Abstract

We study a dynamic, decentralized market environment with asymmetric information and interdependent values between buyers and sellers, and characterize the complete set of equilibria. The model delivers a stark relationship between the severity of the information frictions and the time it takes for the market to clear, or market liquidity. We use this framework to understand how asymmetric information has contributed to the “frozen” credit market at the core of the current financial crisis, and to characterize optimal policy responses to this market failure.
1 Introduction

A central problem in the current financial crisis has been the inability of financial institutions to sell illiquid assets on their balance sheets. More specifically, banks holding large amounts of structured asset-backed securities, such as collateralized debt obligations and credit default swaps, have been mostly unable to find buyers for these assets. This “frozen” market has posed perhaps the greatest risk to the economy as a whole; if financial institutions can not acquire liquid assets (e.g. cash) in exchange for these illiquid assets, they can not make loans. As a result, consumers have more difficulty buying cars and homes, and businesses cannot acquire the financing they need for new investment. This, in turn, can lead to a further decrease in asset prices and a decline in economic growth. Given the danger associated with this downward spiral, the task of identifying the underlying frictions in this market, and understanding the inefficiencies introduced by these frictions, is of crucial importance. Without such an understanding, market participants remain unsure of how this market will behave in the future, and policymakers remain unsure of the optimal form of intervention.

While one could point to a number of potential reasons that trade in this market has broken down, this paper focuses on asymmetric information. The story is simple: at the onset of the financial crisis, it became apparent that many assets being held by financial institutions were worth considerably less than had been previously claimed; they were of low quality or, in the language of Akerlof (1970), they were “lemons.” Of course, these financial institutions also held assets of higher quality, whose fundamental value (though difficult to discern) was likely at or near pre-crisis evaluations. However, as these assets tend to be relatively complex, it was quite difficult to differentiate high quality from low. Thus the market had many of the basic ingredients of Akerlof’s classic “market for lemons”: sellers possessed assets that were heterogeneous in quality, and they were more informed about the quality of their assets than potential buyers. The most basic theory would predict that, in this type of environment, trade can break down completely.

However, there are several important features of this particular market that are not consistent with the assumptions typically embedded in existing models of markets with
asymmetric information. For one, the market is decentralized; in contrast to the standard competitive paradigm, where the law of one price prevails, buyers and sellers in this market typically negotiate bilaterally. Therefore, a model of this market must allow the exchange of different quality assets to take place at potentially different prices. Moreover, the market is inherently dynamic and non-stationary; any serious analysis has to consider the manner in which the composition of assets in the market evolves over time, and how this affects both prices and the incentive of market participants to delay trade. In contrast to a static, centralized market environment, there are two mechanisms that can adjust to facilitate trade in a dynamic, decentralized market: prices and time.

In this paper, we develop a rigorous economic model that captures these important features of the market discussed above. To be more precise, we consider a discrete time, one-time entry model with a continuum of buyers and sellers. Sellers each possess a single good of heterogeneous quality, and this quality is private information. In each period, buyers and sellers are randomly matched, and buyers make a price offer chosen from an exogenously specified set of prices. If a seller accepts, trade ensues and the pair exits; if the seller rejects, the pair remain in the market and are randomly matched again the following period. Finally, we assume that agents are subject to stochastic discount factor shocks in each period, which we interpret as liquidity shocks across agents and over time.¹

Within this environment, we address a variety of questions that are relevant in the current financial crisis. The first of these questions are positive: Will this market eventually clear and, if so, how long will it take? How does this length of time depend on the initial composition of high- and low-quality assets? What are the welfare costs associated with buyers being imperfectly informed? We will study these questions when the degree of informational asymmetry is exogenous, and when it is endogenous; i.e. when financial institutions can choose the quality of the assets they hold, and when potential investors can choose the degree to which they are uninformed about the quality of an asset before engaging in bilateral negotiations.

Then we will turn to normative questions: Can government intervention increase welfare?

¹This assumption also allows us to focus on pure strategies.
If so, what is the optimal policy? This last point is particularly important in light of the variety of policy responses that have been either proposed or implemented since the financial crisis began. For example, one proposed policy has been for the federal government to buy assets directly from the sellers. An alternative, that has been implemented recently, is to essentially subsidize private-sector buyers to purchase assets from sellers. An important, open question that we intend to address is whether one of these policies implies larger efficiency gains than the other.

1.1 Related Literature

Our work builds on the literature that studies dynamic, decentralized markets with asymmetric information and interdependent values. The primary focus of this literature has been to determine what happens to equilibria in a decentralized environment as market frictions vanish.\(^2\) See Inderst (2005) and Moreno and Wooders (2009) for a steady-state analysis of this issue, and Moreno and Wooders (2002) and Blouin (2003) for analysis of this issue in a one-time entry model. Janssen and Roy (2002) also study a dynamic environment with asymmetric information and interdependent values; however, they assume is assumed takes place in a sequence of Walrasian markets. Though the framework we develop shares certain features in common with several of these papers, the focus will be quite different. We are interested in studying the relationship between information frictions and market liquidity (i.e. how long it takes for markets to clear), and the manner in which both market participants and policymakers can respond to overcome these frictions.

Our paper, and those discussed above, are also closely related to the literature that studies sequential bargaining between a single seller and a single buyer in the presence of asymmetric information.\(^3\) Most relevant to the current project is the work of Vincent (1989), Evans (1989), and Deneckere and Liang (2006), who study the dynamic bargaining game in

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\(^2\)Note that this was an exercise first conducted in a perfect information setting by Rubinstein and Wolinsky (1985) and Gale (1986a, 1986b). A parallel literature has emerged that studies dynamic, decentralized markets with imperfect information and private values; see, for example, Satterthwaite and Shneyerov (2007) and the references therein.

\(^3\)Seminal contributions in this literature include Fudenberg et al. (1985) and Gul et al. (1986), among others.
which a seller has private information about the quality of her good, a buyer makes offers in each period, and the buyer’s valuation of the good is correlated with the seller’s valuation. Equilibria in this environment tend to have the property that buyers use time to screen the different types of sellers: initially buyers will make low offers that only very low type sellers would accept.\textsuperscript{4} If the seller rejects such an offer, the buyer learns that the seller is not a very low type, and updates his posterior accordingly. In the following period, his offer increases, and so on.

This notion of using price dispersion over time to overcome the problem of adverse selection is central to our work, as well as the majority of papers cited above.\textsuperscript{5} What is different about the market setting we consider, as opposed to the single buyer/seller setting considered in much of the bargaining literature, is that complementarities can arise in a market setting between e.g. a buyer and other buyers. For example, in our setting multiple equilibria can arise, precisely because a single buyer will have greater incentive to delay trade if other buyers are doing the same. This type of complementarity between agents on the same side of the market is not present in an environment where there is only one agent on each side of the market; this is why there is generally a unique sequential equilibrium in the environments studied by Vincent (1989), Evans (1989), and Deneckere and Liang (2006).

\section{The Model}

Time is discrete, and begins in period $t = 0$. There is an equal measure of infinitely lived buyers and sellers, which we normalize to one. At $t = 0$, each seller possesses a single, indivisible asset, which is either of high (H) quality or low (L) quality. The fraction of sellers with a high quality asset is $q_0 \in (0, 1)$. An asset of quality $j \in \{L, H\}$ yields flow utility $y_j$ to a seller in each period that he holds the asset. When a buyer purchases the asset, we assume that he receives instantaneous utility $u_j$.\textsuperscript{6}

\textsuperscript{4}A “low type seller” is a seller with a good of very low quality.
\textsuperscript{5}This basic idea goes back to, at least, Wilson (1980).
\textsuperscript{6}As in Duffie et al. (2005), our preference specification is such that buyers and sellers receive different levels of utility from holding a particular asset. This can arise for a multitude or reasons: agents can have different levels of risk aversion, financing costs, regulatory requirements, or hedging needs. In addition, the
The discount factor $\delta$ of an agent in each period is an i.i.d. draw from a continuous and strictly increasing c.d.f. $F$ with support on $[0, \tilde{\delta}]$, where $\tilde{\delta} < 1$. This is meant to capture the idea that buyers and sellers have different liquidity needs at different times. At a given time, some sellers may need to sell their asset urgently, while others may be more patient. Likewise, at a given time some buyers may desire consumption urgently, while others may be more patient. Across time, each individual agent may be more or less patient in any given period.\(^7\) The assumption that $F$ is strictly increasing rules out mass points in the distribution of discount factors.

It will be convenient to denote the present discounted lifetime value of a type $j \in \{L, H\}$ asset to a seller, computed before the seller draws his discount factor, by $c_j$, where

$$c_j = \frac{y_j}{1 - \mathbb{E}[\delta]},$$

with $\mathbb{E}[\delta] = \int \delta dF(\delta) < 1$. We normalize $y_L$ to zero, so that $c_L = 0$, and assume that

$$u_H > y_H + \tilde{\delta}c_H > u_L > 0.$$  \hspace{1cm} (2)

The assumptions that $u_H > y_H + \tilde{\delta}c_H$ and $u_L > 0$ assure us that there are gains from trade in every match (i.e. from both low and high quality goods). The assumption that $y_H + \tilde{\delta}c_H > u_L$ generates the lemons problem, as the price that buyers are willing to pay for a low quality asset would not be accepted by a sufficiently patient high quality seller.

In every period, after the agents draw their discount factors, buyers and sellers are randomly and anonymously matched in pairs. Discount factors and the quality of the seller’s asset are private information. Once matched, the buyer can offer one of two prices, which are fixed exogenously: a high price $p_H \in (y_H + \tilde{\delta}c_H, u_H)$ or a low price $p_L \in (0, u_L)$.\(^8\) The seller can accept or reject. If a seller accepts, trade ensues and the pair exits the market; correlation of endowments with asset returns may differ across agents. The current formulation is a reduced-form representation of such differences; see Duffie et al. (2007), Vayanos and Weill (2008), and Gărleanu (2009).

\(^7\) Note that all types of agents draw their discount factors from the same distribution $F$. Though this is non–essential, we think that it is reasonable. For a deeper look at the use of random discount factors, see Higashi et al. (2009).

\(^8\) Exogenous prices in these types of models have been used extensively; see, for example, Wolinsky (1990) and Blouin and Serrano (2001).
there is no entry by additional buyers and sellers. If a seller rejects, no trade occurs and the pair remains in the market. This ensures that there is always an equal measure of buyers and sellers. We assume that $u_H - p_H > u_L - p_L$, so that a buyer would choose to transact with a type $H$ seller if he could choose. We also assume that

$$y_H + \delta p_H \leq p_H.$$  

(3)

As it turns out, (3) implies sellers accept an offer of $p_H$ regardless of their discount factors.

The history for a buyer is the set of all his past discount factors and (rejected) price offers. However, a buyer has no reason to condition behavior on his past history: this history is private information, discount factors are i.i.d., and the probability that he meets his current trading partner in the future is zero, as there is a continuum of agents. Moreover, since there is no aggregate uncertainty, the buyer’s history of past offers is not helpful in learning any information about the aggregate state. Thus, a pure strategy for a buyer is a sequence $p = \{p_t\}_{t=0}^{\infty}$, with $p_t : [0, \delta] \rightarrow \{p_L, p_H\}$ measurable for all $t \geq 0$, such that $p_t(\delta)$ is the price the buyer offers in period $t$ if he is in the market in this period and his discount factor is $\delta$.

A history for a seller is the set of all his past discount factors and all price offers that he has rejected. The same argument as above implies that a seller has no reason to condition behavior on his past history. Thus, a pure strategy for a type $j$ seller (i.e. a seller with a type $j \in \{L, H\}$ asset) is a sequence $a_j = \{a^j_t\}_{t=0}^{\infty}$, with $a^j_t : [0, \bar{\delta}] \times \{p_L, p_H\} \rightarrow \{\text{accept, reject}\}$ measurable for all $t \geq 0$, such that $a^j_t(\delta, p)$ is the seller’s acceptance decision in period $t$ as a function of his discount factor and the price offer he receives.

We consider symmetric pure–strategy equilibria. A strategy profile can then be described by a list $\sigma = (p, a_L, a_H)$. In order to define equilibria we need to specify what happens when there is a zero measure of agents remaining in the market; more specifically, when all remaining agents trade and exit the market in the current period, we must specify the (expected) payoff to an individual should he choose a strategy that results in not trading. In order to avoid imposing ad hoc assumptions, we adopt the following procedure for computing payoffs.

Suppose that, in every period $t$, the probability an agent gets the opportunity to trade
is $\alpha \in (0, 1]$, and that this probability is independent of his discount factor. Thus, in every period $t$, a fraction $\alpha \in (0, 1]$ of the buyers and sellers in the market are matched in pairs, and the remainder do not get the opportunity to trade. The definition of strategies when $\alpha < 1$ is the same as when $\alpha = 1$.\footnote{Now a history for a player also includes the periods in which he was able to trade; for the same reasons given above, a player has no motive to condition his behavior on this information.} However, when $\alpha \in (0, 1)$, in every period $t$ there is always a positive mass of agents who have not traded.

Let us denote by $V^j_t(a|\sigma, \alpha)$ the expected payoff to a seller of type $j \in \{L, H\}$ who is in the market in period $t$ following strategy $a$, given the strategy profile $\sigma$ for all other agents. The payoff $V^j_t$ is computed before the seller gets the draw for his discount factor and learns whether he can trade or not. For $\alpha \in (0, 1)$, $V^j_t$ is well-defined for all $t \geq 0$, and satisfies the following recursion:

$$V^j_t(a|\sigma, \alpha) = (1 - \alpha) \int \left[ y_j + \delta V^{j}_{t+1}(a|\sigma, \alpha) \right] dF(\delta)$$
$$+ \alpha \sum_{i \in \{L, H\}} \xi_t(p_i) \int \left\{ V^j_t(\delta, p_i) p_i + \left[ 1 - V^j_t(\delta, p_i) \right] \left[ y_j + \delta V^{j}_{t+1}(a|\sigma, \alpha) \right] \right\} dF(\delta),$$

(4)

where $\xi_t(p)$ is the fraction of buyers who offer $p \in \{p_L, p_H\}$ in period $t$ and $I_t(\delta, p)$ is the indicator function of the set $\{\delta : a^j_t(\delta, p) = accept\}$. Note that $\xi_t(p)$ is the probability that a buyer who can trade draws a discount factor $\delta$ with $p_t(\delta) = p$. In words, with probability $1 - \alpha$ a seller is not matched in period $t$, enjoys flow utility $y_j$, and proceeds to period $t + 1$. With probability $\alpha$ the seller is matched, in which case either the seller accepts his partner’s offer and exits the market, or rejects the offer and stays in the market.

Similarly, we denote by $V^b_t(p|\sigma, \alpha)$ the expected payoff to a buyer who is in the market in period $t$ following strategy $p$, given the strategy profile $\sigma$ for all other agents. The payoff $V^b_t$ is also computed before the seller gets the draw for his discount factor and learns whether he can trade or not. Again, for $\alpha \in (0, 1)$, $V^b_t$ is well-defined for all $t \geq 0$, and satisfies the
following recursion:

\[ V_t^b(p|\sigma, \alpha) = (1 - \alpha)\delta V_{t+1}^b(p|\sigma, \alpha) + \alpha \sum_{i \in \{L, H\}} \xi_i(p_i) \left\{ q_t \lambda_t^H(p_i) [u_H - p_i] + (1 - q_t) \lambda_t^L(p_i) [u_L - p_i] \right\} 
+ \left[ 1 - q_t \lambda_t^H(p_i) - (1 - q_t) \lambda_t^L(p_i) \right] \delta V_{t+1}^b(p|\sigma, \alpha) \]

\[ = \delta V_{t+1}^b(p|\sigma, \alpha) + \alpha \sum_{i \in \{L, H\}} \xi_i(p_i) \left\{ q_t \lambda_t^H(p_i) [u_H - p_i - \delta V_{t+1}^b(p|\sigma, \alpha)] + (1 - q_t) \lambda_t^L(p_i) [u_L - p_i - \delta V_{t+1}^b(p|\sigma, \alpha)] \right\}, \]

where \( \lambda_t^j(p) \) is the likelihood that a seller of type \( j \in \{L, H\} \) in the market in period \( t \) accepts an offer \( p \in \{p_L, p_H\} \) and \( q_t \) is the fraction of \( H \) sellers in the market in period \( t \). In words, with probability \( 1 - \alpha \) a buyer is not matched in period \( t \), enjoys no utility, and proceeds to period \( t + 1 \). With probability \( \alpha \) a buyer is matched, in which case his partner either accepts his offer (and the buyer exits the market) or rejects his offer (and the buyer stays in the market).

Standard dynamic programming arguments show that for each \( \sigma, a, p, \) and \( t \geq 0 \), the payoffs \( V_t^j(a|\sigma, \alpha) \) and \( V_t^b(p|\sigma, \alpha) \) are continuous functions of \( \alpha \) in the interval \( (0, 1) \). Hence, the limits of both \( V_t^j(a|\sigma, \alpha) \) and \( V_t^b(p|\sigma, \alpha) \) are well-defined as \( \alpha \) converges to one.

**Definition 1.** Let \( \sigma \) be the strategy profile under play. The payoff to a buyer who is in the market in period \( t \) following the strategy \( p \) is \( V_t^b(p|\sigma) = \lim_{\alpha \to 1} V_t^b(p|\sigma, \alpha) \). The payoff to a seller of type \( j \in \{L, H\} \) who is in the market in period \( t \) following the strategy \( a \) is \( V_t^j(a|\sigma) = \lim_{\alpha \to 1} V_t^j(a|\sigma, \alpha) \).

We can now define equilibria in our environment.

**Definition 2.** The strategy profile \( \sigma^* = (p^* = \{p^*_t\}, a^*_L = \{a^*_t^L\}, a^*_H = \{a^*_t^H\}) \) is an equilibrium if for each \( t \geq 0 \) and \( j \in \{L, H\} \), we have that:

(i) \( p^*_t(\delta) \) maximizes

\[ q_t \lambda_t^H(p) [u_H - p - \delta V_{t+1}^b(\sigma^*)] + (1 - q_t) \lambda_t^L(p) [u_L - p - \delta V_{t+1}^b(\sigma^*)] \]

for all \( \delta \in [0, \bar{\delta}] \), where \( V_t^b(\sigma^*) = V_t^b(p^*|\sigma^*) \);
(ii) For each \( p \in \{ p_L, p_H \} \), \( a_i^*(\delta, p) = \text{accept if, and only if,} \)
\[
p \geq y_j + \delta V_{i+1}^j(\sigma^*),
\]
where \( V_i^j(\sigma^*) = V_i^j(a_j^*|\sigma^*) \).

Note that (6) implies that in equilibrium a seller accepts any offer that he is indifferent between accepting and rejecting. This is without loss since \( F \) is continuous, and so the probability that a seller is ever indifferent between accepting and rejecting is zero.

3 Preliminary Results

For a given strategy profile, we say the market “clears” in period \( t \) if all sellers remaining in the market accept the price offer made by the buyers. In this section we establish that the market clears in finite time in every equilibrium and that as long as the market does not clear, the fraction of type \( H \) sellers in the population increases strictly over time.

We first show that the market clears in period \( t \) if, and only if, all buyers in the market offer \( p_H \). From (4) and (6), we have that for any equilibrium \( \sigma^* \),
\[
V_i^j(\sigma^*) = \sum_{j \in \{L, H\}} \xi_i(p_j) \int \max \{ p_j, y_j + \delta V_{i+1}^j(\sigma^*) \} dF(\delta)
\]
for all \( t \geq 0 \). Given (3), it should be obvious that \( V_i^j(\sigma^*) \leq p_H \) for all \( t \geq 0 \), so that all sellers in the market always accept an offer of \( p_H \). Thus, the market clears in period \( t \) if all buyers offer \( p_H \). Now observe that since a seller has the option of always rejecting any offer he receives, \( V_i^j(\sigma^*) \geq c_j \) for all \( t \geq 0 \). Thus, letting \( \tilde{\delta} = (u_L - y_H)/c_H \), we have
\[
y_H + \delta V_{i+1}^H(\sigma^*) \geq y_H + \delta c_H = u_L > p_L.
\]
Therefore, a type \( H \) seller with discount factor \( \delta \geq \tilde{\delta} \) always rejects an offer of \( p_L \). Since \( \tilde{\delta} < \bar{\delta} \) by (2), there is always a strictly positive mass of such sellers. Hence, the market does not clear in period \( t \) if a positive mass of buyers offers \( p_L \).

So, in any equilibrium \( \sigma^* \) the market clears in the first period in which all remaining buyers offer \( p_H \), which we denote by \( T = T(\sigma^*) \); we set \( T = \infty \) if in every period \( t \) a positive
mass of buyers offers \( p_L \). For all \( t < T \), a positive mass of buyers offer \( p_L \), and the fraction of type \( j \) sellers who accept \( p_L \) in \( t \) is \( F \left[ (p_L - y_j)/V_{t+1}^{j}(\sigma^*) \right] \). Since all sellers who receive an offer of \( p_H \) accept the offer and exit the market, we then have that

\[
q_{t+1} = \frac{q_t \left[ 1 - F \left( \frac{p_L - y_H}{V_{t+1}^H(\sigma^*)} \right) \right]}{q_t \left[ 1 - F \left( \frac{p_L - y_H}{V_{t+1}^H(\sigma^*)} \right) \right] + (1 - q_t) \left[ 1 - F \left( \frac{p_L}{V_{t+1}^L(\sigma^*)} \right) \right]}. \tag{7}
\]

Now notice that an option for a type \( H \) seller is to replicate the behavior of a type \( L \) seller. Since \( y_H > y_L \), we then have that \( V_t^H(\sigma^*) \geq V_t^L(\sigma^*) \) for all \( t \geq 0 \).\(^\text{10}\) Hence, since \( F((p_L - y_H)/V_{t+1}^H(\sigma^*)) \leq F(\tilde{\alpha}) < 1 \) and \( F \) is strictly increasing in its support,

\[
F \left( \frac{p_L - y_H}{V_{t+1}^H(\sigma^*)} \right) < F \left( \frac{p_L}{V_{t+1}^L(\sigma^*)} \right), \tag{8}
\]

for all \( t \geq 0 \); that is, whenever buyers offer \( p_L \), the fraction of type \( L \) sellers who accept this offer is larger than the fraction of type \( H \) sellers who accept the same offer. This is a fundamental feature of this environment: type \( H \) sellers are de facto more patient than type \( L \) sellers because their flow payoff from not trading is larger. Looking at the law of motion for \( \{q_t\}_{t=0}^T \), equation (7), a consequence of this fact is that the fraction of type \( H \) sellers in the population increases strictly over time before the market clears.

**Proposition 1.** Let \( q_0 \in (0, 1) \). In any equilibrium, the market clears in finite time.

The proof of Proposition 1 is in the Appendix. The intuition for this result is as follows. Suppose, by contradiction, that there is an equilibrium in which in every period \( t \) the mass of buyers who offer \( p_L \) is positive. We know the sequence \( \{q_t\}_{t=0}^\infty \) is strictly increasing, and thus convergent (since it is bounded above). The limit \( q_\infty \) of this sequence cannot be one, though. Indeed, the payoff from offering \( p_H \) converges to \( u_H - p_H \) as the fraction of type \( H \) sellers in the market converges to one. Since the highest payoff possible for a buyer is \( u_H - p_H \), and buyers discount the future (\( \tilde{\delta} < 1 \)), there is a fraction \( q^* < 1 \) of type \( H \) sellers in the market above which all buyers find it optimal to offer \( p_H \). However, \( q_\infty < 1 \) implies

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\(^\text{10}\)Indeed, \( y_H > y_L \) implies that \( V_t^H(a_L|\sigma^*, \alpha) > V_t^L(a_L|\sigma^*, \alpha) \) for all \( \alpha \in (0, 1) \) and \( t \geq 0 \). Taking the limit as \( \alpha \) converges to one implies the desired result.
that eventually the fraction of type $H$ sellers who accept an offer of $p_L$ is virtually the same as the fraction of type $L$ sellers who accept the same offer, which is not possible by (8).

In what follows, we will present the case where the lemons problems is most severe by assuming that $p_L < y_H$, so that no type $H$ seller accepts $p_L$. Relaxing this assumption does not substantively change any of our results. We also assume that

$$\delta \leq (u_L - p_L)/(u_H - p_H),$$

(9)

which ensures that the buyer would never prefer to simply not make an offer.

4 Characterizing Equilibria

In this section we provide a complete characterization of the set of equilibria. We start with a characterization of the equilibria in which the market clears immediately. We refer to such equilibria as 0–step equilibria. In general, we refer to equilibria in which the market clears in period $k$, with $k \geq 0$, as $k$–step equilibria. Note that if $\sigma^*$ is an equilibrium, then $V_b^t(\sigma^*) > 0$ for all $t \geq 0$. Indeed, an option for a buyer is to offer $p_L$ as long as there is a positive mass of type $L$ sellers in the market. Since $V_L^t(\sigma^*) \leq p_H$, the probability that a type $L$ seller accepts $p_L$ is at least $F(p_L/p_H)$, in which case the buyer’s payoff is $u_L - p_L > 0$.

0–step equilibria.

Denote by $\pi_b^j(q, \delta, v_L, v_H, v_b)$ the payoff to a buyer who offers $p_j$, with $j \in \{L, H\}$, when: (i) the fraction of type $H$ sellers in the market is $q \in (0, 1)$; (ii) the buyer’s discount factor is $\delta$; (iii) the continuation payoff to a seller of type $j$ who chooses not to trade is $v_j \geq c_j$; and (iv) the continuation payoff to the buyer should he not trade is $v_b \in (0, u_H - p_H]$. We know that sellers always accept an offer of $p_H$. So,

$$\pi_b^H(q, \delta, v_L, v_H, v_b) = \pi_b^H(q) = q[u_H - p_H] + (1 - q)[u_L - p_H].$$

We also know that a type $H$ seller always rejects an offer of $p_L$. So,

$$\pi_b^L(q, \delta, v_L, v_H, v_b) = \pi_b^L(q, \delta, v_L, v_b)$$

$$= (1 - q)F\left(\frac{p_L}{v_L}\right)[u_L - p_L] + \left\{q + (1 - q)\left[1 - F\left(\frac{p_L}{v_L}\right)\right]\right\} \delta v_b,$$

(10)
where $F(p_L/v_L)$ is the fraction of type $L$ sellers who accept $p_L$. Note that $\pi^L_b(q, \delta, v_L, v_b)$ is strictly increasing in $v_b$. Since $v_b > 0$, $\pi^L_b(q, \delta, v_L, v_b)$ is also positive and strictly increasing in $\delta$. Moreover, since $v_b \leq u_H - p_H$, (9) implies that $\delta v_b \leq u_L - p_L$, and so $\pi^L_b(q, \delta, v_L, v_b)$ is non-increasing in $v_L$.

Consider now the candidate 0-step equilibrium $\sigma^0$ where all buyers always offer $p_H$, and in every period $t$ the type $j$ sellers accept an offer $p$ if, and only if, $\delta \leq (p - y_j)/p_H$. It is immediate to see that for all $t \geq 0$,

$$V^b_t(\sigma^0) = v^0_b(q_0) = \pi^H_b(q_0) \quad \text{and} \quad V^j_t(\sigma^0) = v^0_j = p_H.$$ 

Recall that equilibrium payoffs are computed before agents draw their discount factors. Note that we have introduced the following notation: in a 0-step equilibrium, the expected payoff to a buyer given $q_0$ is $v^0_b(q_0)$ and the expected payoff to a type $j$ seller is $v^0_j$.$^{11}$

The strategy profile $\sigma^0$ is an equilibrium only if $v^0_b(q_0) > 0$ (for otherwise $V^b_0(\sigma_0) \leq 0$, which cannot happen in equilibrium) and in every period $t$ all buyers find it optimal to offer $p_H$, which is true as long as

$$\pi^H_b(q_0) \geq \pi^L_b(q, \delta, v^0_L, v^0_b(q_0))$$

for all $\delta \in [0, \overline{\delta}]$. Since $v^0_b(q_0) > 0$ implies that $\pi^L_b(q_0, \delta, v^0_L, v^0_b(q_0))$ is strictly increasing in $\delta$, a necessary and sufficient condition for (11) is that

$$\pi^H_b(q) \geq \pi^L_b(q, \overline{\delta}, v^0_L, v^0_b(q)).$$

Condition (12) is slightly more subtle than it may appear. The left side is clearly the payoff to a buyer from offering $p_H$. The right side is the payoff to a buyer from offering $p_L$ in the current period and $p_H$ in the ensuing period, conditional on all other buyers offering $p_H$ in the current period. There are two things to notice. First, that when all other buyers offer $p_H$ and exit the market, the payoff to a buyer who remains in the market and offers $p_H$ in the next period is $v^0_b(q_0)$. This comes directly from our refinement for computing payoffs

$^{11}$In general, we will adopt the convention that a numerical subscript refers to a particular time period, while a numerical superscript refers to the number of periods it takes for the market to clear in equilibrium.
when the mass of agents in the market is zero. Indeed, under $\sigma^0$, when the fraction of buyers and sellers who are matched in each period is $\alpha < 1$, all buyers who get the opportunity to trade exit the market, and so the fraction of type $H$ sellers among the sellers who remain in the market stays the same. Second, as we show in the proof of Proposition 2 below, (12) is the loosest possible constraint on $q_0$ that ensures that a buyer finds it optimal to offer $p_H$ when he believes that all other buyers in the market offer $p_H$ as well.

**Proposition 2.** Let $q^0 \in (0, 1)$ denote the unique value of $q$ satisfying

$$\pi^L_b (q, \tilde{\sigma}, v^0_L, v^0_b(q)) = \pi^H_b (q).$$

There exists a 0-step equilibrium if, and only if, $q_0 \geq q^0$.

The sketch of the proof of Proposition 2 is as follows; the details are in the Appendix. We first show that (12) is satisfied, and so $\sigma^0$ is an equilibrium if, and only if, $q \geq q^0$. This follows from the fact that the payoff difference $\pi^H_b (q) - \pi^L_b (q, \tilde{\sigma}, v^0_L, v^0_b(q))$ is strictly increasing and continuous in $q$. We then show that there is no 0-step equilibrium if $q_0 < q^0$. Indeed, an option for a buyer is to offer $p_H$ in every period. It is possible to show the buyer’s payoff from doing so is at least $v^0_b(q_0)$ no matter the strategy profile under play. This follows from our refinement for computing payoffs and the fact that the fraction of type $H$ sellers in the market does not decrease over time as long as there is a positive mass of agents in the market. Thus, $V^b_1(\tilde{\sigma}) \geq v^0_b(q_0)$ if $\tilde{\sigma}$ is to be an equilibrium. Since $V^L_1(\tilde{\sigma}) \leq p_H$ and $q_0 < q^0$ implies that $\pi^L_b (q_0, \tilde{\sigma}, v^0_L, v^0_b(q_0)) > \pi^H_b (q_0)$, we then have

$$\pi^L_b (q, \tilde{\sigma}, V^L_1(\tilde{\sigma}), V^H_b(\tilde{\sigma})) \geq \pi^L_b (q, \tilde{\sigma}, v^0_L, v^0_b(q)) > \pi^H_b (q).$$

for all $q_0 < q^0$. Thus, there exists $\delta' < \tilde{\delta}$ such that it can *not* be optimal for a buyer with discount factor in $(\delta', \tilde{\delta}]$ to offer $p_H$ at $t = 0$ if $q_0 < q^0$, so that the market clearing immediately cannot be an equilibrium outcome.

Notice that (12) defines $q^0$ as a function of the distribution $F$ of discount factors. The next result describes how $q^0$ reacts to changes in $F$. 

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Corollary 1. Consider two distributions $F_1$ and $F_2$ for the discount factors that have the same support and let the associated cutoffs be $q^0(F_1)$ and $q^0(F_2)$, respectively. If $F_2$ dominates $F_1$ in the first order stochastic sense, then $q^0(F_1) > q^0(F_2)$.

In words, Corollary 1 implies that as sellers become more impatient (keeping the highest discount factor the same), the cutoff $q^0$ increases and thus the set of 0–step equilibria shrinks. The reason is simple: as sellers become more impatient, buyers have greater incentive to offer $p_L$, since it becomes more likely to be accepted, as opposed to $p_H$, which would clear the market. The highest value of $q^0$ is achieved when $\delta \leq p_L/p_H$, in which case $q^0$ is the unique solution to $u_L - p_L = \delta v_0^0(q)$.

Notice that $q_0u_H + (1 - q_0)u_L \geq p_H > y_H + \delta_s c_H$ for any $q_0$ in the interval $[q^0, 1)$, since a buyer is only willing to offer $p_H$ if his payoff from doing so is non–negative. Hence, $p_H$ corresponds to a market clearing price in a Walrasian equilibrium. Thus, when the lemons problem is relatively small, i.e., when the fraction of type $H$ sellers is sufficiently large, the market behaves as if it were Walrasian.

We will now show that when the lemons problem gets severe, the market outcomes no longer resemble those of a centralized Walrasian market; instead, these markets appear more like standard decentralized search markets, in the sense that it takes time for buyers and sellers to trade, and they do so at potentially different prices. The following convention will be useful. For any strategy profile $\sigma = \{\sigma_t\}_{t=0}^{\infty}$, let $\sigma^+ = \{\sigma_t'\}_{t=0}^{\infty}$ be the strategy profile such that $\sigma_t' = \sigma_{t+1}$ for all $t \geq 0$. Also, for any set $S$, let $1_S$ be the indicator function of $S$.

1–step equilibria.

As the next step, we characterize the set of 1–step equilibria. For this, let

$$q^+(q, v_L) = \frac{q}{q + (1 - q) [1 - F(p_L/v_L)]},$$

with $q \in (0, 1)$. By construction, $q^+(q, v_L)$ is the fraction of type $H$ sellers in the market in the next period if in the current period this fraction is $q$, a positive mass of buyers offer $p_L$, and the continuation payoff to a type $L$ seller in case he rejects a price offer is $v_L$. Since
\( v_L \leq p_H, F(p_L/v_L) \geq F(p_L/p_H) > 0 \), and so \( q^+(q, v_L) > q \) for all \( q \in (0, 1) \). Also notice that \( q^+(q, v_L) \) is strictly increasing in \( q \) if \( p_L/v_L < \bar{\delta} \) and that \( q^+(q, v_L) \equiv 1 \) if \( p_L/v_L \geq \bar{\delta} \).

Consider a strategy profile \( \sigma^1 \) such that a positive mass of buyers offer \( p_L \) in \( t = 0 \) and all buyers offer \( p_H \) in \( t = 1 \). In order for \( \sigma^1 \) to be an equilibrium, it must be that \( \sigma^{1+} \) is a 0–step equilibrium when the initial fraction of type \( H \) sellers is \( q^+(q_0, v^0_L) \) and a positive mass of buyers find it optimal to offer \( p_L \) in \( t = 0 \) when the market clears in \( t = 1 \).\(^{12}\) Hence, the following two conditions are necessary and sufficient for \( \sigma^1 \) to be an equilibrium:

\[
\begin{align*}
q^+(q_0, v^0_L) &\geq q^0; \\
\pi^H_b(q_0) &< \pi^L_b(q_0, \bar{\delta}, v^0_L, v^0_b[q^+(q_0, v^0_L)]).
\end{align*}
\]

Condition (14) follows from Proposition 2. It ensures that the fraction of type \( H \) sellers in \( t = 1 \) is high enough for market clearing in this period to be an equilibrium outcome. Since \( q^+(q_0, v^0_L) \geq q^0 \) implies that \( v^0_b[q^+(q_0, v^0_L)] \geq 0, \pi^L_b(q_0, \bar{\delta}, v^0_L, v^0_b[q^+(q_0, v^0_L)]) \) is strictly increasing in \( \bar{\delta} \). Thus, the incentive of a buyer to offer \( p_L \) in \( t = 0 \) when a positive mass of the other buyers also offer \( p_L \) and the market clears in \( t = 1 \) increases with the buyer’s patience. Condition (15) then ensures that a positive mass of buyers indeed find it optimal to offer \( p_L \) in \( t = 0 \) when the strategy profile under play is \( \sigma^1 \); if it is optimal for the most patient buyer to offer \( p_H \) in \( t = 0 \), then every other buyer prefers to offer \( p_H \) as well.

Let \( \eta^1(q, \bar{\delta}) = \pi^H_b(q) - \pi^L_b(q, \bar{\delta}, v^0_L, v^0_b[q^+(q, v^0_L)]) \). In a 1–step equilibrium, a buyer finds it optimal to offer \( p_H \) in \( t = 0 \) if, and only if, \( \eta^1(q_0, \bar{\delta}) \geq 0 \). The payoff to a buyer in a 1–step equilibrium is

\[
v^1_b(q_0) = \int \max \{\pi^H_b(q_0), \pi^L_b(q_0, v^0_L, v^0_b[q^+(q_0, v^0_L)])\} dF(\bar{\delta}) \geq \pi^H_b(q_0),
\]

while the payoff to a type \( L \) seller is

\[
v^1_L(q_0) = \xi^1(q_0)p_H + (1 - \xi^1(q_0)) \int \max \{p_L, \bar{\delta}v^0_L\} dF(\bar{\delta}) \leq v^0_L,
\]

where \( \xi^1(q_0) \) is the probability that a buyer offers \( p_H \) in \( t = 0 \). Note that

\[
\xi^1(q_0) = \int 1_{\{\bar{\delta}, \eta^1(q_0, \bar{\delta}) \geq 0\}} dF(\bar{\delta}).
\]

\(^{12}\)It must also be the case that the type \( j \) sellers accept an offer of \( p \) in \( t = 0 \) if, and only if, \( \bar{\delta} \leq (p-y_j)/p_H \). This optimal behavior of sellers will be implicitly assumed throughout the analysis.
The proof of the next result is in the Appendix.

**Proposition 3.** There exist $0 \leq q^1 < q^0 < \bar{q}^1 < 1$ such that a 1-step equilibrium exists if, and only if, $q_0 \in [q^1, \bar{q}^1)$. The upper cutoff $\bar{q}^1$ is the unique value of $q$ for which

$$\pi^H_b(q) = \pi^L_b(q; v^0_L, v^0_{b_0}[q^+(q, v^0_L)]).$$

If $p_L/v^0_L \geq \delta$, the lower cutoff $q^1$ is zero. Otherwise, $q^1$ is the unique value of $q$ such that

$$q^+(q, v^0_L) = q^0.$$

In words, if $q_0 = \bar{q}^1$, then the most patient buyer is exactly indifferent between offering $p_L$ and $p_H$ when a positive mass of the other buyers are offering $p_L$; for any $q_0 > \bar{q}^1$ the payoff to such a buyer from immediately trading at price $p_H$ is greater than the payoff from offering $p_L$ and not trading with positive probability, in which case the buyer trades at price $p_H$ in the ensuing period (with a higher fraction of type $H$ sellers in the market). If even the most patient type $L$ seller would rather accept an offer of $p_L$ today than wait one period for an offer of $p_H$, i.e., if $p_L/v^0_L \geq \delta$, then we have $q_1 = 0$. Otherwise, $q^1$ is such that if $q_0 = q^1$ and a positive mass of buyers offer $p_L$, then the fraction of high quality sellers in the next period will be exactly $q^0$, the minimum value of the fraction of type $H$ sellers needed for market clearing. Note that $q^1 > 0$ if $p_L/v^0_L < \delta$.

The fact that $q^0 < \bar{q}^1$ implies that there are multiple equilibria when $q_0 \in [q^0, \bar{q}^1)$. In this region, if all other buyers are offering $p_H$, it is optimal for an individual buyer to offer $p_H$. However, if a positive mass of other buyers are offering $p_L$, the market does not clear at $t = 0$ and the payoff to trading at $t = 1$ increases, rendering it optimal for patient buyers to offer $p_L$ and incur a chance that they trade only in the next period.

Since $q^+(q_0, v^0_L)$ is continuous in $q_0$, it is easy to see that $v^1_b$ is continuous in $q_0$. In the Appendix we show that the probability $\xi^1(q_0)$ that a buyer offers $p_H$ in $t = 0$ in a 1-step equilibrium is continuous and strictly increasing in $q_0$, and it converges to one as $q_0$ increases to $\bar{q}^1$. Thus, $v^1_b$ converges to $v^0_b(\bar{q}^1)$ as $q_0$ increases to $\bar{q}^1$. Moreover, by (16), $v^1_L$ is continuous and strictly increasing in $q_0$, and it converges to $v^0_L$ as $q_0$ increases to $\bar{q}^1$. In what follows, we write $v^1_L(\bar{q}^1)$ to denote the limit of $v^0_L(q_0)$ as $q_0$ increases to $\bar{q}^1$. 17
Corollary 2. The probability $\xi^1(q_0)$ that a buyer offers $p_H$ in $t = 0$ in a 1–step equilibrium is continuous and strictly increasing in $q_0$, and it converges to one as $q_0$ increases to $\bar{q}^1$.

$k$–step equilibria

We are now in position to provide a complete characterization of the $k$–step equilibria for all $k \geq 2$. Recall from Proposition 3 that if $p_L/p_H \geq \delta$, then the only equilibria possible are 0–step and 1–step equilibria. Given this, we assume that $p_L/p_H < \delta$ in what follows. It is helpful to first characterize the 2–step equilibria.

Consider a strategy profile $\sigma^2$ such that a positive mass of buyers offer $p_L$ in the first two periods, and then all buyers offer $p_H$. By construction, the fraction of type $H$ sellers in the market in $t = 1$ is $\hat{q} = q^+ (q_0, V^L_1(\sigma^2)) > q_0$. So, in order for $\sigma^2$ to be an equilibrium, it must be that $\sigma^{2+}$ is a 1–step equilibrium when the initial fraction of type $H$ sellers is $\hat{q}$ and a positive mass of buyers find it optimal to offer $p_L$ in $t = 0$ when behavior from $t = 1$ is given by $\sigma^{2+}$. Hence, the following three conditions are necessary and sufficient for $\sigma^2$ to be an equilibrium:

\begin{align}
\hat{q} &\in [q^1, \bar{q}^1]; \\
\pi^H_b(q_0) &< \pi^L_b(q_0, \delta, v^L_1(\hat{q}), v^1_b(\hat{q})); \\
q^+ (q_0, v^L_1(\hat{q})) &= \hat{q}.
\end{align}

Condition (17) follows from Proposition 3. Since $v^L_b(\hat{q}) > 0$ implies that $\pi^L_b(q_0, \delta, v^1_L(\hat{q}), v^1_b(\hat{q}))$ is strictly increasing in $\delta$, a positive mass of buyers finds it optimal to offer $p_L$ in $t = 0$ only if the most patient buyer prefers to do so. Condition (18) ensures this is the case. Finally, (19) implies that if the type $L$ sellers expect their continuation payoff to be that of a 1–step equilibrium where the initial fraction of type $H$ sellers is $\hat{q}$, then the fraction of type $L$ sellers who accept an offer of $p_L$ in $t = 0$ is such that this conjecture is correct.

We first show that (17) implies (18), so that conditions (17) and (19) completely determine the range of initial values of $q_0$ for which there exists a 2–step equilibrium. Before we start, notice that

$$\left\{ q + (1 - q) \left[ 1 - F \left( \frac{p_L}{v_L} \right) \right] \right\} q^+(q, v_L) = q$$
for all \( q \in (0, 1) \) and \( v_L \geq p_L \). Hence,

\[
\pi_b^L \left( q, \delta, v_L, \pi_b^H \left[ q^+ (q, v_L) \right] \right) = \delta \pi_b^H (q) + (1 - q) F \left( \frac{p_L}{v_L} \right) [u_L - p_L - \delta (u_L - p_H)] \tag{20}
\]

for all \( q \in (0, 1) \) and \( \delta \in [0, \overline{\delta}] \). Equation (20) is useful in what follows.

Suppose \( \hat{q} \in [\underline{q}^1, \overline{q}^1] \). In order to prove that (18) is satisfied, it is sufficient to show

\[
\pi_b^H (\hat{q}) - \pi_b^H (q_0) \geq \pi_b^L \left( \hat{q}, \overline{\delta}, v_L^0, v_0^0 \left[ q^+ (\hat{q}, v_L^0) \right] \right) - \pi_b^L \left( q_0, \overline{\delta}, v_L^1 (\hat{q}), v_1^1 (\hat{q}) \right). \tag{21}
\]

Condition (21) implies that the incentive of the most patient buyer to choose \( p_L \) in \( t = 0 \) is even greater than his incentive to choose \( p_L \) in \( t = 1 \), when the fraction of type \( H \) sellers in the market is \( \hat{q} > q_0 \). First, note that

\[
\pi_b^L \left( \hat{q}, \overline{\delta}, v_L^0, v_0^0 \left[ q^+ (\hat{q}, v_L^0) \right] \right) = \pi_b^L \left( \hat{q}, \overline{\delta}, v_L^0, \pi_b^H \left[ q^+ (\hat{q}, v_L^0) \right] \right) = \delta \pi_b^H (\hat{q}) + (1 - \hat{q}) F \left( \frac{p_L}{v_L^0} \right) [u_L - p_L - \overline{\delta} (u_L - p_H)].
\]

Second, since \( v_1^1 (\hat{q}) \geq \pi_b^H (\hat{q}) \), we have

\[
\pi_b^L \left( q_0, \overline{\delta}, v_L^1 (\hat{q}), v_1^1 (\hat{q}) \right) \geq \pi_b^L \left( q_0, \overline{\delta}, v_L^1 (\hat{q}), \pi_b^H (\hat{q}) \right) = \delta \pi_b^H (q_0) + (1 - q_0) F \left( \frac{p_L}{v_L^1 (\hat{q})} \right) [u_L - p_L - \overline{\delta} (u_L - p_H)];
\]

the second equality follows from (19) and (20). Therefore,

\[
\pi_b^L \left( \hat{q}, \overline{\delta}, v_L^0, v_0^0 \left[ q^+ (\hat{q}, v_L^0) \right] \right) - \pi_b^L \left( q_0, \overline{\delta}, v_L^1 (\hat{q}), v_1^1 (\hat{q}) \right)
\]

\[
\leq \overline{\delta} \left[ \pi_b^H (\hat{q}) - \pi_b^H (q_0) \right] + \left\{ (1 - \hat{q}) F \left( \frac{p_L}{v_L^0} \right) - (1 - q_0) F \left( \frac{p_L}{v_L^1 (\hat{q})} \right) \right\} [u_L - p_L - \overline{\delta} (u_L - p_H)].
\]

Since \( v_1^1 > v_L^1 (\hat{q}) \) for all \( \hat{q} \in [\underline{q}^1, \overline{q}^1] \), \( u_L < p_H \), and \( \hat{q} > q_0 \), the second term on the right–hand side of the above inequality is negative, which confirms (21).

Now define \( \underline{q}^2 \) to be such that: (i) \( \underline{q}^2 = 0 \) if \( \overline{\delta} \leq p_L/v_L^1 (\underline{q}^1) \); (ii) \( q^+ (\underline{q}^2, v_L^1 (\underline{q}^1)) = \underline{q}^1 \) if \( \overline{\delta} > p_L/v_L^1 (\underline{q}^1) \). Since \( \overline{\delta} > p_L/v_L^1 (\underline{q}^1) \) implies that \( q^+ (q, v_L^1 (\underline{q}^1)) \) is strictly increasing in \( q \), we have that \( \underline{q}^2 \) is well–defined and unique. Notice that \( p_L/p_H < \overline{\delta} \) implies that \( \underline{q}^1 > 0 \), so that \( \underline{q}^2 \) is smaller than \( \underline{q}^1 \) whether (i) or (ii) holds. Moreover, let \( \overline{q}^2 \) be such that \( q^+ (\overline{q}^2, v_L^1 (\overline{q}^1)) = \overline{q}^1 \). Since \( q^+ (q, v_L^1 (\overline{q}^1)) = q^+ (q, v_0^0) \) is strictly increasing in \( q \), we have that \( \overline{q}^2 \) is well–defined and unique, and that \( \underline{q}^2 < \overline{q}^1 \). To finish, observe that \( \overline{q}^1 > \underline{q}^0 \) implies that

\[
q^+ (\overline{q}^2, v_L^0) = \overline{q}^1 > \underline{q}^0 = q^+ (\underline{q}^1, v_L^0),
\]
where the last equality follows from the assumption that $p_L/p_H < \delta$. Thus, $q^2 > q^1$.

**Proposition 4.** There exists a 2–step equilibrium if, and only if, $q_0 \in [q^2, q^1)$.

It follows from the proof of Proposition 4 that for each $q_0 \in [q^2, q^1)$, the fraction $q_1$ of type $H$ sellers in the market in $t = 1$ in a 2–step equilibrium starting at $q_0$ is uniquely defined. Denote this fraction by $Q_+(q_0)$ and let $\eta^2(q_0, \delta) = \pi^H_b(q_0) - \pi^L_b(q_0, \delta, v^1_L(Q_+(q_0)), v^1_H(Q_+(q_0)))$. In a 2–step equilibrium, a buyer finds it optimal to offer $p_H$ in $t = 0$ if, and only if, $\eta^2(q_0, \delta) \geq 0$. The payoff to a buyer in a 2–step equilibrium is

$$v^2_b(q_0) = \int \max \{ \pi^H_b(q_0), \pi^L_b(q_0, v^1_L(Q_+(q_0)), v^1_H(Q_+(q_0))) \} dF(\delta) \geq \pi^H_b(q_0),$$

while the payoff to a type $L$ seller is

$$v^2_L(q_0) = \xi^2(q_0)p_H + (1 - \xi^2(q_0)) \int \max \{ p_L, \delta v^1_L(Q_+(q_0)) \} dF(\delta) \leq v^1_L(Q_+(q_0)), \quad (22)$$

where $\xi^2(q_0)$ is the probability that a buyer offers $p_H$ in $t = 0$. Note that

$$\xi^2(q_0) = \int 1_{\{\delta: \eta^2(q_0, \delta) \geq 0\}} dF(\delta).$$

The proof of the next result is in the Appendix.

**Corollary 3.** Suppose that $\delta > p_L/v^1_L(q^1)$. Then: (i) $v^2_b$ is continuous in $q_0$, and it converges to $v^1_b(q^2)$ as $q_0$ increases to $q^2$; (ii) $v^2_L$ is continuous and strictly increasing in $q_0$, and it converges to $v^1_L(q^2)$ as $q_0$ increases to $q^2$.

We can now state and prove a complete characterization of the $k$–step equilibria for all $k \geq 2$.

**Proposition 5.** Suppose that $p_L/p_H < \delta$. There exist $N \geq 2$ and sequences $\{q^k\}_{k=0}^N$ and $\{\theta^k\}_{k=0}^N$, with $\theta^N = 0$, $\theta^0 = 1$, and $q^k < q^{k-1} < q^k$ for all $k \in \{1, \ldots, N\}$, such that a $k$–step equilibrium exists if, and only if, $q_0 \in [q^k, q^k)$. The payoffs for buyers and sellers are uniquely defined in every equilibrium. Let $v^k_b(q_0)$ and $v^k_L(q_0)$ be, respectively, the payoffs for buyers and sellers in a $k$–step equilibrium. Then:

1. $v^0_b(q_0) = q_0[u_H - p_H] + (1 - q_0)[u_L - p_H]$ and $v^0_L(q_0) \equiv p_L$;
2. For all $k \in \{1, \ldots, N-1\}$, $v_b^k$ is continuous in $[q^k, \bar{q}^k)$, and such that $v_b^k$ converges to $v_b^{k-1}(\bar{q}^k)$ as $q_0$ increases to $\bar{q}^k$;

3. For all $k \in \{1, \ldots, N-1\}$, $v_L^k$ is continuous and strictly increasing in $[q^k, \bar{q}^k)$, and such that $v_L^k$ converges to $v_L^{k-1}(\bar{q}^k)$ as $q_0$ increases to $\bar{q}^k$. Moreover, $p_L/v_L^{N-1}(\bar{q}^{N-1}) < \delta$;

4. The upper cutoff $\bar{q}^1$ is the unique value of $q$ for which $\pi_H^0(q) = \pi_L^0(q, v_L^0, v_b^0[q^+(q, v_L^0)])$. For all $k \in \{2, \ldots, N\}$, the cutoffs $\bar{q}^k$ are such that

$$q^+(\bar{q}^k, v_L^{k-1}(\bar{q}^{k-1})) = \bar{q}^{k-1};$$

5. For all $k \in \{1, \ldots, N-1\}$, the cutoffs $\bar{q}^k$ are such that

$$q^+(\bar{q}^k, v_L^{k-1}(\bar{q}^{k-1})) = \bar{q}^{k-1}.$$

The proof is in the appendix.

5 Information Frictions and Liquidity

Given the cutoff $\bar{q}^0$ and the sequence of cutoffs $\{\bar{q}^k, \bar{q}^k\}$ for $k \geq 1$, with $\bar{q}^k < \bar{q}^{k-1} \leq \bar{q}^k$, we have provided a complete characterization of all equilibria for any $q_0 \in (0, 1)$. Though $\bar{q}^1 > \bar{q}^0 > \bar{q}^1$ implies the existence of multiple equilibria for some values of $q_0$, it also implies a natural monotonicity: for any $q_0 \in (0, 1)$, if there exists an equilibrium in which the market clears at time $T$, then for any $q_0' > q_0$ there exists an equilibrium in which the market clears at some time $T' \leq T$. In order to study the relationship between the severity of the information frictions (i.e. the fraction of low quality sellers, $1 - q_0$) and market liquidity (i.e. the amount of time it takes for a market to clear, $T$) we focus on the equilibria that take the smallest number of periods before the market clears.\footnote{This particular selection device is not essential for our comparative statics; given the monotonicity of equilibria, so long as we are consistent with the equilibria we select, our results will remain unchanged. However, this seems the most natural.}

Below we illustrate this relationship.
6 Appendix

6.1 Proof of Proposition 1

Let $\sigma^*$ be an equilibrium and assume, by contradiction, that $T = \infty$. First notice that there is $q^* \in (0, 1)$ such that

$$q^*[u_H - p_H] + (1 - q^*)[u_L - p_H] = \delta[u_H - p_H].$$

Since the highest payoff possible for a buyer is $u_H - p_H$, the definition of $q^*$ implies that if the fraction of type $H$ sellers in the market is above $q^*$, then all buyers offer $p_H$ and the market clears. Thus, for all $t \geq 0$, the fraction $q_t$ of type $H$ sellers in the market in period $t$ is bounded above by $q^*$, and so is the limit $q_\infty$ of the sequence $\{q_t\}$. Now notice that the sequences $\{V_t^L(\sigma^*)\}$ and $\{V_t^H(\sigma^*)\}$ are bounded, and so have convergent subsequences. Dropping subscripts if necessary, we can assume that both sequences converge. Denote their respective limits by $V_\infty^L$ and $V_\infty^H$ and note that $V_\infty^H \geq V_\infty^L$, given that $V_t^H(\sigma^*) \geq V_t^L(\sigma^*)$ for
all $t$. Since the c.d.f. $F$ is continuous, the law of motion (7) for $q_t$ implies that

$$q_\infty = \frac{q_\infty \left[ 1 - F \left( \frac{p_L - y_H}{V_\infty^H} \right) \right]}{q_\infty \left[ 1 - F \left( \frac{p_L - y_H}{V_\infty^H} \right) \right] + (1 - q_\infty) \left[ 1 - F \left( \frac{p_L}{V_\infty^L} \right) \right]}$$

from which we obtain that

$$q_\infty \left[ 1 - F \left( \frac{p_L - y_H}{V_\infty^H} \right) \right] + (1 - q_\infty) \left[ 1 - F \left( \frac{p_L}{V_\infty^L} \right) \right] = 1 - F \left( \frac{p_L - y_H}{V_\infty^H} \right).$$

However, $q_\infty < 1$, and so the last equation implies that $F(p_L/V_\infty^L) = F((p_L - y_H)/V_\infty^H)$, a contradiction since $(p_L - y_H)/V_\infty^H < p_L/V_\infty^L$. Thus, the market must clear in finite time.

### 6.2 Proof of Proposition 2

Simple algebra shows that $\eta^0(q) = \pi_b^H(q) - \pi_b^L(q, \tilde{\sigma}, v_L^0, v_0^0(q))$ is strictly increasing in $q$. Since $\eta^0$ is continuous, $\eta^0(0) < 0$, and $\eta^0(1) > 0$, there is a unique value of $q$ in $(0, 1)$, that we denote by $q^0$, such that $\eta^0(q^0) \geq 0$ if, and only if $q \geq q^0$. Since $v_b^0(q) \leq u_H - p_H$, (9) then implies that $\pi_b^L(q, \tilde{\sigma}, v_L^0, v_0^0(q)) > 0$, and so $v_b^H(q_0) = \pi_b^H(q_0) > 0$. This assures that the strategy profile $\sigma^0$ is an equilibrium if $q_0 \geq q^0$.

As the next step, suppose that $q_0 < q^0$ and consider a candidate 0–step equilibrium $\hat{\sigma}$ with the necessary property that all buyers offer $p_H$ in $t = 0$. One alternative for a buyer is to offer $p_H$ in every period. Let $\hat{\sigma}$ denote this strategy. If $\hat{\sigma}$ is to be an equilibrium, then it must be that $V_t^b(\hat{\sigma}) \geq V_t^b(\hat{\sigma} | \hat{\sigma})$ for all $t \geq 0$. Now observe that when the probability that an agent can trade in a period is $\alpha \in (0, 1)$,

$$V_t^b(\hat{\sigma} | \alpha) = \sum_{\tau=1}^{\infty} \alpha(1 - \alpha)^{\tau-1}(E[\delta])^{\tau-\tau}v_b^0(q_{t+\tau-1}^\alpha),$$

where $q_{t+\tau-1}^\alpha$ is the fraction of type $H$ sellers in the market in period $t + \tau - 1$. It is easy to see that

$$q_{t+1}^\alpha = \frac{q_t^\alpha[1 - \alpha + \alpha \eta_t^H]}{q_t^\alpha[1 - \alpha + \alpha \eta_t^H] + (1 - q_t^\alpha)[1 - \alpha + \alpha \eta_t^L]},$$

where $\xi_t(p_L)$ is the probability that a buyer who gets the opportunity to trade in period $t$ offers $p_L$, $\eta_t^H = \xi_t(p_L)[1 - F((p_L - y_H)/V_{t+1}^H(\hat{\sigma} | \alpha))$, and $\eta_t^L = \xi_t(p_L)[1 - F(p_L/V_{t+1}^L(\hat{\sigma} | \alpha))]$. 


Since \( V^H_{t+1}(\hat{\sigma}|\alpha) > V^L_{t+1}(\hat{\sigma}|\alpha) \) if \( \hat{\sigma} \) is to be an equilibrium (for a type \( H \) seller can imitate a type \( L \) seller and \( y_H > y_L \)), the sequence \( \{q^0_t\}_{t=0}^\infty \) is non-decreasing. Hence,
\[
V^b_t(\hat{p}|\hat{\sigma}, \alpha) \geq \sum_{\tau=1}^{\infty} \alpha (1 - \alpha)^{\tau-1}(\mathbb{E}[\delta])^{\tau-1}v^0_b(q_0),
\]
which implies that \( V^b_t(\hat{p}|\hat{\sigma}) \geq v^0_b(q_0) \). From the main text we know that this last fact implies that a positive mass of buyers do not find it optimal to offer \( p_H \) in \( t = 0 \), so that the market clearing immediately cannot be an equilibrium outcome.

### 6.3 Proof of Corollary 1

By construction, \( \bar{q}^0 \) is the value of \( q \) that satisfies
\[
(1 - q)F\left(\frac{p_L}{p_H}\right)\left[u_L - p_L - \delta v^0_b(q)\right] - (1 - \delta)v^0_b(q) = 0.
\]
It is straightforward to show that an increase in \( F(p_L/p_H) \) leads to an increase in \( \bar{q}^0 \).

### 6.4 Proof of Proposition 3

Recall that \( q^+(q, v^0_L) \) is strictly increasing in \( q \) when \( p_L/v^0_L < \delta \) and that \( q^+(q, v^0_L) \equiv 1 \) otherwise. From this it is immediate to see that there exists \( q^1 < \bar{q}^0 \) such that \( q^+(q_0, v^0_L) \geq q^1 \) if, and only if, \( q_0 \in [q^1, 1) \). Note that \( q^1 = 0 \) if \( p_L/v^0_L \geq \delta \) and \( q^1 \) is such that \( q^+(q^1, v^0_L) = \bar{q}^0 \) otherwise. Now recall that \( \eta^l(q, \delta) = \pi^H_b(q) - \pi^L_b(q, \delta, v^0_L, v^0_b[q^+(q, v^0_L)]) \). Straightforward algebra shows that
\[
\frac{\partial \eta^l}{\partial q}(q, \delta) = F\left(\frac{p_L}{p_H}\right)\left[u_L - p_L - \delta v^0_b[q^+(q, p_H)]\right] + (u_H - u_L)\left[1 - \delta\left[q + (1 - q)\left[1 - F\left(\frac{p_L}{p_H}\right)\right]\right]\right] \frac{\partial q}{\partial q},
\]
from which we can conclude, by (9) and the fact that \( \partial q^+/\partial q \geq 0 \), that \( \partial \eta^l/\partial q > 0 \) regardless of the value of \( p_L/p_H \). Since \( \eta^l(0, \delta) < 0 \) and \( \eta^l(1, \delta) > 0 \), there exists \( \bar{q}^l \in (0, 1) \) such that \( \eta^l(q, \delta) < 0 \) if, and only if, \( q_0 \in [0, \bar{q}^l) \). Hence, \( \pi^H_b(q_0) < \pi^L_b(q_0, \delta, v^0_L, v^0_b[q^+(q_0, v^0_L)]) \) if, and only if \( q_0 \in [0, \bar{q}^l) \). To finish, observe that since \( v^0_b[q^+(q, p_H)] > v^0_b(q) \) for all \( q \in (0, 1) \),
\[
\pi^L_b\left(q_0, \delta, v^0_L, v^0_b[q^+(q^0, p_H)]\right) > \pi^L_b\left(q_0, \delta, v^0_L, v^0_b(q^0)\right) = \pi^H_b(q^0).
\]
Thus, \( \eta^l(q_0, \delta) < 0 \), from which we obtain that \( \bar{q}^l > \bar{q}^0 \).
6.5 Proof of Corollary 2

Since \( \pi_b(q_0, \delta, v_L^1, v_L^0[q^+(q_0, \delta_L)]) \) is strictly increasing in \( \delta \), the function \( \eta^1 \) is strictly decreasing in \( \delta \). Now recall from the proof of Proposition 3 that \( \eta^1 \) is strictly increasing in \( q \). Thus, for each \( q_0 \in [\bar{q}^1, \bar{q}^2] \), there is a unique \( \delta' = \delta'(q_0) \in [0, \bar{\delta}) \) such that \( \eta^1(q_0, \delta') = 0 \). By construction, \( \delta' \) is the cutoff discount factor below which a buyer finds it optimal to offer \( p_H \) in \( t = 0 \). Hence, the probability \( \xi^1(q_0) \) that a buyer offers \( p_H \) in \( t = 0 \) is equal to \( F(\delta'(q_0)) \). Since \( \eta^1 \) is jointly continuous, it is easy to see that \( \delta' \) depends continuously on \( q_0 \). Moreover, the cutoff \( \delta' \) is strictly increasing in \( q_0 \), as \( \eta^1 \) is strictly increasing in \( q \). The desired result follows from the fact that the c.d.f. \( F \) is continuous and strictly increasing and \( \delta'(\bar{q}^1) = \bar{\delta} \) (given that \( \eta^1(\bar{q}^1, \bar{\delta}) = 0 \)).

6.6 Proof of Proposition 4

We need to show that (17) and (19) are satisfied if, and only if, \( q_0 \in [\bar{q}^2, \bar{q}^2] \). Consider first the case where \( \bar{\delta} > p_L/v_L^1(\bar{q}^1) \). The desired result is true if \( q_0 = \bar{q}^2 \) by construction. We claim that for all \( q_0 \in (\bar{q}^2, \bar{q}^2) \), there exists a unique \( \hat{q} \in (\bar{q}^1, \bar{q}^1) \) such that \( q^+(q_0, v_L^1(\hat{q})) = \hat{q} \), so that (17) and (19) are satisfied for all \( q_0 \in (\bar{q}^2, \bar{q}^2) \). For this notice that if \( p_L/v_L < \bar{\delta} \), then \( q^+(q_0, v_L) \) is jointly continuous, strictly increasing in \( q_0 \), and strictly decreasing in \( v_L \). Thus, for all \( \hat{q} \in [\bar{q}^1, \bar{q}^1] \), \( q^+(q_0, v_L^1(\hat{q})) \) is continuous, strictly decreasing in \( \hat{q} \), and strictly increasing in \( q_0 \). Hence,

\[
q^+(q_0, v_L^1(\bar{q}^1)) > q^+(q_2, v_L^1(q_2)) = \bar{q}^1 \quad \text{and} \quad q^+(q_0, v_L^1(\bar{q})) < q^+(\bar{q}^2, v_L^1(\bar{q})) = \bar{q}^1
\]

for all \( q_0 \in (\bar{q}^2, \bar{q}^2) \), which implies the desired result. Finally, notice that for all \( \hat{q} \in [\bar{q}^2, \bar{q}^2] \), \( q_0 = \bar{q}^2 \) implies that \( q^+(q_0, v_L^1(\hat{q})) \geq q^+(\bar{q}^2, v_L^0) = \bar{q}^1 \) and \( q_0 < \bar{q}^2 \) implies that \( q^+(q_0, v_L^1(\hat{q})) < q^+(\bar{q}^2, v_L^1(\bar{q})) = \bar{q}^1 \), and so (17) and (19) cannot be satisfied for \( q_0 \notin [\bar{q}^2, \bar{q}^2] \).

Consider now the case where \( \bar{\delta} \leq p_L/v_L^1(\bar{q}^1) \). First notice that the same argument as above shows that (17) and (19) cannot be satisfied if \( q_0 \geq \bar{q}^2 \). So, we are done if we show that for all \( q_0 \in (0, \bar{q}^2) \) there exists \( \hat{q} \in [\bar{q}^1, \bar{q}^1] \) with \( q^+(q_0, v_L^1(\hat{q})) = \hat{q} \). For this, let \( \underline{q}^1 \) be the unique value of \( q \in (\bar{q}^1, \bar{q}^1) \) for which \( \bar{\delta} = p_L/v_L^1(q) \). Note that \( \underline{q}^1 \) is well-defined since
$p_L/p_H < \delta$ and $v^1_L(q_0)$ is continuous and strictly increasing in $[q^1, \overline{q}]$, with $v^1_L(\overline{q}) = v^0_L$. Now, for each $\tilde{q} \in (\overline{q}, \overline{q}^1)$, let

$$q^-(\tilde{q}) = \frac{\tilde{q} \left[ 1 - F\left( \frac{p_L}{v^1_L(\tilde{q})} \right) \right]}{1 - \tilde{q}F\left( \frac{p_L}{v^1_L(\tilde{q})} \right)}.$$

By construction, $q^+(q^-(\tilde{q}), v^1_L(\tilde{q})) = \tilde{q}$. To finish, notice that $q^-(\tilde{q})$ is continuous, converges to zero as $\tilde{q}$ converges to $\overline{q}$, and converges to $\overline{q}$ as $\tilde{q}$ converges to $\overline{q}^1$.

### 6.7 Proof of Corollary 3

First notice that $Q^1_+$ is continuous and strictly increasing, and that $q^1_0$ converges to $\overline{q}^2$. Indeed, the argument in the proof of Proposition 4 shows that if $\delta > p_L/v^1_L(q^1)$, then $q^+(q_0, v^1_L(Q^1_+(q_0))) = Q^1_+(q_0)$ for all $q_0 \in [q^2, \overline{q}^2]$. Since $q^+(q_0, v_L)$ is jointly continuous in $q_0$ and $v_L$ if $\delta > p_L/v_L$ and $v^1_L(\hat{q})$ is continuous in $\hat{q}$, it is easy to show that $Q^1_+(q_0)$ is continuous in $q_0$, and that its limit as $q_0$ increases to $\overline{q}^2$ is $\overline{q}^1$. Now let $q_0, q'_0 \in [q^2, \overline{q}^2]$ be such that $q'_0 > q_0$ and suppose, towards a contradiction, that $Q^1_+(q_0) \geq Q^1_+(q'_0)$.

Since $v^1_L(\tilde{q})$ is strictly increasing, $v^1_L(Q^1_+(q_0)) \geq v^1_L(Q^1_+(q'_0))$. Moreover, $q^+(q_0, v_L)$ is strictly increasing in $q_0$ and strictly decreasing in $v_L$ if $\delta > p_L/v_L$. Thus,

$$Q^1_+(q_0) = q^+(q_0, v^1_L(Q^1_+(q_0))) \geq q^+(q_0, v^1_L(Q^1_+(q_0))) > q^+(q_0, v^1_L(Q^1_+(q_0))) = Q^1_+(q_0),$$

a contradiction.

The continuity of $v^2_b$ follows from the continuity of $Q^1_+$ and the continuity of $v^1_b$ and $v^1_L$. Since $v^1_L(\overline{q}^1) = v^0_L$ and $v^1_b(\overline{q}^1) = v^0_b(\overline{q}^1) = v^0_b[\eta^+(\overline{q}^2, v^0_L)]$, we then have that

$$\lim_{q_0 \rightarrow \overline{q}^2} v^2_b(q_0) = \int_0^\delta \max\{\pi^H_b(\overline{q}^2), \pi^L_b(\overline{q}^2, v^0_L, v^0_b[\eta^+(\overline{q}^2, v^0_L)])\} dF(\delta) = v^1_L(\overline{q}^2).$$

Now observe that an argument similar to the one used in the proof of Proposition 3 shows that for each $q_0 \in [q^2, \overline{q}^2]$ there is a unique $\delta' = \delta'(q_0) \in [0, \overline{\delta})$, depending continuously on $q_0$, such that $\eta^2(q_0, \delta) \geq 0$ if, and only if $\delta \leq \delta'(q_0)$. Thus, $\xi^2(q_0) = F(\delta'(q_0))$ is continuous and strictly increasing in $q_0$, from which we obtain that $v^2_L$ is continuous and strictly increasing.
in $q_0$. To finish, notice that

$$\lim_{q_0 \to \bar{q}^2} \eta^2(q_0, \delta) = \pi^H_b(\bar{q}^2) - \pi^L_b(\bar{q}^2, \delta, v^1_L(\bar{q}^1), v^1_L(\bar{q}^1))$$

$$= \pi^H_b(\bar{q}^2) - \pi^L_b(\bar{q}^2, \delta, v^1_L, v^0_b[q^+(\bar{q}^2, v^0_b)]) = \eta^1(\bar{q}^1, \delta),$$

so that $\lim_{q_0 \to \bar{q}^2} \xi^2(\bar{q}^2) = \xi^1(\bar{q}^2)$, from which we can conclude that

$$\lim_{q_0 \to \bar{q}^2} v^2_L(q_0) = \xi^1(\bar{q}^2) p_H + (1 - \xi^1(\bar{q}^2)) \int_0^{\bar{q}} \max \{ p_L, \delta v^0_L \} dF(\delta) = v^1_L(\bar{q}^1).$$

### 6.8 Sketch of Proof of Proposition 5

The proof is by induction in the number of steps it takes for the market to clear. We know from Propositions 2 to 4 and Corollary 2 that if $K = 2$, then:

A. There exist \{q^k\}_{k=0}^K and \{\bar{q}^k\}_{k=0}^K, with $q^K \geq 0$, $q^0 = 1$, and $q^k < q^{k-1} < \bar{q}^k$ for all $k \in \{1, \ldots, K\}$, such that a $k$-step equilibrium, with $k \in \{0, \ldots, K\}$, exists if, and only if, $q_0 \in [q^k, \bar{q}^k]$;

B. For each $k \in \{0, \ldots, K\}$, the payoffs $v^k_L(q_0)$ and $v^k_L(q_0)$ satisfy 1, 2, and 3 in the statement of Proposition 5. Moreover, $p_L/v^L_K-1(\bar{q}^{K-1}) < \bar{\delta}$;

C. The cutoffs $\bar{q}^1$ to $\bar{q}^K$ satisfy condition 4 in the statement of Proposition 5;

D. The cutoffs $\underline{q}^1$ to $\underline{q}^{K-1}$ satisfy condition 5 in the statement of Proposition 5.

We also know that if $p_L/v^L_K(q^{K-1}) \geq \bar{\delta}$, then $\underline{q}^K = 0$, in which case $N = K$ and the Proposition is proved. Let then $p_L/v^L_K(q^1) < \bar{\delta}$ and suppose, by induction, that A to D are satisfied for some $K = K' \geq 2$. If $\underline{q}^{K'} = 0$, then the Proposition is true with $N = K'$.

Suppose then that $\underline{q}^{K'} > 0$, which is only possible if $p_L/v^L_{K'}(\underline{q}^{K'-1}) < \bar{\delta}$. We are done if we show that A to D are satisfied when $K = K' + 1$ and that $\underline{q}^{K'+1} = 0$ if $p_L/v^L_{K'}(\underline{q}^{K'}) \geq \bar{\delta}$.

Let $\underline{q}^{K'+1}$ be such that $\underline{q}^{K'+1} = 0$ if $p_L/v^L_{K'}(\underline{q}^{K'}) \geq \bar{\delta}$ and $q^+(\underline{q}^{K'+1}, v^L_{K'}(\underline{q}^{K'})) = \underline{q}^{K'}$ otherwise. Moreover, let $\bar{q}^{K'+1}$ be such that $q^+(\bar{q}^{K'+1}, v^L_{K'}(\bar{q}^{K'})) = \bar{q}^{K'}$. The same argument used in the paragraph leading to Proposition 4 to show that $\underline{q}^2$ and $\bar{q}^2$ are well-defined, with $\underline{q}^2 < \underline{q}^1$ and $\bar{q}^2 < \bar{q}^1$, also shows that $\underline{q}^{K+1}$ and $\bar{q}^{K+1}$ are well-defined, with $\underline{q}^{K+1} < \underline{q}^{K'}$ and $\bar{q}^{K+1} < \bar{q}^{K'}$. Since B implies that $v^L_{K'}(\bar{q}^{K'}) = v^L_{K'-1}(\bar{q}^{K'})$ and $v^L_{K'-1}(\underline{q}^{K'}) < v^L_{K'-1}(\underline{q}^{K'})$,
A implies that $\bar{q}^{K'} > q^{K' - 1}$, we also have that

$$\bar{q}^{K'} = q^+(\bar{q}^{K' + 1}, v^{K' - 1}_L(\bar{q}^{K'})) > q^{K' - 1} = q^+(q^{K'}, v^{K' - 1}_L(q^{K'})) > q^+(q^{K'}, v^{K' - 1}_L(\bar{q}^{K'})),$$

and so $\bar{q}^{K' + 1} > q^{K'}$ as well. An argument similar to the one used to prove Proposition 4 shows that there exists a $(K' + 1)$-step equilibrium if, and only if, $q_0 \in [q^{K' + 1}, \bar{q}^{K' + 1})$.

Thus, $A$ is satisfied when $K = K' + 1$. Likewise, an argument similar to the one used to prove Corollary 3 shows that $B$ is also satisfied when $K = K' + 1$; notice that $v^{K'}_L(\bar{q}^{K'}) = v^{K' - 1}_L(\bar{q}^{K'}) > v^{K' - 1}_L(q^{K'} - 1)$, and so $p^L/v^{K'}_L(\bar{q}^{K'}) < \delta$. Since, by construction, $C$ and $D$ are satisfied when $K' = K + 1$, the desired result follows.

### 6.9 Omitted Details from the Proof of Proposition 5

***We need to prove that $q^+(q_0, v^K_L) \in [q^K, \bar{q}^K]$ implies that

$$\pi^H_b(q_0) \leq \pi^L_b(q_0, v^K_L, v^K_b[q^+(q_0, v^K_L)]).$$

(23)

For this, let $q' = q^+(q_0, v^K_L)$ and $q'' = q^+(q', v^K_L)$. First note that

$$\pi^L_b(q_0, v^K_L, v^K_b(q'))$$

$$= (1 - q_0)F\left(\frac{p^L}{v^K_L}\right)[u_L - p_L] + \delta_b\left\{q_0 + (1 - q_0)\left[1 - F\left(\frac{p^L}{v^K_L}\right)\right]\right\}v^K_b(q')$$

$$= (1 - q_0)F\left(\frac{p^L}{v^K_L}\right)[u_L - p_L] + \delta_b\left\{q_0 + (1 - q_0)\left[1 - F\left(\frac{p^L}{v^K_L}\right)\right]\right\}v^K_b(q')$$

$$+ \delta_b\left\{q_0 + (1 - q_0)\left[1 - F\left(\frac{p^L}{v^K_L}\right)\right]\right\}[v^K_b(q') - v^K_0(q')]$$

$$= \delta_bv^K_0(q_0) + (1 - q_0)F\left(\frac{p^L}{v^K_L}\right)[u_L - p_L - \delta_b(u_L - p_H)]$$

$$+ \delta_b\left\{q_0 + (1 - q_0)\left[1 - F\left(\frac{p^L}{v^K_L}\right)\right]\right\}[v^K_b(q') - v^K_0(q')]$$

where the last equality follows from (20). Similarly, one can show that

$$\pi^L_b(q', v^{K' - 1}_L, v^{K' - 1}_b(q''))$$

$$= \delta_bv^{K' - 1}_0(q') + (1 - q')F\left(\frac{p^L}{v^{K' - 1}_L}\right)[u_L - p_L - \delta_b(u_L - p_H)]$$

$$+ \delta_b\left\{q' + (1 - q')\left[1 - F\left(\frac{p^L}{v^{K' - 1}_L}\right)\right]\right\}[v^{K' - 1}_b(q'') - v^{K' - 1}_b(q'')].$$

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Now observe, by (??), that

\[
\pi_b^L (q_0, v_L^K, v_b^K(q')) \geq \delta_b v_b^0(q_0) + (1 - q_0)F \left( \frac{p_L}{v_K^L} \right) \left[ u_L - p_L - \delta_b (u_L - p_H) \right] \\
+ \delta_b \left\{ q_0 + (1 - q_0) \left[ 1 - F \left( \frac{p_L}{v_K^L} \right) \right] \right\} [v_b^{K-1}(q'') - v_b^0(q')] .
\]

Hence,

\[
\pi_b^L (q', v_L^{K-1}, v_b^{K-1}(q'')) - \pi_b^L (q_0, v_L^K, v_b^K(q')) \leq \delta_b [v_b^0(q') - v_b^0(q_0)] \\
+ \left\{ u_L - p_L - \delta_b (u_L - p_H - [v_b^{K-1}(q'') - v_b^0(q'')]) \right\} \left\{ (1 - q')F \left( \frac{p_L}{v_L^{K-1}} \right) \\
- (1 - q_0)F \left( \frac{p_L}{v_K^L} \right) \right\} . \tag{24}
\]

Now observe that \( q' > q_0, v_L^K < v_L^{K-1} \), and

\[
(1 - q')F \left( \frac{p_L}{v_L^{K-1}} \right) - (1 - q_0)F \left( \frac{p_L}{v_K^L} \right) < 0.
\]

In addition, \( u_L < p_H \) and \( u_L - p_L \geq \delta_b (u_H - p_H) > \delta_b [v_b^{K-1}(q'') - v_b^0(q'')] \). Therefore, (24) implies that

\[
\pi_b^L (q', v_L^{K-1}, v_b^{K-1}(q'')) - \pi_B (q_0, v_L^K, v_b^K(q')) < v_b^0(q') - v_b^0(q_0) \\
= \pi_b^H (q') - \pi_b^H (q_0).
\]

Since \( \pi_b^L (q', v_L^{K-1}, v_b^{K-1}(q'')) \geq \pi_b^H (q') \) by the induction hypothesis, we can then conclude that (23) is satisfied. \(\blacksquare\)
References


