers. Given that there is no capital, there is no real intertemporal link between periods. Therefore, we can look at a static planner problem which only includes periods $s = 1, 2, 3$.

Each household is characterized by a pair $(\theta, \tilde{\theta})$, where the first element represents the shock in the producer's island and the second represents the one in the consumer's island. An allocation is given by consumption functions $\{c_s(\theta, \tilde{\theta})\}_{s \in \{1, 2, 3\}}$ and an effort function $n(\theta, \tilde{\theta})$. The planner chooses an allocation that maximizes the ex-ante utility of the representative household

$$\int_0^{\bar{\theta}} \int_0^{\bar{\theta}} \left(u(c_1(\theta, \tilde{\theta})) - v(n(\theta, \tilde{\theta})) + U(c_2(\theta, \tilde{\theta})) + c_3(\theta, \tilde{\theta}) \right) dF(\tilde{\theta})dF(\theta),$$

subject to the economy's resource constraints. In period 1 there is one resource constraint for each island θ

$$\int_0^{\bar{\theta}} c_1(\tilde{\theta}, \theta) dF(\tilde{\theta}) \leq \int_0^{\bar{\theta}} \theta n(\theta, \tilde{\theta}) dF(\tilde{\theta}).$$

In period $s = 2, 3$ the resource constraint is

$$\int_0^{\bar{\theta}} \int_0^{\bar{\theta}} c_s(\theta, \tilde{\theta}) dF(\tilde{\theta})dF(\theta) \leq e_s.$$

The following proposition characterizes the optimal allocation.

Proposition 1 *The optimal allocation in period 1 is given by*

$$\begin{aligned} c_1(\tilde{\theta}, \theta) &= \theta n^{FB}(\theta), \\ n(\theta, \tilde{\theta}) &= n^{FB}(\theta), \end{aligned}$$

for each pair $(\theta, \tilde{\theta})$, where $n^{FB}(\theta)$ satisfies

$$\theta u'(\theta n^{FB}(\theta)) = v'(n^{FB}(\theta))$$

for all θ . Consumption in period 2 is given by

$$c_2(\theta, \tilde{\theta}) = e_2.$$

Due to the separability of the utility function, the optimal effort of a producer located in island θ depends only on the productivity θ and it does not depend on the shock $\tilde{\theta}$ in the island visited by the respective consumer. A useful consequence of this result is the following corollary.

Corollary 1 *The first-best level of output in island θ is independent of the economy-wide distribution of productivity shocks F .*

Moreover, at the optimum, c_2 is constant across households, that is, households are fully insured against the household specific shocks θ and $\tilde{\theta}$. Finally, notice that, given linearity, the consumption level in period 3 is indeterminate.

3 Equilibrium

We turn now to the definition and characterization of the competitive equilibrium. In this section we will look at a steady state equilibrium with a constant distribution for the productivity shocks θ_t . In the next section we will introduce aggregate shifts in the distribution F . We begin by characterizing optimal individual behavior for given prices. In each period t , the function $p_{1,t}(\theta_t)$ denotes the nominal price in period 1 in the island with shock θ_t , and $p_{2,t}$ and $p_{3,t}$ are the nominal prices in periods 2 and 3.

Consider a household with an initial stock of money m_t in period t . The consumer chooses $c_1(\tilde{\theta}_t)$, only based on the productivity shock observed in the island where he is located, $\tilde{\theta}_t$, while the producer chooses his labor effort $n(\theta_t)$ only based on the productivity in his island,

θ_t . Given that all exchanges are anonymous, agents have to use cash to finance their purchases, and, moreover, cash holdings are restricted to be non-negative. In period 1 the consumer budget constraint and the liquidity constraint are then

$$\begin{aligned} m_{1,t}(\tilde{\theta}_t) + p_{1,t}(\tilde{\theta}_t)c_1(\tilde{\theta}_t) &\leq m_t, \\ m_{1,t}(\tilde{\theta}_t) &\geq 0. \end{aligned}$$

At the end of period 1, the consumer and the producer get back together, therefore the cash available to consumers at the beginning of period 2 includes the producer's revenue from the previous period. The budget constraint and the liquidity constraint are now

$$\begin{aligned} m_{2,t}(\theta_t, \tilde{\theta}_t) + p_{2,t}c_{2,t}(\theta_t, \tilde{\theta}_t) &\leq m_{1,t}(\tilde{\theta}_t) + p_{1,t}(\theta_t)y_1(\theta_t), \\ m_{2,t}(\theta_t, \tilde{\theta}_t) &\geq 0. \end{aligned}$$

Finally, in period 3, the consumer and the producer are located in the same island and they only need to finance the net expenditure $c_{3,t} - e_3$. The constraints are now

$$\begin{aligned} m_{3,t}(\theta_t, \tilde{\theta}_t) + p_{3,t}c_{3,t}(\theta_t, \tilde{\theta}_t) &\leq p_{3,t}e_3 + m_{2,t}(\theta_t, \tilde{\theta}_t) + p_{2,t}e_2 - T, \\ m_{3,t}(\theta_t, \tilde{\theta}_t) &\geq 0. \end{aligned}$$

Let $V_t(m_t)$ denote the expected utility at the beginning of period t of a household with initial nominal balances m_t . The household problem is characterized by the Bellman equation

$$V_t(m_t) = \max_{\{c_s\}, \{m_s\}, n} \mathbb{E}[u(c_{1,t}) - v(n_t) + U(c_{2,t}) + c_{3,t} + \beta V_{t+1}(Rm_{3,t})], \quad (1)$$

subject to the budget constraints and liquidity constraints introduced above and the technological constraint

$$y_{1,t}(\theta_t) = \theta_t n_t(\theta_t),$$

and where $\{c_{s,t}\}$, $\{m_{s,t}\}$ and n_t are functions of the shocks θ_t and $\tilde{\theta}_t$ in the manner described above.

We are now in a position to define a competitive equilibrium. Let H_t denote the cross-sectional distribution of cash balances at the beginning of period t , with support \mathcal{M}_t . With a slight abuse of notation denote by $c_{1,t}(\theta_t, m_t)$, $c_{2,t}(\theta_t, \tilde{\theta}_t, m_t)$, etc., the optimal decisions of a household with initial nominal balances m_t .

Definition 1 A competitive equilibrium is given by a sequence of prices $\left\{ \{p_{1,t}(\theta_t)\}_{\theta_t \in \Theta}, p_{2,t}, p_{3,t} \right\}$, a sequence of allocations $\{n_t(\theta_t, m_t)\}_{\theta_t \in \Theta, m_t \in \mathcal{M}_t}$, $\{c_{1,t}(\theta_t, m_t)\}_{\theta_t \in \Theta, m_t \in \mathcal{M}_t}$, and $\{c_{2,t}(\theta_t, \tilde{\theta}_t, m_t), c_{3,t}(\theta_t, \tilde{\theta}_t, m_t)\}_{\theta_t \in \Theta, \tilde{\theta}_t \in \Theta, m_t \in \mathcal{M}_t}$ and a sequence of money distributions $\{H_t\}$, such that:

1. The allocations are solutions to (1) for each t and $m_t \in \mathcal{M}_t$.
2. Markets clear

$$\begin{aligned} \int m_t dH_t &= M, \\ \int c_{1,t}(\tilde{\theta}_t, m_t) dH_t &= \int \tilde{\theta}_t n_t(\tilde{\theta}_t, m_t) dH_t \quad \forall \tilde{\theta}_t \in \Theta, \\ \int \int \int c_{s,t}(\theta_t, \tilde{\theta}_t, m_t) dF dF dH_t &= e_s, \quad s = 1, 2. \end{aligned}$$

3. The sequence $\{H_t\}$ is consistent with the transition probability for money holdings derived from individual behavior.

In this definition, we omit money balances from the description of the allocation and we omit the money market equilibrium conditions for $m_{1,t}$, $m_{2,t}$ and $m_{3,t}$. Market clearing in the goods markets ensures that the money markets clear in each period.² The final money balances $m_{3,t}$ are derived from the consumer budget constraints and, for each distribution H_t , give us the distribution of $m_{t+1} = Rm_{3,t}$. This distribution must be equal to H_{t+1} to ensure that condition 3 in the definition above is met.

From now on, we focus on steady states where nominal prices and allocations are constant over time and where the cross sectional distribution of money holdings is degenerate, i.e., all agents begin each period t with $m_t = M$. As in Lagos and Wright (2005), competitive equilibria of this simple form exist because agents have linear utility in period 3 and the value function $V_t(m_t)$ is concave. In equilibrium, all agents will adjust their consumption in period 3, so as to reach the same level of $m_{3,t}$, irrespective of the history of their shocks. For any initial distribution H_0 of money holdings, the economy will converge in one period to this steady state.³

²This is just Walras' Law.

³The assumption of a large e_3 guarantees that this transition is feasible for any H_0 .

3.1 Unconstrained equilibrium

Now we analyze the effects of different monetary regimes on equilibrium behavior. We consider two polar cases. First, we look at the case where the government sets a rate of return R high enough that the economy is able to achieve the first-best allocation. This requires $R = 1/\beta$, that is, a monetary policy that follows the Friedman rule. Second, we look at the case where the rate of return is so low that the allocation in periods 1 and 2 looks like the allocation in a static economy with uninsurable income shocks. We show that this case arises whenever $R \leq \hat{R}$, for a given cutoff $\hat{R} \in (0, 1/\beta)$.

We begin by considering “unconstrained equilibria,” that is, equilibria where the liquidity constraints are never binding. The following proposition shows that when $R = 1/\beta$ such an equilibrium exists and achieves the efficient allocation of resources.⁴ Equilibrium prices are

$$p_1(\theta) = \kappa u'(\theta n^{FB}(\theta)) \text{ for all } \theta, \quad (2)$$

$$p_2 = \kappa U'(e_2), \quad (3)$$

$$p_3 = \kappa, \quad (4)$$

for a positive constant κ .

Proposition 2 *Suppose that $R = 1/\beta$ and $M > 0$. Then, there is a $\hat{\kappa}$ such that if $\kappa \in (0, \hat{\kappa}]$, there exists an unconstrained equilibrium with equilibrium prices (2)-(4). For all $\kappa \in (0, \hat{\kappa}]$ the equilibrium allocation is identical to the first-best allocation.*

Notice that, under the Friedman rule, the price level is indeterminate, since the first-best allocation can be supported by a different set of prices for any $\kappa \in (0, \hat{\kappa}]$. To resolve the indeterminacy, it is possible to define a monetary regime in terms of the pair (τ, M) where $\tau \equiv T/p_3$, instead of defining it in terms of the pair (R, M) . Once we fix a value for the real tax τ , the price level is pinned down by the condition $p_3 = T/\tau$ and the rate of return R is endogenously determined. Under this alternative approach, we can show that there exists a cutoff $\hat{\tau}^U < e_3$ such that if $\tau \in [\hat{\tau}^U, e_3)$, then there exists a unique unconstrained equilibrium with $R = 1/\beta$. A high level of real taxation corresponds to a high real value of the liquid asset M circulating in the economy. To sustain the first-best level of risk sharing the government has to commit enough fiscal resources to ensure that public liquidity pays the real rate of return $R = 1/\beta$.

⁴This result is related to efficiency results in Rocheteau and Wright (2005).

A further remark on the equilibrium distribution of nominal balances. If $R = 1/\beta$ and $\kappa < \hat{\kappa}$, this distribution is not uniquely pinned down. In this case the value function $V(m)$ is locally linear at $m = M$, and agents are indifferent between consuming a bit less (more) in period 3 and increasing (decreasing) their money balances m_3 .⁵ Therefore, there are also equilibria where agents choose different values for m_3 and the distribution of money balances is non-degenerate. These equilibria are identical to the one described in terms of consumption in periods 1 and 2, and in terms of labor effort. They only differ for the distribution of c_3 , and therefore are ex-ante equivalent in terms of welfare.

3.2 Fully constrained equilibrium

We now turn to the case where agents are constrained. In particular, we focus on the case where the liquidity constraint is always binding in period 2, i.e., $m_2(\theta, \tilde{\theta}) = 0$ for all $\theta, \tilde{\theta}$. The following proposition shows that this type of equilibrium arises when R is sufficiently low. We call this type of equilibrium a “fully constrained equilibrium.”

Proposition 3 *There exists a cutoff $\hat{R} \in (0, 1/\beta)$ such that, if $R \leq \hat{R}$, then there is an equilibrium with: (i) $m_1(\theta) > 0$ for all θ , (ii) $m_2(\theta, \tilde{\theta}) = 0$ for all $\theta, \tilde{\theta}$.*

Since the liquidity constraint is binding for all agents at date 2, the price level p_2 is determined by a simple “quantity-theory” equation, that is,

$$p_2 e_2 = M. \tag{5}$$

Since all agents begin period 1 with $m = M$ and end period 2 with $m_2(\theta, \tilde{\theta}) = 0$, the consumer’s budget constraints in periods 1 and 2 can be aggregated to give

$$p_2 c_2(\theta, \tilde{\theta}) = p_2 e_2 - p_1(\tilde{\theta}) c_1(\tilde{\theta}) + p_1(\theta) \theta n(\theta). \tag{6}$$

Then, the equilibrium in periods 1 and 2 is formally equivalent to the equilibrium of a two-period economy where agents face uninsurable income shocks. We define the real wage in period 1, in terms of period 2 consumption, as

$$w(\theta) \equiv \theta p_1(\theta) / p_2.$$

⁵They are not indifferent to large changes in m_3 , because their liquidity constraints can be binding in the following periods 1 or 2.

Substituting the market clearing condition $c_1(\tilde{\theta}) = \tilde{\theta}n(\tilde{\theta})$ and using (5), we can then write the following expression for the consumer Euler equation between periods 1 and 2

$$\theta u'(\theta n(\theta)) = w(\theta) \int_0^{\bar{\theta}} U'(e_2 - w(\theta)n(\theta) + w(\tilde{\theta})n(\tilde{\theta}))dF(\theta) \text{ for all } \theta. \quad (7)$$

Moreover, the optimality condition for labor supply can be written as

$$v'(n(\theta)) = w(\theta) \int_0^{\bar{\theta}} U'(e_2 - w(\tilde{\theta})n(\tilde{\theta}) + w(\theta)n(\theta))dF(\tilde{\theta}) \text{ for all } \theta. \quad (8)$$

Equations (7) and (8) are two functional equations in $w(\theta)$ and $n(\theta)$. The proof of Proposition 3, in the appendix, shows that there is a unique pair of functions $(w(\theta), n(\theta))$ that solves (7) and (8).

To ensure that the allocations derived for periods 1 and 2 are part of a full dynamic equilibrium, we need to ensure that the constraint $m_2(\theta, \tilde{\theta}) \geq 0$ is indeed binding for all pairs of shocks θ and $\tilde{\theta}$. The proof of Proposition 3 shows that this condition is met when $R \leq \hat{R}$. An interesting feature of this equilibrium is that, as long as $R \in (0, \hat{R})$, the choice of R has no real effects. The only real variables affected by the choice of R are the value of real balances in period 3 and the value of the real tax T/p_3 . As in the case of Proposition 2, also Proposition 3 can be restated in fiscal terms. Namely, there is a cutoff $\hat{\tau}^C$, such that if $\tau \leq \hat{\tau}^C$, then there is a fully constrained equilibrium with $R \leq \hat{R}$. If the government commits limited fiscal resources to sustain the value of public liquidity, then agents have limited liquid assets to insure against temporary income shocks.

The crucial difference between this case and the unconstrained case in Proposition 2 is that consumers in period 1 are concerned about the value of their income $w(\theta)n(\theta)$, when making their spending decisions. In particular, we can show that the income of the producer in island θ , equal to $w(\theta)n(\theta)$, is increasing in θ .⁶

Remark 1 *In a fully constrained equilibrium, the function $w(\theta)n(\theta)$ is monotone increasing in θ .*

The income risk, due to the volatility of θ , induces a precautionary behavior on the consumers' side. This has two consequences. First, if a consumer expects higher realizations of θ , he will increase consumption in period 1, since he is less concerned about having low liquid

⁶This remark is an immediate corollary of Lemma 2 in the Appendix.

reserves in period 2. This will increase output in the market where the consumer is located. Second, there is a general equilibrium feed-back involving the spending decisions of consumers. If consumers in all other markets are increasing their demand, the expected output produced by any producer increases. Hence, the consumer in island θ will expect even higher income and he will increase his own demand even further. Through this channel the initial increase in demand is magnified. This two mechanisms are going to be crucial in the next section.

4 Aggregate Shocks

The main question we want to address is how an economy with binding liquidity constraints responds to an aggregate productivity shock, and whether it responds more or less than an unconstrained economy. Let us focus on the effects of a shift in the distribution of the productivity shocks θ , which can be interpreted as an unexpected aggregate shock.⁷ Specifically, we consider a shift that reduces the probability of low shocks and increases the probability of high shocks, that is, an increase in the sense of first-order stochastic dominance.

4.1 A decomposition

Suppose that the cross-sectional distribution of productivity shocks depends continuously on the parameter $\zeta \in [0, \bar{\zeta}]$. In particular, assume $F(\theta; \zeta)$ is non-increasing in ζ for each θ , so that a distribution with a higher ζ first-order stochastically dominates a distribution with lower ζ .⁸ Let $y_1(\theta; \zeta)$ denote the output in island θ when the aggregate shock is ζ . Aggregate output in period 1 is

$$Y_1 \equiv \int_0^{\bar{\theta}} y_1(\theta; \zeta) dF(\theta; \zeta).$$

We focus on the proportional response of output to aggregate shocks, that is on $d \ln Y_1 / d\zeta$, given that different monetary regimes are also characterized by differences in levels. This measure can be decomposed as follows

$$\frac{d \ln Y_1}{d\zeta} = \frac{\int_0^{\bar{\theta}} y_1(\theta; \zeta) \frac{\partial f(\theta; \zeta)}{\partial \zeta} d\theta}{Y_1} + \frac{\int_0^{\bar{\theta}} \frac{\partial y_1(\theta; \zeta)}{\partial \zeta} dF(\theta; \zeta)}{Y_1}. \quad (9)$$

The first member on the right-hand side represents the mechanical effect of having a larger number of islands with high productivity. This effect is positive both in an unconstrained

⁷At the expense of further notation, the results in this section and in the previous one can be extended to a model with an explicit treatment of aggregate shocks. We only need to assume that the shocks are small enough that the economy is either always in an unconstrained equilibrium or always in a fully constrained equilibrium.

⁸Let the support of the distribution be $[0, \bar{\theta}]$, independently of ζ .

economy and in a constrained economy, given that, output is increasing in θ in both regimes. We call this effect the *own-productivity effect*. The second member captures the endogenous response of output for each given level of θ . We will now focus on deriving and comparing the second effect in the two regimes introduced above, the Friedman rule regime with abundant supply of liquid assets and the regime with $R < \hat{R}$ where a fully constrained equilibrium arises.

4.2 Expected income effects

First, consider the case $R = 1/\beta$. We already know from Proposition 1 that the optimal level of labor supply in island θ is independent of the economy-wide distribution of productivity. Therefore, under the Friedman rule $\partial y_1(\theta; \zeta) / \partial \zeta = 0$ and the second effect in (9) is zero.

Next, consider the case of a constrained economy, assuming $R < \hat{R}$.⁹ The next proposition shows that in this case the second effect in (9) is positive, as, for any given θ , output in period 1 increases with ζ . We call this the *expected income effect*.

Proposition 4 *Consider a constrained economy, with $R < \hat{R}$. For each $\theta > 0$, the output $y_1(\theta; \zeta)$ is increasing in ζ .*

To convey the intuition behind this effect, it is useful to consider the following partial equilibrium exercise. Let us focus on the market θ . The consumers' demand is given by

$$\theta u'(c_1(\theta; \zeta)) = w(\theta; \zeta) \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta}; \zeta)) dF(\theta), \quad (10)$$

while labor supply is given by

$$v'(n(\theta; \zeta)) = w(\theta; \zeta) \int_0^{\bar{\theta}} U'(c_2(\tilde{\theta}, \theta; \zeta)) dF(\tilde{\theta}), \quad (11)$$

market clearing requires $c_1(\theta; \zeta) = \theta n(\theta; \zeta)$. The demand equation (10) defines implicitly $c_1(\theta; \zeta)$ as a decreasing function of $w(\theta; \zeta)$, while the supply equation (11) defines implicitly $n(\theta; \zeta)$ as an increasing function of $w(\theta; \zeta)$.¹⁰ In a fully constrained equilibrium agents are not fully insured and both $c_2(\theta, \tilde{\theta}; \zeta)$ and $c_2(\tilde{\theta}, \theta; \zeta)$ are different for different $\tilde{\theta}$, with

$$\begin{aligned} c_2(\theta, \tilde{\theta}; \zeta) &= e_2 - w(\theta; \zeta) n(\theta; \zeta) + w(\tilde{\theta}; \zeta) n(\tilde{\theta}; \zeta), \\ c_2(\tilde{\theta}, \theta; \zeta) &= e_2 - w(\tilde{\theta}; \zeta) n(\tilde{\theta}; \zeta) + w(\theta; \zeta) n(\theta; \zeta). \end{aligned}$$

⁹Notice that \hat{R} depends on ζ continuously. For small changes in ζ , $R < \hat{R}$ ensures that the economy remains in a constrained equilibrium.

¹⁰The last claim follows from the assumption on the elasticity of the utility function U .

We observed above that $w(\theta; \zeta) n(\theta; \zeta)$ is increasing in θ (see Remark 1). It follows that $U'(c_2(\theta, \tilde{\theta}; \zeta))$ is decreasing in θ , while $U'(c_2(\tilde{\theta}, \theta; \zeta))$ is increasing in θ . Hence, when ζ increases the integral on the right-hand side of (10) becomes smaller, while the integral on the right-hand side of (11) becomes bigger.¹¹ This implies that both the demand and the supply curves are going to shift to the right. On the demand side, the intuition is that when a liquidity constrained consumer expects higher income, his marginal value of money decreases. Then, he reduces his precautionary reserves and increases consumption in period 1 and, by market clearing, increases labor demand. On the supply side, when a producer expects higher consumption, then, for a given wage, he is going to work more, increasing labor supply. These two effects combined imply that equilibrium output is going to increase unambiguously.

On top of this partial equilibrium mechanism, there is a general equilibrium feed-back effect. The output increase due to increased spending by other consumers and increased production by other producers further increase expected income in period 2 and lead to a further increase in spending and production in period 1.

The question that remains to be addressed is whether, overall, a constrained economy reacts more or less to an aggregate shock than an economy with full insurance. We have shown that the expected income effect present in a fully constrained equilibrium tends to magnify the output response to aggregate shocks in the constrained economy. Going back to equation (9), the second term of the decomposition is zero in the unconstrained case and positive in the constrained one. However, we do not know the relative magnitude of the own-productivity effect, the first term in the decomposition, which we know is positive in both cases. In order to compare the total output response of the two economies, we turn to some examples.

4.3 An example with a binary shock

Consider the case a simple example with a binary shock, that is, $\theta \in \{0, \bar{\theta}\}$. Denote by π the probability of $\theta = \bar{\theta}$. In this case, π takes the place of ζ , since a higher π corresponds to first-order stochastically dominant distribution. Let C and U denote, respectively, the fully constrained and the unconstrained monetary regime.

Proposition 5 *Suppose $u(c) = c$, $U(c) = c^{1-\sigma}/(1-\sigma)$, $v(n) = n^{1+\eta}/(1+\eta)$ and $\theta \in \{0, \bar{\theta}\}$, then $d \ln Y^C/d\pi > d \ln Y^U/d\pi$.*

¹¹The fact that the integral decreases follows from the fact that an increase in ζ leads to a shift of the distribution of θ in the sense of first order stochastic dominance.

This proposition shows that, in the example considered, the output response to a positive aggregate shock is always higher when the liquidity constraint of the agents in the economy is binding. In particular, with these specific functional forms, the own-productivity effect, adjusted for the output levels, is identical in the two economies, so that what matters is only the expected income effect, that is, the second term in equation (9), which we have already shown is always positive.

When $R = 1/\beta$, the economy achieves the unconstrained equilibrium with

$$n^U(\theta; \pi) = \theta^{\frac{1}{\eta}} \text{ and } Y_1^U(\pi) = \pi \bar{\theta}^{\frac{\eta+1}{\eta}}.$$

When $R < \hat{R}$, the economy achieves the fully constrained equilibrium characterized by aggregate output equal to

$$Y_1^C = \pi \bar{\theta} n^C(\bar{\theta}; \pi)$$

where $n^C(\bar{\theta}; \pi)$ solves, together with $w^C(\bar{\theta}; \pi)$, the following system of equations:

$$\begin{aligned} \bar{\theta} &= w^C(\bar{\theta}; \pi) e_2^{-\sigma} \pi + w^C(\bar{\theta}; \pi) (e_2 - w^C(\bar{\theta}; \pi) n^C(\bar{\theta}; \pi))^{-\sigma} (1 - \pi), \\ n^C(\bar{\theta}; \pi)^\eta &= w^C(\bar{\theta}; \pi) e_2^{-\sigma} \pi + w^C(\bar{\theta}; \pi) (e_2 + w^C(\bar{\theta}; \pi) n^C(\bar{\theta}; \pi))^{-\sigma} (1 - \pi). \end{aligned}$$

The output response to aggregate shocks for the two economies is

$$\frac{d \ln Y_1^U}{d\pi} = \frac{1}{\pi} \text{ and } \frac{d \ln Y_1^{LC}}{d\pi} = \frac{1}{\pi} + \frac{1}{n^C(\bar{\theta}; \pi)} \frac{\partial n^C(\bar{\theta}; \pi)}{\partial \pi},$$

where the own-productivity effect is identical in the two economies and equal to $1/\pi$. Proposition 4 implies that $\partial n^C(\bar{\theta}; \pi) / \partial \pi > 0$. This immediately implies that the fully constrained economy is going to react more to an aggregate shock.

4.4 Some quantitative implications

Now we turn to some basic numerical examples that show that, under a reasonable parametrization, the amplification effect identified above is sizeable and depends crucially on two things: (i) the level of idiosyncratic risk in the economy and (ii) the type of aggregate shocks hitting the economy. To compute these examples we generalize our theory to monetary regimes with $R \in (\hat{R}, 1/\beta)$, where the liquidity constraints are occasionally binding.¹² The formal characterization of the equilibrium for this intermediate region is in the appendix, where we also describe our computational strategy.

¹²In Appendix A, you can find the characterization of a general equilibrium where $R \in (\hat{R}, 1/\beta)$.

We interpret each sequence of three subperiods as a quarter, and set the discount factor β at 0.96. We assume that θ has a discrete uniform distribution on $\{0, \dots, \theta^k, \dots, \bar{\theta}\}$, and that the labor cost function takes the iso-elastic form $v(n) = n^{1+\eta}/(1+\eta)$ where η represents the Frisch elasticity of labor supply. We choose an elasticity $\eta = 1$. Moreover, we assume CRRA utility both in periods 1 and 2, that is, we set $u(c) = c^{1-\gamma}/(1-\gamma)$ and $U(c) = c^{1-\sigma}/(1-\sigma)$. We set $\sigma = 2$. We set the number of realizations of the productivity shock at $K = 5$, and choose γ , $\bar{\theta}$ and e_3 in order to match the following two statistics. First, we want to obtain a volatility of income around 0.2, to match (on the conservative side) the values for the temporary component of income uncertainty obtained in Hubbard et al. (1994), Gourinchas and Parker (2002) and Storseletten, Telmer and Yaron (2004). Second, we match the observed demand function for real money balances. The real rate of return on money balances is captured by the inverse of the expected inflation rate. By assuming a constant real interest rate on illiquid assets, the real return on money balances is inversely related to the nominal interest rate. An advantage of this approach is that there is a vast literature on the estimation of aggregate money demand. In particular, we can compare our calibration to the results in Lucas (2000), Lagos and Wright (2004), and Craig and Rocheteau (2007). In particular, we look at the relation between money velocity and the nominal interest rate. Money velocity is defined as the ratio of money supply M1 (currency and demand deposits) to nominal gross domestic product. The nominal interest rate is measured by the short-term commercial paper rate.¹³ Next figure shows the relation implied by the model (solid line) and compares it with the relation in the data (crossed scatterplot).

The first type of aggregate shock we consider is a shock that reduces the probability of $\theta = 0$ and increases the probability of all positive realizations of θ . We choose an aggregate shock which increases output by 10% under the Friedman rule. Figure 1 shows the output response to this shift for different levels of R .

Notice that the output response to aggregate shocks is almost 50% higher in the fully constrained economy, with $R \leq \hat{R}$. Moreover, there is a monotone relation between R and the effect of the aggregate shock.

The next figure illustrates the effects identified in Proposition 4. The two panels illustrate how output varies across islands in the two polar regimes. In the panel on the right we show output under the Friedman rule. In this case, as argued above, output per island is independent

¹³The data are the same used in Lagos and Wright (2004).

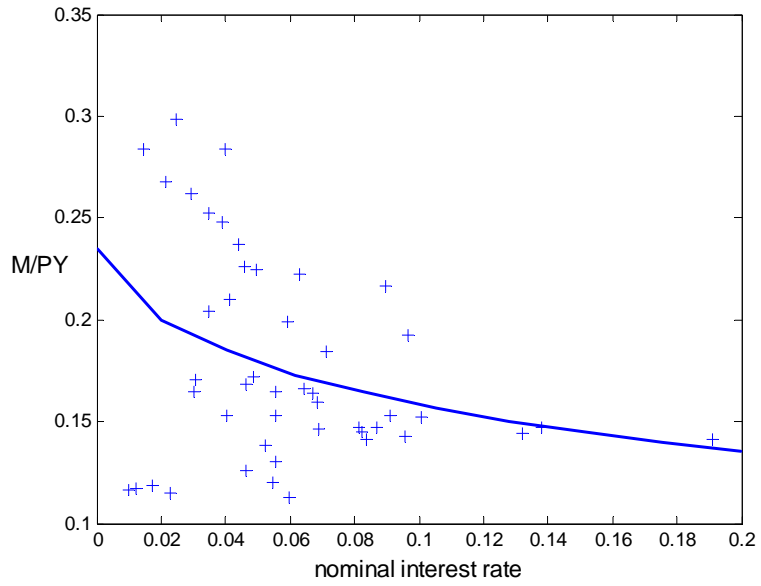


Figure 1:

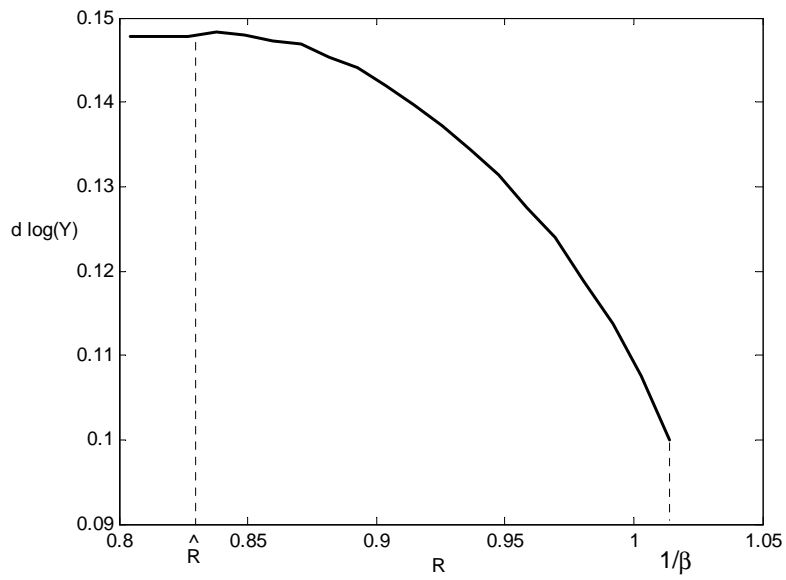


Figure 2: Monetary regimes and aggregate volatility

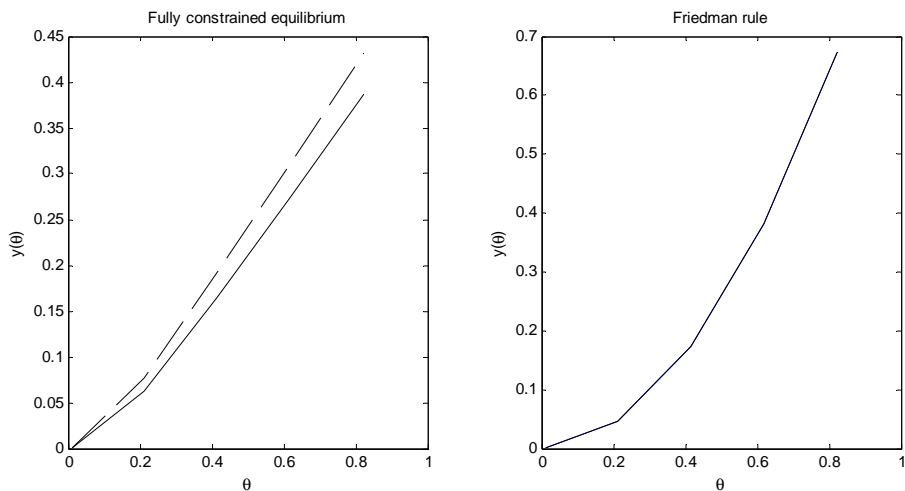


Figure 3: The expected income effect

of the aggregate shock. Therefore, only one line is present in this panel. On the other hand, in the left panel we show output per island under the low aggregate shock (solid line) and under the high aggregate shock (dashed line). As argued in Proposition 4, there is a positive effect of the aggregate shock on the output in each island.

[To be completed.]

5 Conclusions

In this paper we have analyzed how different monetary regimes can affect the response of an economy to aggregate shocks. When liquid assets are scarce, agents are liquidity constrained, the response of the economy tends to be magnified. In particular, when a positive aggregate shock hits the economy, consumers have higher income expectations, they need to build up less precautionary reserves and increase consumption spending. This feeds-back into higher income expectations of other agents and amplifies further the spending response.

Our mechanism is driven by the combination of decentralized trade, risk aversion and idiosyncratic uncertainty. All these three ingredients are necessary. In fact, the amplification effect described would disappear in a representative agent version of the model with no idiosyncratic risk, even if we keep an anonymous setup with decentralized trading.

To keep the model analytically tractable, we have assumed linear preferences in period 3. This allows agents, in the unconstrained regime, to fully insure against negative income

shocks in period 2 by adjusting their period 3 consumption. In a model with concave utility in all periods, this type of adjustment would only be possible if shocks are temporary. In that case, the consumer would be able to smooth a negative income shock in period 2, by lowering consumption by a small amount in all future periods, so as to go back to the initial money balances. Our simplifying assumptions allows agents to adjust in one shot. However, it would be interesting to study a more general environment, and to explore the full dynamics of our mechanism, with different degrees of persistence for the shocks.

Appendix

Proof of Proposition 1

Write the planner maximization problem as

$$\max_{\{c_s\}, \{n\}} \int_0^{\bar{\theta}} \int_0^{\bar{\theta}} \left(u(c_1(\theta, \tilde{\theta})) - v(n(\theta, \tilde{\theta})) + U(c_2(\theta, \tilde{\theta})) + c_3(\theta, \tilde{\theta}) \right) dF(\tilde{\theta})dF(\theta),$$

subject to

$$\begin{aligned} \int_0^{\bar{\theta}} c_1(\tilde{\theta}, \theta) dF(\tilde{\theta}) &\leq \int_0^{\bar{\theta}} \theta n(\theta, \tilde{\theta}) dF(\tilde{\theta}), \text{ for all } \theta \\ \int_0^{\bar{\theta}} \int_0^{\bar{\theta}} c_s(\theta, \tilde{\theta}) dF(\tilde{\theta})dF(\theta) &\leq e_s \text{ for } s = 2, 3. \end{aligned}$$

For each $\theta > 0$ we obtain the first order conditions

$$\begin{aligned} u'(c_1(\tilde{\theta}, \theta))f(\theta) &= \lambda_1(\theta), \\ v'(n(\theta, \tilde{\theta}))f(\theta) &= \theta\lambda_1(\theta), \end{aligned}$$

where $\lambda_1(\theta)$ is the Lagrange multipliers for the resource constraints in island θ . These conditions show that $c_1(\tilde{\theta}, \theta)$ and $n(\theta, \tilde{\theta})$ are independent of $\tilde{\theta}$ so, with a slight abuse of notation, the resource constraint becomes $c_1(\theta) = \theta n(\theta)$. Substituting back in the first order condition shows that $n(\theta)$ must satisfy

$$v'(n(\theta)) = \theta u'(\theta n(\theta)).$$

The first order condition with respect to $c_2(\theta, \tilde{\theta})$ gives $U'(c_2(\theta, \tilde{\theta})) = \lambda_2$, which implies that c_2 is constant across households. The resource constraint requires $c_2(\theta, \tilde{\theta}) = e_2$. Notice that the allocation of $c_3(\theta, \tilde{\theta})$ is indeterminate.

Household Optimization

In this section, we derive the optimality conditions for an individual household in a stationary equilibrium, where prices are equal to $p_1(\theta)$, p_2 and p_3 . Consider a household with initial balances m , then the household Bellman equation can be written as follows

$$V(m) = \max_{\{c_s\}, \{m_s\}, n} \int_0^{\bar{\theta}} \int_0^{\bar{\theta}} \left[u(c_1(\tilde{\theta})) - v(n(\theta)) + U(c_2(\theta, \tilde{\theta})) + c_3(\theta, \tilde{\theta}) + \beta V(Rm_3(\theta, \tilde{\theta})) \right] dF(\theta)dF(\tilde{\theta}),$$

s.t.

$$\begin{aligned} m_1(\tilde{\theta}) + p_1(\tilde{\theta})c_1(\tilde{\theta}) &\leq m, \\ m_2(\theta, \tilde{\theta}) + p_2c_2(\theta, \tilde{\theta}) &\leq m_1(\tilde{\theta}) + p_1(\theta)\theta n(\theta), \\ m_3(\theta, \tilde{\theta}) + p_3c_3(\theta, \tilde{\theta}) &\leq p_3e_3 + m_2(\theta, \tilde{\theta}) + p_2e_2 - T, \\ m_1(\tilde{\theta}) \geq 0, m_2(\theta, \tilde{\theta}) \geq 0, m_3(\theta, \tilde{\theta}) &\geq 0. \end{aligned}$$

From the first order conditions and the envelope condition we get the labor supply equation

$$v'(n(\theta)) = \frac{\theta p_1(\theta)}{p_2} \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta})) dF(\tilde{\theta}) \text{ for all } \theta, \quad (12)$$

and the three Euler equations (with the respective complementary slackness conditions)

$$u'(c_1(\tilde{\theta})) \geq \frac{p_1(\tilde{\theta})}{p_2} \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta})) dF(\theta) \quad (m_1(\tilde{\theta}) \geq 0) \text{ for all } \tilde{\theta}, \quad (13)$$

$$U'(c_2(\theta, \tilde{\theta})) \geq \frac{p_2}{p_3} \quad (m_2(\theta, \tilde{\theta}) \geq 0) \text{ for all } \theta \text{ and } \tilde{\theta}, \quad (14)$$

$$1 \geq \beta R p_3 \int_0^{\bar{\theta}} \frac{u'(c_1(\theta'))}{p_1(\theta')} dF(\theta') \quad (m_3(\theta, \tilde{\theta}) \geq 0) \text{ for all } \theta \text{ and } \tilde{\theta}. \quad (15)$$

Proof of Proposition 2

We proceed by guessing and verifying that there is an unconstrained equilibrium that implements the first-best under the prices (2)-(4). To check optimality, substitute the first-best allocation derived in Proposition 1 in the first-order conditions (12)-(15), under the conjecture that the non-negativity constraints for money holdings are never binding. We obtain

$$v'(n^{FB}(\theta)) = \frac{\theta p_1(\theta)}{p_2} U'(e_2),$$

and

$$u'(\tilde{\theta} n^{FB}(\tilde{\theta})) = \frac{p_1(\tilde{\theta})}{p_2} U'(e_2) \text{ for all } \tilde{\theta},$$

$$U'(e_2) = \frac{p_2}{p_3},$$

$$1 = \beta R p_3 \int_0^{\bar{\theta}} \frac{u'(\theta n^{FB}(\theta))}{p_1(\theta)} dF(\theta).$$

Substituting the prices (2)-(4) and the interest rate $R = 1/\beta$ verifies that these conditions hold. It remains to derive money holdings and check that they are indeed non-negative. Substituting the first-best allocation and the prices (2)-(4) in the consumer's budget constraints we get

$$m_1(\tilde{\theta}) = m - \kappa u'(\tilde{\theta} n^{FB}(\tilde{\theta})) \tilde{\theta} n^{FB}(\tilde{\theta}), \quad (16)$$

$$m_2(\theta, \tilde{\theta}) = m_1(\tilde{\theta}) + \kappa u'(\tilde{\theta} n^{FB}(\tilde{\theta})) \tilde{\theta} n^{FB}(\tilde{\theta}) - \kappa U'(e_2) e_2, \quad (17)$$

$$m_3(\theta, \tilde{\theta}) = m_2(\theta, \tilde{\theta}) + \kappa U'(e_2) e_2 + \kappa(e_3 - c_3(\theta, \tilde{\theta})) - T, \quad (18)$$

where the prices are given by (2)-(4). To have a stationary, degenerate distribution of money balances set $m = M$ and $m_3(\theta, \tilde{\theta}) = M/R$. Then, for each pair $\theta, \tilde{\theta}$ (16)-(18) can be solved to derive $m_1(\tilde{\theta})$, $m_2(\theta, \tilde{\theta})$ and $c_3(\theta, \tilde{\theta})$. With a slight abuse of notation, denote the solutions for money balances as $m_1(\tilde{\theta}; \kappa)$ and $m_2(\theta, \tilde{\theta}; \kappa)$. Then, we have $\lim_{\kappa \rightarrow 0} m_1(\tilde{\theta}; \kappa) = \lim_{\kappa \rightarrow 0} m_2(\theta, \tilde{\theta}; \kappa) = M > 0$. By continuity, we can show that there exists a $\hat{\kappa}$ such that, if $\kappa \leq \hat{\kappa}$ then $m_1(\tilde{\theta}; \kappa)$, $m_2(\theta, \tilde{\theta}; \kappa)$ are non-negative for all pairs $\theta, \tilde{\theta}$. Finally, using the government budget constraint, it is easy to show that the solution for $c_3(\theta, \tilde{\theta})$ is independent of the κ chosen. The assumption that e_3 is large ensures that $c_3(\theta, \tilde{\theta})$ is always non-negative.

Preliminary Results for Proposition 3

In order to prove Proposition 3 it is useful to prove several preliminary lemmas. These results will also be useful to prove Proposition 4.

Consider an island θ , and suppose both producers and consumers in this island believe that nominal spending in all other islands $\tilde{\theta}$ is equal to $w(\tilde{\theta})n(\tilde{\theta}) = x(\tilde{\theta})$, where $x(\cdot)$ is a measurable function

$x : [0, \bar{\theta}] \rightarrow [0, e_2]$. The next two lemmas characterize the equilibrium in island θ , for given beliefs $x(\cdot)$. Then, we will construct a fixed point problem that finds the equilibrium function $x(\cdot)$.

Lemma 1 *Given $\theta > 0$ and a measurable function $x : [0, \bar{\theta}] \rightarrow [0, e_2]$, there exists a unique pair (w, n) which solves the system of equations*

$$\theta u'(\theta n) - w \int_0^{\bar{\theta}} U'(e_2 - wn + x(\tilde{\theta})) dF(\tilde{\theta}) = 0, \quad (19)$$

$$v'(n) - w \int_0^{\bar{\theta}} U'(e_2 - x(\tilde{\theta}) + wn) dF(\tilde{\theta}) = 0. \quad (20)$$

Proof. We will proceed in two steps, first we prove existence, then we show uniqueness.

Step 1. For a given $w \in (0, \infty)$ let $n^D(w)$ denote the n which solves (19) and $n^S(w)$ the n which solves (20). A solution to the system (19)-(20) will be found by looking for a w which solves

$$n^D(w) = n^S(w). \quad (21)$$

It is straightforward to prove that $n^D(w)$ and $n^S(w)$ are continuous functions on $(0, \infty)$. We now prove that they satisfy the two conditions

$$\limsup_{w \rightarrow 0} n^D(w) = \infty, \quad (22)$$

$$\limsup_{w \rightarrow \infty} wn^S(w) = \infty. \quad (23)$$

To prove (22) consider two cases. First, suppose $\limsup_{w \rightarrow 0} wn^D(w) < e_2$. Then there is an $M < e_2$ and an $\epsilon > 0$ such that if $w \in (0, \epsilon)$, then $wn^D(w) < M$. Since U' is decreasing, this implies that

$$\int_0^{\bar{\theta}} U'(e_2 - wn^D(w) + x(\tilde{\theta})) dF(\tilde{\theta}) < U'(e_2 - M).$$

Then

$$\lim_{w \rightarrow 0} w \int_0^{\bar{\theta}} U'(e_2 - wn^D(w) + x(\tilde{\theta})) dF(\tilde{\theta}) = 0,$$

which implies that $\lim_{w \rightarrow 0} \theta n^D(w) = 0$ and, hence, $\lim_{w \rightarrow 0} n^D(w) = \infty$. Second, suppose $\limsup_{w \rightarrow 0} wn^D(w) = e_2$. Then, we immediately have $\limsup_{w \rightarrow 0} n^D(w) = \infty$. To prove (23), suppose, by contradiction, that there is a Z and a \bar{w} such that $wn^S(w) \leq Z$ for all $w \geq \bar{w}$. Then

$$\int_0^{\bar{\theta}} U'(e_2 - x(\tilde{\theta}) + wn^S(w)) dF(\tilde{\theta}) \geq U'(e_2 + Z) > 0,$$

which implies that

$$\lim_{w \rightarrow \infty} w \int_0^{\bar{\theta}} U'(e_2 - x(\tilde{\theta}) + wn^S(w)) dF(\tilde{\theta}) = \infty. \quad (24)$$

Moreover since $0 \leq wn^S(w) \leq Z$ for all $w \geq \bar{w}$, it follows that $\lim_{w \rightarrow \infty} n^S(w) = 0$ and thus

$$\lim_{w \rightarrow \infty} v'(n^S(w)) < \infty. \quad (25)$$

Combining (24) and (25) with equation (20) gives a contradiction. Next, notice that

$$n^S(w) \leq \bar{n} \quad (26)$$

for all $w > 0$. The non-negativity of c_2 implies that

$$e_2 - wn^D(w) + x(\theta) \geq 0$$

for all $\theta > 0$ and all $w > 0$, which implies that

$$wn^D(w) \leq 2e_2. \quad (27)$$

Then from (22) and (26) we get

$$\limsup_{w \rightarrow 0} (n^D(w) - n^S(w)) = \infty$$

and from (23) and (27)

$$\limsup_{w \rightarrow \infty} (wn^S(w) - wn^D(w)) = \infty.$$

It follows that there exist $w_1, w_2 \in (0, \infty)$ such that

$$n^D(w_1) - n^S(w_1) > 0, \quad n^D(w_2) - n^S(w_2) < 0.$$

By the intermediate value theorem there exists a w which solves (21).

Step 2. The implicit function theorem implies that the derivatives of the functions $n^D(w)$ and $n^S(w)$ defined above are

$$\frac{dn^D(w)}{dw} = \frac{\int_0^{\bar{\theta}} U' (e_2 - wn + x(\tilde{\theta})) dF(\tilde{\theta}) - wn \int_0^{\bar{\theta}} U'' (e_2 - wn + x(\tilde{\theta})) dF(\tilde{\theta})}{\theta^2 u''(\theta n) + w^2 \int_0^{\bar{\theta}} U'' (e_2 - wn + x(\tilde{\theta})) dF(\tilde{\theta})},$$

and

$$\frac{dn^S(w)}{dw} = \frac{\int_0^{\bar{\theta}} U' (e_2 - x(\tilde{\theta}) + wn) dF(\tilde{\theta}) + wn \int_0^{\bar{\theta}} U'' (e_2 - x(\tilde{\theta}) + wn) dF(\tilde{\theta})}{v''(n) - w^2 \int_0^{\bar{\theta}} U'' (e_2 - x(\tilde{\theta}) + wn) dF(\tilde{\theta})}.$$

Let (\hat{w}, \hat{n}) be a solution to the system (19)-(20). To show uniqueness, it is sufficient to show that $dn^D(\hat{w})/d\hat{w} < dn^S(\hat{w})/d\hat{w}$. Using the two expressions above, this condition can be rewritten as

$$\begin{aligned} & - \left[\theta^2 u''(\theta n) + w^2 \int_0^{\bar{\theta}} U'' (e_2 - wn + x(\tilde{\theta})) dF(\tilde{\theta}) \right] \int_0^{\bar{\theta}} U' (e_2 - x(\tilde{\theta}) + wn) dF(\tilde{\theta}) + \\ & + v''(n) \left[\int_0^{\bar{\theta}} U' (e_2 - wn + x(\tilde{\theta})) dF(\tilde{\theta}) - wn \int_0^{\bar{\theta}} U'' (e_2 - wn + x(\tilde{\theta})) dF(\tilde{\theta}) \right] + \\ & - \left[\theta^2 u''(\theta n)n + w \int_0^{\bar{\theta}} U' (e_2 - wn + x(\tilde{\theta})) dF(\tilde{\theta}) \right] w \int_0^{\bar{\theta}} U'' (e_2 - x(\tilde{\theta}) + wn) dF(\tilde{\theta}) > 0, \end{aligned}$$

with $w = \hat{w}$ and $n = \hat{n}$. The first two terms of this expression are positive. Using (19) and the assumption that u has elasticity of substitution smaller or equal than 1, we obtain

$$-\hat{n}\theta^2 u''(\theta \hat{n}) \leq \theta u'(\theta \hat{n}) = \hat{w} \int_0^{\bar{\theta}} U' (e_2 - \hat{w}\hat{n} + x(\tilde{\theta})) dF(\tilde{\theta}).$$

This implies that also the last term is positive, completing the argument. ■

Lemma 2 Given a measurable function $x : [0, \bar{\theta}] \rightarrow [0, e_2]$ that satisfies $x(0) = 0$, let $(w(\theta), n(\theta))$ be the unique pair solving the system (19)-(20). Let $y(\theta) = w(\theta)n(\theta)$. The function $y(\theta)$ is monotone increasing on $(0, \bar{\theta}]$.

Proof. Define the two functions

$$\begin{aligned} f_1(y, w; \theta) &\equiv \theta u'(\theta y/w) - w \int U' \left(e_2 - y + x(\tilde{\theta}) \right) dF(\tilde{\theta}), \\ f_2(y, w; \theta) &\equiv v'(y/w) - w \int U' \left(e_2 - x(\tilde{\theta}) + y \right) dF(\tilde{\theta}). \end{aligned}$$

Lemma 1 ensures that there exists a unique pair (w, y) which solves

$$\begin{aligned} f_1(y(\theta), w(\theta); \theta) &= 0, \\ f_2(y(\theta), w(\theta); \theta) &= 0. \end{aligned}$$

Applying the implicit function theorem, one obtains

$$y'(\theta) = \frac{\frac{\partial f_1}{\partial w} \frac{\partial f_2}{\partial \theta} - \frac{\partial f_2}{\partial w} \frac{\partial f_1}{\partial \theta}}{\frac{\partial f_1}{\partial y} \frac{\partial f_2}{\partial w} - \frac{\partial f_1}{\partial w} \frac{\partial f_2}{\partial y}}. \quad (28)$$

To prove the lemma it is sufficient to show that $y'(\theta) > 0$ for all $\theta \in (0, \bar{\theta}]$. Let us use y and w as shorthand for $y(\theta)$ and $w(\theta)$, and notice that $\int U' \left(e_2 - x(\tilde{\theta}) + y \right) dF(\tilde{\theta}) = v'(y/w)/w$ and $\int U' \left(e_2 - y + x(\tilde{\theta}) \right) dF(\tilde{\theta}) = \theta u'(\theta y/w)/w$. Then, the numerator on the right-hand side of (28) can be written as

$$\frac{1}{w} \left(v' \left(\frac{y}{w} \right) + \frac{y}{w} v'' \left(\frac{y}{w} \right) \right) \left(u' \left(\theta \frac{y}{w} \right) + \theta \frac{y}{w} u'' \left(\theta \frac{y}{w} \right) \right),$$

and the denominator as

$$\begin{aligned} & -\frac{1}{w} \left(v' \left(\frac{y}{w} \right) + \frac{y}{w} v'' \left(\frac{y}{w} \right) \right) \left[\frac{\theta^2}{w} u'' \left(\theta \frac{y}{w} \right) + w \int U'' \left(e_2 - x(\tilde{\theta}) + y \right) dF(\tilde{\theta}) \right] + \\ & + \frac{\theta}{w} \left(u' \left(\theta \frac{y}{w} \right) + \theta \frac{y}{w} u'' \left(\theta \frac{y}{w} \right) \right) \left[\frac{1}{w} v'' \left(\frac{y}{w} \right) - w \int U'' \left(e_2 - x(\tilde{\theta}) + y \right) dF(\tilde{\theta}) \right]. \end{aligned} \quad (29)$$

The assumption that u has a coefficient of relative risk aversion less or equal than one, ensures that both numerator and denominator are positive, completing the proof. ■

Let X be the space of measurable, bounded functions $x : [0, \bar{\theta}] \rightarrow [0, e_2]$ that satisfy $x(0) = 0$. Lemma 1 allows us to define the following map from the space X into itself.

Definition 2 For any $x \in X$ and $\theta > 0$, let $(w(\theta), n(\theta))$ be the unique pair solving the system (19)-(20). Define a map $T : X \rightarrow X$ as follows. Set $(Tx)(\theta) = w(\theta)n(\theta)$ for $\theta > 0$ and $(Tx)(0) = 0$.

The following lemmas prove monotonicity and discounting for the map T . These properties will be used to find a fixed point of T . In turns, this fixed point will be used to construct the equilibrium in Proposition 3.

Lemma 3 Take any $x^0, x^1 \in X$, with $x^1(\theta) \geq x^0(\theta)$ almost surely. Then $(Tx^1)(\theta) \geq (Tx^0)(\theta)$ for all θ .

Proof. Define

$$x(\tilde{\theta}, \alpha) \equiv x^0(\tilde{\theta}) + \alpha(x^1(\tilde{\theta}) - x^0(\tilde{\theta})),$$

for each $\tilde{\theta} \in [0, \bar{\theta}]$ and $\alpha \in [0, 1]$. Fix a value for θ and define the two functions

$$\begin{aligned} f_1(y, w; \alpha) &\equiv \theta u'(\theta y/w) - w \int U'(e_2 - y + x(\tilde{\theta}, \alpha)) dF(\tilde{\theta}), \\ f_2(y, w; \alpha) &\equiv v'(y/w) - w \int U'(e_2 - x(\tilde{\theta}, \alpha) + y) dF(\tilde{\theta}). \end{aligned}$$

Applying Lemma 1, for each $\alpha \in [0, 1]$ we can find a unique pair $(y(\alpha), w(\alpha))$ that satisfies

$$\begin{aligned} f_1(y(\alpha), w(\alpha); \alpha) &= 0, \\ f_2(y(\alpha), w(\alpha); \alpha) &= 0. \end{aligned}$$

We are abusing notation in the definition of $f_1(\cdot, \cdot; \alpha)$, $f_2(\cdot, \cdot; \alpha)$, $y(\alpha)$, and $w(\alpha)$, given that the same symbols were used above to define functions of θ . Here we keep θ constant throughout the proof, so no confusion can arise. Applying the implicit function theorem, we obtain

$$y'(\alpha) = \frac{\frac{\partial f_1}{\partial w} \frac{\partial f_2}{\partial \alpha} - \frac{\partial f_2}{\partial w} \frac{\partial f_1}{\partial \alpha}}{\frac{\partial f_1}{\partial y} \frac{\partial f_2}{\partial w} - \frac{\partial f_1}{\partial w} \frac{\partial f_2}{\partial y}}.$$

To prove our statement it is sufficient to show that $y'(\alpha) \geq 0$ for all $\alpha \in [0, 1]$. Let us use y and w as shorthand for $y(\alpha)$ and $w(\alpha)$, and notice that $\int U'(e_2 - x(\tilde{\theta}, \alpha) + y) dF(\tilde{\theta}) = v'(y/w)/w$ and $\int U'(e_2 - y + x(\tilde{\theta}, \alpha)) dF(\tilde{\theta}) = \theta u'(\theta y/w)/w$. Then, the numerator on the right-hand side of (??) can be written as

$$\begin{aligned} &\frac{\theta}{w} \left(u'(\theta \frac{y}{w}) + \theta \frac{y}{w} u''(\theta \frac{y}{w}) \right) w \int U''(e_2 - x(\tilde{\theta}, \alpha) + y) \Delta(\tilde{\theta}) dF(\tilde{\theta}) + \\ &-\frac{1}{w} \left(v'(\frac{y}{w}) + \frac{y}{w} v''(\frac{y}{w}) \right) w \int U''(e_2 - y + x(\tilde{\theta}, \alpha)) \Delta(\tilde{\theta}) dF(\tilde{\theta}), \end{aligned}$$

where $\Delta(\tilde{\theta}) \equiv x^1(\tilde{\theta}) - x^0(\tilde{\theta}) \geq 0$. The denominator takes the same form as in (29). The assumption that u has a coefficient of relative risk aversion less or equal than one, ensures that both numerator and denominator are positive, completing the proof. ■

Before proving the discounting property, it is convenient to restrict the space X to the space \tilde{X} of functions bounded in $[0, \bar{y}]$ where $\bar{y} < e_2$. The following lemma shows that the map T maps \tilde{X} into itself, and that any fixed point of T in X must lie in \tilde{X} .

Lemma 4 *There exists a $\bar{y} < e_2$, such that if $x \in X$ then $(Tx)(\theta) \leq \bar{y}$ for all θ .*

Proof. Set $\bar{x}(0) = 0$ and $\bar{x}(\theta) = e_2$ for all $\theta > 0$. Setting $x(\cdot) = \bar{x}(\cdot)$ and $\theta = \bar{\theta}$, equations (19)-(20) take the form

$$\begin{aligned} \bar{\theta} u'(\bar{\theta} n) &= wF(0)U'(e_2 - wn) + w(1 - F(0))U'(2e_2 - wn), \\ v'(n) &= wF(0)U'(e_2 + wn) + w(1 - F(0))U'(wn). \end{aligned}$$

Let (\hat{w}, \hat{n}) denote the pair solving these equations, and let $\bar{y} = \hat{w}\hat{n}$. Since $F'(0) > 0$ and U satisfies the Inada condition $\lim_{c \rightarrow 0} U'(c) = \infty$, it follows that $\bar{y} < e_2$.

Now take any $x \in X$. Since $x(\theta) \leq \bar{x}(\theta)$ for all θ , Lemma 3 implies that $(Tx)(\theta) \leq (T\bar{x})(\theta)$. Moreover, Lemma 2 implies that $(T\bar{x})(\theta) \leq (T\bar{x})(\bar{\theta}) = \bar{y}$. Combining these inequalities we obtain $(Tx)(\theta) \leq \bar{y}$. ■

Lemma 5 *There exists a $\delta \in (0, 1)$ such that the map T satisfies the following property. Take any $x^0 \in \tilde{X}$ and any $a > 0$, let $x^1(\theta) = x^0(\theta) + a$. Then, the follow inequality holds for all θ*

$$|(Tx^1)(\theta) - (Tx^0)(\theta)| \leq \delta a.$$

Proof. Notice that the pair of functions x^0, x^1 satisfy the assumptions of Lemma 3. Therefore, we can proceed as in the proof of that lemma, with $\Delta(\theta) = a$ for all θ , and, after some algebra, obtain

$$y'(\alpha) = \frac{\left(1 + \frac{\theta n u''(\theta n)}{u'(\theta n)}\right) A + \left(1 + \frac{n v''(n)}{v'(n)}\right) B}{\left(1 + \frac{\theta n u''(\theta n)}{u'(\theta n)}\right) \left(A + \frac{n v''(n)}{v'(n)}\right) + \left(1 + \frac{n v''(n)}{v'(n)}\right) \left(A - \frac{\theta n u''(\theta n)}{u'(\theta n)}\right)} a, \quad (30)$$

where n is shorthand for $y(\alpha)/w(\alpha)$ and

$$A = -\frac{y(\alpha) \int U''(e_2 - x(\tilde{\theta}, \alpha) + y(\alpha)) dF(\tilde{\theta})}{\int U'(e_2 - x(\tilde{\theta}, \alpha) + y(\alpha)) dF(\tilde{\theta})},$$

$$B = -\frac{y(\alpha) \int U''(e_2 - y(\alpha) + x(\tilde{\theta}, \alpha)) dF(\tilde{\theta})}{\int U'(e_2 - y(\alpha) + x(\tilde{\theta}, \alpha)) dF(\tilde{\theta})}.$$

Now, given that $y(\alpha), x(\tilde{\theta}, \alpha) \in [0, \bar{y}]$ and $\bar{y} < e_2$, and given that U is increasing and has continuous second derivative on $(0, \infty)$, it follows that both A and B are bounded above. Let M be an upper bound for both A and B . Recall that $\underline{\eta} > 0$ is a lower bound for $n v''(n)/v'(n)$. Then we set

$$\delta = \frac{1}{1 + \frac{1}{M} \frac{\underline{\eta}}{2 + \underline{\eta}}} < 1.$$

We now proceed to prove the inequality

$$y'(\alpha) \leq \delta a.$$

Given that $1 + \theta n u''(\theta n)/u'(\theta n) \geq 0$, $n v''(n)/v'(n) \geq 0$ and $-\theta n u''(\theta n)/u'(\theta n) \geq 0$, the expression on the right-hand side of (30) is increasing in both A and B . Thus, we have the following inequality

$$y'(\alpha) \leq \frac{\left(1 + \frac{\theta n u''(\theta n)}{u'(\theta n)}\right) M + \left(1 + \frac{n v''(n)}{v'(n)}\right) M}{\left(1 + \frac{\theta n u''(\theta n)}{u'(\theta n)}\right) \left(M + \frac{n v''(n)}{v'(n)}\right) + \left(1 + \frac{n v''(n)}{v'(n)}\right) \left(M - \frac{\theta n u''(\theta n)}{u'(\theta n)}\right)} a.$$

Therefore, we need to prove that

$$1 + \frac{\theta n u''(\theta n)}{u'(\theta n)} + 1 + \frac{n v''(n)}{v'(n)} \leq \delta \left(\left(1 + \frac{\theta n u''(\theta n)}{u'(\theta n)}\right) \left(1 + \frac{n v''(n)}{v'(n) M}\right) + \left(1 + \frac{n v''(n)}{v'(n)}\right) \left(1 - \frac{\theta n u''(\theta n)}{u'(\theta n) M}\right) \right).$$

By definition, $(1 - \delta)/\delta = \underline{\eta}/(M(2 + \underline{\eta}))$. Hence, the last inequality is equivalent to

$$\frac{\frac{n v''(n)}{v'(n) M} - \frac{\theta n u''(\theta n)}{u'(\theta n) M}}{1 + \frac{\theta n u''(\theta n)}{u'(\theta n)} + 1 + \frac{n v''(n)}{v'(n)}} \geq \frac{1}{M} \frac{\underline{\eta}}{2 + \underline{\eta}}.$$

This inequality follows from the facts that: (i) $\theta n u''(\theta n)/u'(\theta n) \in [-1, 0]$, (ii) $n v''(n)/v'(n) \geq \underline{\eta}$, and (iii) $\underline{\eta}/(2 + \underline{\eta})$ is an increasing function for $\underline{\eta} \geq 0$.

Having shown that $y'(\alpha) \leq \delta a$ for all $\alpha \in [0, 1]$, we can integrate both sides and obtain

$$(Tx^1)(\theta) - (Tx^0)(\theta) = y(1) - y(0) \leq \delta a,$$

completing the proof. ■

Proof of Proposition 3

The discussion in the text shows that if there exists an equilibrium with $m_2(\theta, \tilde{\theta}) = 0$ for all θ and $\tilde{\theta}$, then the equilibrium labor supply and the real wage schedules, respectively $n(\theta)$ and $w(\theta)$, must solve the functional equations (7) and (8). To find the pair $(w(\theta), n(\theta))$ we first find a fixed point of the map T defined above (see Definition 2). Lemmas 3 and 5 show that T is a map from a space of bounded functions into itself and satisfies the assumptions of Blackwell's Theorem. Therefore, a fixed point exists and is unique. Let x denote the fixed point, Lemma 1 shows that we can find two functions $w(\theta)$ and $n(\theta)$ that satisfy (19)-(20). Since x is a fixed point of T we have $x(\theta) = w(\theta)n(\theta)$, and substituting in (19)-(20) shows that (7) and (8) are satisfied.

Having found candidate functions $w(\theta)$ and $n(\theta)$, it remains to check that $m_2(\theta, \tilde{\theta}) = 0$ is indeed optimal. To do so, we need to verify that the Euler equation in period 2, (14), is satisfied for any pair $(\theta, \tilde{\theta})$. Notice that Lemma 2 implies that $c_2(\theta, \tilde{\theta})$ is increasing in θ and decreasing in $\tilde{\theta}$, given that $c_2(\theta, \theta) = e_2 - w(\tilde{\theta})n(\tilde{\theta}) + w(\theta)n(\theta)$. Therefore, a necessary and sufficient condition for (14) to hold for all $(\theta, \tilde{\theta})$ is that

$$U'(c_2(\underline{\theta}, \bar{\theta})) \geq \frac{p_2}{p_3}. \quad (31)$$

To check this condition we need to derive the equilibrium value of p_3 . Given that $m_1 > 0$ and $m_3 > 0$, the consumer Euler equations in period 1 and 3, (13) and (15), must be satisfied with equality. Combining them, we obtain:

$$\frac{p_2}{p_3} = \int_0^{\bar{\theta}} \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta})) dF(\theta) dF(\tilde{\theta}), \quad (32)$$

which gives us the equilibrium value of p_3 . Substituting in (31) we get:

$$U'(c_2(\underline{\theta}, \bar{\theta})) \geq \beta R \int_0^{\bar{\theta}} \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta})) dF(\theta) dF(\tilde{\theta}).$$

This requires that $R \leq \hat{R}$ where

$$\hat{R} \equiv \frac{1}{\beta} \frac{U'(c_2(\underline{\theta}, \bar{\theta}))}{\int_0^{\bar{\theta}} \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta})) dF(\theta) dF(\tilde{\theta})}.$$

To derive an upper bound on real taxes in period 3 note that, in a constrained equilibrium, real balances in period 3 are:

$$\frac{M}{p_3} = \frac{p_2}{p_3} \frac{M}{p_2} = \xi$$

where

$$\xi \equiv e_2 \int_0^{\bar{\theta}} \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta})) dF(\theta) dF(\tilde{\theta}).$$

This implies that the map between R and real taxation is the following, and is independent of M ,

$$\tau = \xi(R - 1).$$

This is an increasing function. Moreover, when $R = \hat{R}$ we obtain:

$$\hat{\tau}^C \equiv \frac{e_2}{\beta} \left[U'(c_2(\underline{\theta}, \bar{\theta})) - \beta \int_0^{\bar{\theta}} \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta})) dF(\theta) dF(\tilde{\theta}) \right].$$

Therefore, if $\tau \leq \hat{\tau}^C$, we have an equilibrium with $R \leq \hat{R}$.

Proof of Proposition 4

The proof proceeds in three steps. The first two steps prove that, for each θ , the nominal income in island θ is increasing with the aggregate shock ζ , i.e., $w(\theta; \zeta) n(\theta; \zeta)$ is increasing in ζ . Using this result, the third step shows that $n(\theta; \zeta)$ is increasing in ζ .

Consider two values ζ^I and ζ^{II} , with $\zeta^{II} > \zeta^I$, and suppose the economy is in a fully constrained equilibrium under both ζ^I and ζ^{II} . Denote, respectively, by T_I and T_{II} the maps defined in Definition 2 under the distributions $F(\theta; \zeta_I)$ and $F(\theta; \zeta_{II})$. Let x^I and x^{II} be the fixed points of T_I and T_{II} .

Step 1. Let $x^0 = T_{II}x^I$. In this step, we want to prove that $x^0(\theta) > x^I(\theta)$ for all $\theta > 0$. Fix $\theta > 0$ and define the functions

$$\begin{aligned} f_1(y, w; \zeta) &\equiv \theta u'(\theta y/w) - w \int U'(e_2 - y + x^I(\tilde{\theta})) dF(\tilde{\theta}; \zeta), \\ f_2(y, w; \zeta) &\equiv v'(y/w) - w \int U'(e_2 - x^I(\tilde{\theta}) + y) dF(\tilde{\theta}; \zeta), \end{aligned}$$

for each $\zeta \in [\zeta^I, \zeta^{II}]$. Applying Lemma 1, we can find a unique pair $(y(\zeta), w(\zeta))$ that satisfies

$$\begin{aligned} f_1(y(\zeta), w(\zeta); \zeta) &= 0, \\ f_2(y(\zeta), w(\zeta); \zeta) &= 0. \end{aligned}$$

Once more, we are abusing notation in the definition of $f_1(\cdot, \cdot; \zeta)$, $f_2(\cdot, \cdot; \zeta)$, $y(\zeta)$, and $w(\zeta)$. However, as θ is kept constant, there is no room for confusion. Notice that $x^I(\theta) = y(\zeta^I)$, since x^I is a fixed point of T_I , and $x^0(\theta) = y(\zeta^{II})$, by construction. Therefore, we need to show that

$$y(\zeta^{II}) > y(\zeta^I).$$

Applying the implicit function theorem, we obtain

$$y'(\zeta) = \frac{\frac{\partial f_1}{\partial w} \frac{\partial f_2}{\partial \zeta} - \frac{\partial f_2}{\partial w} \frac{\partial f_1}{\partial \zeta}}{\frac{\partial f_1}{\partial y} \frac{\partial f_2}{\partial w} - \frac{\partial f_1}{\partial w} \frac{\partial f_2}{\partial y}}. \quad (33)$$

To prove our statement it is sufficient to show that $y'(\zeta) > 0$ for all $\zeta \in [\zeta^I, \zeta^{II}]$. First, we show that $\partial f_1 / \partial \zeta > 0$ and $\partial f_2 / \partial \zeta < 0$. Notice that $x^I(\tilde{\theta})$ is monotone increasing in $\tilde{\theta}$, by applying Lemma 2. Then, the concavity of U implies that $U'(e_2 - y + x^I(\tilde{\theta}))$ is decreasing in $\tilde{\theta}$ and $U'(e_2 - x^I(\tilde{\theta}) + y)$ is increasing in $\tilde{\theta}$. By the properties of first-order stochastic dominance, $\int U'(e_2 - y + x^I(\tilde{\theta})) dF(\tilde{\theta}; \zeta)$ is a decreasing function of ζ and $\int U'(e_2 - x^I(\tilde{\theta}) + y) dF(\tilde{\theta}; \zeta)$ is an increasing function of ζ . This immediately implies that $\partial f_1 / \partial \zeta > 0$ and $\partial f_2 / \partial \zeta < 0$. The numerator on the right-hand side of (33) is

$$\frac{\theta}{w} \left(u' \left(\theta \frac{y}{w} \right) + \theta \frac{y}{w} u'' \left(\theta \frac{y}{w} \right) \right) \frac{\partial f_1}{\partial \zeta} - \frac{1}{w} \left(v' \left(\frac{y}{w} \right) + \frac{y}{w} v'' \left(\frac{y}{w} \right) \right) \frac{\partial f_2}{\partial \zeta}.$$

The denominator takes the same form as (29). Once more, the assumption that u has a coefficient of relative risk aversion less or equal than one ensures that both numerator and denominator are positive, completing the argument for this step.

Step 2. Define the sequence of functions (x^0, x^1, \dots) in X , using the recursion

$$x^{j+1} = T_{II}x^j.$$

Since, by step 1, $x^0 \geq x^I$ and T_{II} is a monotone operator (Lemma 3), it follows that this sequence is monotone, with $x^{j+1} \geq x^j$. Moreover, T_{II} is a contraction, so this sequence has a limit point, which

coincides with the fixed point x^{II} . This implies that $x^{II} \geq x^0$ and, together with the result in step 1, shows that $x^{II}(\theta) > x^I(\theta)$ for all $\theta > 0$.

Step 3. Let $n^I(\theta)$ and $n^{II}(\theta)$ denote the equilibrium labor supply functions for ζ^I and ζ^{II} . Fix a θ and define the functions

$$\begin{aligned} g_1(y, n; \alpha) &\equiv \theta n u'(\theta n) - y \int U'(e_2 - y + x(\tilde{\theta}, \alpha)) dF(\tilde{\theta}), \\ g_2(y, n; \alpha) &\equiv n v'(n) - y \int U'(e_2 - x(\tilde{\theta}, \alpha) + y) dF(\tilde{\theta}). \end{aligned}$$

where

$$x(\tilde{\theta}, \alpha) \equiv x^I(\tilde{\theta}) + \alpha \Delta(\tilde{\theta}),$$

where $\Delta(\tilde{\theta}) \equiv x^{II}(\tilde{\theta}) - x^I(\tilde{\theta})$. With the usual abuse of notation, we can define the functions $y(\alpha)$ and $n(\alpha)$ implicitly using

$$\begin{aligned} g_1(y(\alpha), n(\alpha); \alpha) &= 0, \\ g_2(y(\alpha), n(\alpha); \alpha) &= 0. \end{aligned}$$

Applying the implicit function theorem, we get

$$n'(\alpha) = \frac{\frac{\partial g_1}{\partial y} \frac{\partial g_2}{\partial \alpha} - \frac{\partial g_2}{\partial y} \frac{\partial g_1}{\partial \alpha}}{\frac{\partial g_1}{\partial n} \frac{\partial g_2}{\partial y} - \frac{\partial g_1}{\partial y} \frac{\partial g_2}{\partial n}}.$$

We will show that $n'(\alpha) > 0$ by showing that both numerator and denominator of this expression are positive. To evaluate the numerator, notice that

$$\begin{aligned} \frac{\partial g_1}{\partial y} &= - \int U'(e_2 - y + x(\tilde{\theta}, \alpha)) dF(\tilde{\theta}) + y \int U''(e_2 - y + x(\tilde{\theta}, \alpha)) dF(\tilde{\theta}) < 0, \\ \frac{\partial g_2}{\partial y} &= - \int U'(e_2 - x(\tilde{\theta}, \alpha) + y) dF(\tilde{\theta}) - y \int U''(e_2 - x(\tilde{\theta}, \alpha) + y) dF(\tilde{\theta}) \leq \\ &\leq - \int \left[U'(e_2 - x(\tilde{\theta}, \alpha) + y) + (e_2 - x(\tilde{\theta}, \alpha) + y) U''(e_2 - x(\tilde{\theta}, \alpha) + y) \right] dF(\tilde{\theta}) \leq 0, \end{aligned}$$

where the last inequality follows from the assumption that U has a coefficient of relative risk aversion smaller or equal than one (this is the only place where this assumption is used). Furthermore, notice that

$$\begin{aligned} \frac{\partial g_1}{\partial \alpha} &= -y \int U''(e_2 - y + x(\tilde{\theta}, \alpha)) \Delta(\tilde{\theta}) dF(\tilde{\theta}) > 0 \\ \frac{\partial g_2}{\partial \alpha} &= y \int U''(e_2 - x(\tilde{\theta}, \alpha) + y) \Delta(\tilde{\theta}) dF(\tilde{\theta}) < 0. \end{aligned}$$

Putting together the four inequalities just derived shows that the numerator is positive. After some algebra, we can show that the denominator is equal to

$$\begin{aligned} &-\theta (u'(\theta n) + \theta n u''(\theta n)) y \int U''(e_2 - x(\tilde{\theta}, \alpha) + y) dF(\tilde{\theta}) - \frac{n v'(n)}{y} \theta (\theta n u''(\theta n)) + \\ &-y \int U''(e_2 - y + x(\tilde{\theta}, \alpha)) dF(\tilde{\theta}) (v'(n) + n v''(n)) + \frac{\theta n u'(\theta n)}{y} n v''(n) > 0. \end{aligned}$$

This completes the proof.

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