ACQUIRING AND AGGREGATING COSTLY INFORMATION FROM SOURCES OF DIFFERING QUALITY*

ABSTRACT. Useful information is often widely dispersed, costly to produce, and more reliable if gathered from multiple sources. We investigate the problem of selecting the optimal set of sources when sources know more than we do about their own costs and the reliability of their own information. We design a system of rewards in which lower cost, higher precision sources self-select to produce the best reports they can, and to "put their money where their mouth is" by placing bets on the accuracy of their reports. The size of the bets is positively related to the sources' (perceptions of their own) accuracy, and this allows us to weight the reports in a statistically efficient fashion.

1. INTRODUCTION

Aggregating dispersed information can produce accurate predictions. Accurate predictions can be extremely useful for decision making. When organizations become decentralized or information is otherwise highly dispersed, elicitation and aggregation may be difficult. We study the possibility of designing efficient information elicitation and aggregation mechanisms, both for information dispersed within an organization, and for information dispersed outside the boundaries of an organization.

There is commercially useful information that is essentially free. We do not study it. Rather, we study information that people need to spend resources, measured in money or time, to acquire and assimilate. We assume that the costs and reliability of information are both variable and unobserveable.

Our analysis has four parts: (1) recruiting those sources that have the optimal combination of costs and information quality; (2) compensating the recruited sources for acquiring and processing their information; (3) providing the recruited sources with incentives to truthfully reveal not only their information but also its quality; and finally, (4) estimating the quality of our forecast. The first three parts of the analysis are, to varying degrees, novel. The fourth part leans heavily on normal noise and quadratic loss functions.

(1) One of our major objectives is to recruit those agents with the appropriate quality of information and costs, thereby endogenizing participation in a setting where universal participation is not desirable. By contrast, in most mechanism design applications with multiple agents, the participation of agents is assumed to be desirable so that agents' participation decisions can be addressed by a single, usually simple, individual rationality constraint.

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(2) Our second objective is to endogenize the acquisition of knowledge in a setting where this acquisition is costly. By contrast, most of the previous studies of information elicitation assume that agents' information is provided to them exogeneously.[1]

(3) We treat the problem of recruiting sources and then providing them with incentives to truthfully reveal both their information and its quality. In solving this two-dimensional elicitation problem, we provide conditions under which there is a positive answer to Chen and Plott's (2002) question about information aggregation mechanisms,

> Can (they) not only produce a prediction but also simultaneously help management ascertain which participants have information? That is, can (they) be designed to attract those with good information and discourage those with bad information?

By contrast, in the extensive work on probabilistic elicitation, we have found very little that explicitly considers the problem of evaluating the quality of elicited information.[2]

(4) With the quality of information differs across sources, it becomes more important to provide an estimate of the quality of the elicited information. To this purpose, we use elicited forecast precisions as heteroskedasticity weights to provide both a forecast and an estimate of the precision of the resulting forecast. The use of precisions and the classical weights is formally justified by the assumption of additive, Gaussian noise and a quadratic loss function.

1.1. **Eliciting information and its accuracy.** Much research on information elicitation focuses on "proper scoring rules." These are reward functions that elicit a single agent's true probability forecast of a (future) event.[3] A proper scoring rule that works for any

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[1] A notable exception is Osband (1989), who studied the problem of finding the single person/source with the lowest cost per replicable, stochastically independent observation. Once that person/source was identified, the remaining problem is motivating them to make the correct number of independent observations.

[2] For example, Friedman (1983) discusses methods of elicitation that are flawed because they motivate the forecaster to report densities that overweight the mode of their forecast distribution and thereby "understate his perceived uncertainty (p. 452)." The distinction is between mis-reporting some measure of the diffuseness of a forecast density in the process of reporting a probability density, Friedman's worry, and reporting the quality of the information on which the best forecast is based, our worry. Put another way, we are more interested in the true forecast of someone knowledgeable than in the true forecast of someone ignorant. The literature makes no distinction between knowledgeable and ignorant sources.

[3] Brier (1950) seems to have been the first. McCarthy (1956) both characterized proper scoring rules and coined the term. Hendrickson and Buehler (1971) extended McCarthy's logic to density functions on infinite sets of outcomes. Savage (1971) synthesized and extended the previous literature. Kadane and Winkler (1988) point out the strength of the implicit "no stakes" condition needed for elicitation, a topic we will return to when evaluating the strength and applicability of the assumptions we make.

expected utility maximizing individual can be used on all the individuals in any group. More difficult is the problem defining and eliciting the quality of the information.

Building on the work of Schmeidler (1989) and Gilboa and Schmeidler (1989), it is now common to model low quality information about probability distributions, thought of as ambiguity, as a set of distributions. There are many approaches to decision making with low quality information. From the elicitation point of view, what matters is Chambers' (2005) result that there exists no proper scoring rule for any of these approaches.

Circumventing this problem, the forecasting and signal processing literature has long measured the quality of information as the *ex ante* variance of its error. When there is a long enough history of forecasts, one can estimate the variance of the errors and perform the standard heteroskedasticity corrections (*viz.* any good linear regression textbook). As Clemen's (1989) survey and Winkler and Clemen's (1992) analysis indicates, this is generally helpful so long as one avoids using the estimated correlation of the errors in determining the weights to be given to the different forecasts.

We study situations in which there is not a long enough history to estimate variances. As a result, we must elicit not only our sources' forecasts, but also their knowledge of its quality. This is complicated by needing to motivate the right subset of the potential sources to incur costs to acquire their information.

1.2. **The costs and quality of information.** People have different qualities of information and different costs of processing said information. These differences arise from different experience, background, access to, and competence with, information. Further, these differences are themselves variable across time and topics, meaning that who is better informed varies. Our first design problem is identifying and motivating those with the right costs and precisions.

Our recruitment contract balances two regularities. First, people who know that their information will be more valuable are willing to spend more resources to acquire it. Second, the more precise is a person's information, the more they are willing to bet on it, that is, they are willing to put their money where their mouth is.

We combine these regularities, offering a recruitment contract with higher expected payoffs to higher precision sources. The higher expected payoffs arise (mostly) from the larger bets that a higher precision source is willing to make. Novel to our approach is the elicitation of the quality of information. We then weight the sources' opinions as a function of the revealed precision.

1.3. **Aggregating probabilistic information.** One can aggregate probabilistic information that is spread across a group for two, tightly related purposes: making a decision that affects the entire group; and for producing an accurate forecast.

For group decision purposes in the presence of the same utility function but different personalistic probabilities, Savage (1972, 1954, Ch. 10) proposes a group minimax rule with admissibility/Pareto tie-breaking if necessary. For the same purposes, Stone (1961) named and studied optimality properties of *linear opinion pools*, convex combinations of personalistic probabilities, an idea that Bacharach (1979) traces to Laplace's work in the early 19'th century. A variant of this is the *logarithmic opinion pool*, where logarithms of individuals' probabilities are weighted, then exponentiated. From Savage (1971), weights should be assigned to sources according to their past experience or, in some other fashion, to "give each the weight you think appropriate."

In their survey to aggregating probalistic information for producing forecasts, Clemen and Winkler (1999) separate mathematical and behavioral approaches to aggregation methods. The mathematical approach axiomatizes or otherwise examines the optimality properties of various intuitively appealing procedures. Achieving useful group consensus is more important in the design of behavior methodologies. In general, Clemen and Winkler find that multiple sources improve forecasts and make consensus more difficult to achieve.

1.4. **Market mechanisms.** Market mechanisms also provide a method of "putting your money where your mouth is." The idea of prices carrying information goes back to von Hayek's (1945) response to the arguments that a planner can substitute for the market by using anything equivalent to marginal rates of substitution between goods.[4] For Hayek, the crucial aspect of an economic order is that the use of "knowledge of the circumstances of which we must make use never exists in concentrated or integrated form, but solely as the dispersed bits of incomplete and frequently contradictory knowledge which all of the separate individuals possess." From this starting point, he argued (*ibid.*, p. 526) that, "We must look at the price system as such a mechanism for communicating information if we want to understand its real function — ".

When the goods being priced by a market are securities that payoff only when a specific event happens, a *prediction market*, the information being communicated contains information about others' beliefs and the strength of those beliefs. When the markets are open for several periods, participants iteratively process this information about others' beliefs. As first noted by DeGroot (1974) and Aumann (1976), and then in a variety of fashions by Bacharach (1979), Geanakoplos and Polemarchakis (1982), McKelvey and Page (1986), interative updating procedures aggregate peoples' sampling information into a coherent posterior.[5] These logics underlies the electronic future markets for predicting events that

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[4] The arguments were variously given by Leon Trotsky, Oskar Lange, and Abba Lerner.

[5] Based on this type of logic, Hanson (2003) designs a prediction mechanism intermediate between a market and a proper scoring rule.

have recently proliferated and have generated a great deal of public and academic interest, and the internal prediction markets designed by Chen and Plott (2002).

There is a body of research emphasizing the prediction abilities of futures markets.[6] A more recent theoretical analysis by Manski (2006) shows that, in general, the equilibrium price of a prediction-market contract can be a quantile of a budget-weighted distribution of traders beliefs, that prices generally only yield a bound on the mean belief help by traders, and that it may be that the equilibrium price is immune to belief revision. In a setting where equilibrium prices may be fully revealing, Serrano-Padial (2006) shows that a low proportion of the market being noise/liquidity traders is compatible with revealing prices, while moderate levels are not. Despite these advances, it is not fully understood how reliable such markets are, nor what determines their reliability.[7]

Laboratory research on market structure has identified several systematic connections between market micro-structures and their often quite large effects on equilibrium price of traders' experience, degree of common knowledge (e.g. Forsythe, Palfrey, and Plott 1982, Plott and Sunder 1988, Forsythe and Lundholm, 1990, Sunder 1992). This kind of uncertainty about the underlying structure limits the use of such prediction markets for decisions. In response to this kind of uncertainty, Roust and Plott (2006) study the behavior of a two-stage parimutuel betting system designed for information aggregation. They find systematic differences in group behavior, mainly lower trading volume, when the experimentally controlled quality of the information is lower.

1.5. **Doubts about market mechanisms.** There are two, related theoretical doubts about the reliability of prediction markets. The first is a conceptual problem pointed out by Grossman and Stiglitz (1980). If the market price in a prediction is a good indicator of the future event, as it would be if it aggregated all available information, then no rational person would have an incentive to pay even a trivial cost acquiring their own information and take part in the market since their private information will be incorporated into the market price in equilibrium and they cannot expect to gain from that price.[8]

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[6] Wolfers and Zitzewitz (2004) is a good review

[7] Examples of active trading markets are Iowa Electronic Markets (`http://www.biz.uiowa.edu/iem`), Hollywood Stock Exchange (`http://www.hsx.com`), Tradesports (`http:// www.tradesports.com`), and Intrade (`http://www.intrade.com`). Recent "failures" in the Iowa Electronic Markets were its prediction that John Kerry's chance of winning the 2004 Democratic Caucuses was less than 35%, and its short-lasting prediction, apparently reacting to some small number of polls, that John Kerry's chance of winning the election was well over 60%. Tradesports' estimated probability of for finding weapons of mass destruction in Iran five months after the March 2003 U.S. invasion regularly jumped 10% in a day, with 20% jumps being frequent early on.

[8] Muendler (2007) explicitly adds an inter-temporal aspect to Grossman and Stiglitz's model and shows that it may be the case one investor will pay the cost of acquiring information, and that this can be sufficient for full revelation of the information.

More generally, Milgrom and Stokey (1982) point out that the information that someone else is willing to make a bet against you reveals information making you less willing to bet against them. Your initial best estimate of the odds is based on your private information. Knowing that someone else, on the basis of their own private information, is willing to bet against you means that you know that their private information conflicts with yours. This should decrease your willingness to bet. To resolve these conceptual problems, outside subsidies must be provided in a form that guarantees that the agents have incentives to trade, especially when people incur cost of acquiring information.

There is a third theoretical doubt, a type of "Lucas critique." If a prediction market becomes reliable, and this reliability changes policy or politics, this creates strategic incentives to manipulate the market. If the strategic incentives are strong enough, they could offset any monetary losses incurred by the manipulators.

1.6. **A betting mechanism.** We propose a novel betting mechanism in which the principal who needs the forecast will subsidize the bettors to motivate information acquisition. In our mechanism, the prediction is reliable in the sense the principal can estimate the accuracy of the prediction from the bets made in the market. We conduct our analysis within a single-principal-multi-agent framework to resolve the following questions.

(1) How do we give agents incentives to truthfully reveal both their private information and the quality of their information?
(2) What do these incentives cost?
(3) How do we trade off the incentive costs and the benefits of the information we gather?

In our proposed betting mechanism, the principal asks agents to report their forecasts and place money, as a bet, on their reports. After the uncertainty is realized, the agents payoffs depend on how close their reports are to the realization of the variable being predicted and on money they bet on their report. With such a mechanism, the agents cannot simply claim expertise, they can only demonstrate it by putting their money where their mouth is.

We present a family of reward functions with two crucial properties. First, conditional on deciding to incur the cost to gather information, each agent's dominant strategy is to report their true forecast. Second, agents bet an amount that is monotonically increasing in the precision of their information. Within this class, we solve for the optimal reward function.

1.7. **Organizational matters.** In section 2, we introduce the model setup and examine the existence of solutions. We also give the necessary conditions, in the form of a functional equation, for an optimum recruitment contract. Section 3 considers, in some detail, a case in which the functional equation is very informative, and the following examines several of the results that are more general. The last section discusses the implications of this work and several possible extensions. Proofs are in the Appendix.

2. Optimal Recruitment of Sources

For decision purposes, a firm, the principal, needs a forecast, the more accurate the better, of a random variable, X, to be realized in the future. For example, X could be the costs of an environmental remediation, next year's per-acre yield for rice grown in the Sacramento Delta, the level of Dell's sales next quarter, or the box office take of a movie. The firm offers, to each agent i in a set of N possible sources, a contract seeking forecasts. The agents may already be employees of the principal, a closed organizational system, or they may be independent contractors, an open organizational system.

The contract specifies rewards that are contingent on the submission of a report (forecast) r_i, the amount of money, $b_i \geq 0$, that the agents are willing to bet on the accuracy of their own forecast, and the realization, x, of the random variable X. The contract may involve the principal providing some or all of the money that agent i can use as part or all of b_i.

After examining the contract, the agents decide, on the basis of their own costs, $c_i \geq 0$, and knowledge of the precision, $\tau_i \geq 0$, of their own sources and abilities, whether or not to accept the contract. If they accept the contract and do the work required to access and interpret information from their private source(s), they incur the cost c_i. Whether or not they have done this work and incurred the cost, they may submit a report, r_i, and bet b_i on its accuracy. Otherwise, they receive their reservation utility.

Based on the random number of submitted reports and their inferred accuracy, the principal makes his/her decision, the random variable is realized, and the principal and the agents receive their contingent rewards. In general, the principal wants to attract highly reliable sources at low cost. The first problem the principal faces is to give sources the incentives to access and interpret their private information, to report it accurately, and to reliably indicate its quality. Once this is done, the tradeoff is between the cost of attracting sources and the risk of having very few sources.

We show that an optimal recruitment contract exists and give the associated FOC.

2.1. The Forecasting Problem.

The random variable, X, to be forecast, is normally distributed with mean s_0 and variance $1/\tau_0$, written $X \sim N(s_0, 1/\tau_0)$. The inverse of the variance, τ_0, is the "precision," in this case, the initial precision. If $X = x$ and the forecast, $\widehat{X}$, is equal to $\widehat{x}$, the firm's payoff is $v - p(\widehat{x} - x)^2$. Here v is the value to the firm when the forecast is extremely precise, and $p(\widehat{x} - x)^2$ is a quadratic penalty term for forecast error.

To each i in a set of N risk neutral, potential sources, the principal offers a reward schedule $f(b_i, r_i, x)$, depending on i's bet b_i, their report, r_i, and x, the realization of X. The sources, knowing their own costs, c_i, and their own precision, τ_i, decide whether or not to accept the contract. We assume that the sources' costs and precision, $(\tau_i, c_i)_{i\in N}$, are an iid collection of random vectors. The joint distribution of the vector $(\tau_i, c_i)_{i\in N}$, is given by Q.

If a source accepts the contract, they choose whether or not to incur the cost c_i. If, and only if, they incur c_i, they receive a signal $s_i = X + \varepsilon_i$. They then submit their report, r_i, and place their bet, b_i, on its accuracy. When $X = x$ is realized, $f(b_i, r_i, x) - b_i$ is their payoff.

The random variables $(X, (\varepsilon_i)_{i\in N})$ are independent after conditioning on the vector $(\tau_i, c_i)_{i\in N}$. The ε_i are distributed $N(0, 1/\tau_i)$. Given truthful reporting of signals and precision by each $i \in S \subset N$, the statistical forecasting problem and its payoffs are immediate.

Lemma 1. *When agents $i \in S \subset N$ truthfully report s_i and τ_i to the firm, the optimal predictor is $\widehat{X}(S) = (\tau_0 s_0 + \sum_{i\in S} \tau_i s_i)/\tau(S)$ where $\tau(S) := \tau_0 + \sum_{i\in S} \tau_i$ is the prediction precision, and the associated prediction payoffs are $v - \frac{p}{\tau(S)}$.*

The principal's total payoffs are their prediction payoffs minus the incentive costs, which are the payments to the sources. It is generally not optimal to have $S = N$. The marginal benefit of precision, $\tau(S)$ decreases with S, while, given *ex ante* identical sources, marginal incentive costs are constant in S.

2.2. **Incentive Costs and Recruitment Contracts.** To save on incentive costs while benefitting from the increased precision, the principal wants to attract the lower cost, higher precision sources to accept the contract and observe their signals. We design a reward system $f(\cdot,\cdot,\cdot)$, depending on a source's bets, b, a source's report, r, and the realization, x. The reward system has the property that a source with precision τ_i is willing to accept the contract and incur costs up to some $c(\tau_i)$, truthfully report s_i, and place a bet b_i that is an increasing function of τ_i.

This means that the **recruitment contract**, $c(\cdot)$, is the pivotal part of our analysis. The set of sources, $S = S(c)$, that submit reports is that random set with $0 \le c_i \le c(\tau_i)$. We study monotonic recruitment contracts, the ones for which the more valuable, higher precision sources are willing to incur higher costs.[9] Within the class of monotonic contracts, the optimal recruitment contract, $c^*(\cdot)$, is chosen to trade off the gains from increased precision against the incentive costs.

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[9] Often, this monotonicity is a result of optimality. We will return to this in some detail below.

The tradeoffs are different in closed organizational systems, where the sources are employees, than in open organizational systems, where the sources are independent contractors. This is because the costs that employees incur may have direct implications for the organization, e.g. if an employee spends less time and effort on other work.

Given a recruitment contract $c(\cdot)$, a random subset of the sources, $S = S(c) \subset N$, will satisfy $c_i \leq c(\tau_i)$. From Lemma 1, this yields prediction payoffs of $v - \frac{p}{\tau(S(c))}$. In an open organizational structure, the incentive costs will be $\sum_{i \in S} c(\tau_i)$. Combining and letting "E" be "expectation of," this yields the **open structure expected profit function**,

$$\Pi^{open}(c) = v - E\left[\tfrac{p}{\tau(S(c))} - \textstyle\sum_{i \in S(c)} c(\tau_i)\right], \tag{1}$$

which represents the principal's expected profits when gathering information with is dispersed outside of an organization.

When information is dispersed with an organization run by the principal, some fraction, γ, of the sources' costs may be directly borne by the principal. This would happen if, for example, the sources are employees and some part of the sources' costs are also opportunity costs, to the principal, of the time and effort spent. In this case, the principal's incentive costs will be $\sum_{i \in S(c)} (\gamma c_i + c(\tau_i))$ rather than $\sum_{i \in S(c)} c(\tau_i)$, yielding

$$\Pi^{emp}(c) = v - E\left[\tfrac{p}{\tau(S(c))} - \textstyle\sum_{i \in S(c)} (\gamma c_i + c(\tau_i))\right]. \tag{2}$$

2.3. **The Contracts.** Contracts, $f(b, r, x)$, depend on the bets, b, the reports, r, and the realization, x, of X. If a source of precision τ accepts the contract, incurs their costs c, and receives their signal, $s = X + \varepsilon$, their indirect utility function is

$$V(s, \tau) = \max_{b,r} E\left(f(b, r, X) - b \big| s, \tau\right). \tag{3}$$

If V does not depend on s, $V(s, \tau) = V(\tau)$, then any source with costs $c_i \leq V(\tau_i)$ will accept the contract, and collection the information rents $V(\tau_i) - c_i$.

Definition. *A contract $f(\cdot, \cdot, \cdot)$ is **directly revealing** if the strategy, $(b^*(s, \tau), r^*(s, \tau))$, that solves Problem (3) is $(b^*(s, \tau), r^*(s, \tau)) = (\tau, E[X \big| s, \tau])$, and it **implements the recruitment contract** $c(\cdot)$ if, for all s, $V(s, \tau) = c(\tau)$.*

Our contracts take the form $f(b, r, x) = g(b) - h(b)(r - x)^2$ with $g(\cdot)$, $h(\cdot)$ strictly positive for strictly positive b. For $h(b) > 0$, reporting $r \neq E[X \big| s, \tau]$ is stochastically dominated. This simplifies the sources' problem in (3) to

$$V(\tau) = \max_b \; \eta(b) := E\left[g(b) - h(b)\tfrac{1}{\tau_0 + \tau} \big| \tau\right]. \tag{4}$$

Given that the sources will bet b on their report, $g(b)$ reimburses the sources for their costs, gives them money to bet, and includes an extra term that adjusts their willingness to take the risk involved in the $-h(b)\frac{1}{\tau_0 + \tau}$ term. Specifically, set $g^*(b) = c(b) + b + (\tau_0 + b)c'(b)$

and $h^*(b) = (\tau_0 + b)^2 c'(b)$. Here is another place that we use monotonicity — to keep $h^* > 0$, which is needed to have sources optimal bets monotone in their precision, we require $c' > 0$.

Proposition 1. *Assuming that* $c(\cdot)$ *is twice continuously differentiable and the first order condition* $\eta'(b) = 0$ *holds at any maximum for* (4)*, then if* $g(b) - h(b)(r - x)^2$ *is directly revealing and implements the recruitment contract* $c(\cdot)$*, then* $g = g^*$ *and* $h = h^*$*.*

For concave $c(\cdot)$, $\eta'(b) = 0$ characterizes the optimum if the curvature of $c(\cdot)$ is not too large, specifically, if $\left|\frac{c''(b)}{c'(b)}\right| < \frac{2}{(\tau_0+b)}$. As we will see in Lemma 3 (below), these concavity and curvature conditions are often satisfied. We wil also see that if a potential source does not incur the cost c_i, then the best strategy will be to bet $b_i^* = 0$, obtaining no benefits.

2.4. **The Optimal Selection of Sources.** If all sources with cost $c_i \leq c(\tau_i)$ accept the contract, the principal's forecast will be based on the reports of the set $S(c) = \{i \in N : c_i \leq c(\tau_i)\}$. This leads to profits $\Pi^{open}(S(c))$ in the open organizational structure or $\Pi^{emp}(S(c))$ in the case of auditable or directly borne costs (eqn's (1) and (2)). The principal's problem is then

$$\max_{c \in \mathcal{C}} \ \Pi^{open}(S(c)) \quad \text{or} \quad \max_{c \in \mathcal{C}} \ \Pi^{emp}(S(c)), \tag{5}$$

where $\mathcal{C}$ is the set of non-decreasing recruitment contracts.

Non-decreasing contracts are intuitive, higher precision sources are worth more to the principal. Further, as we will discuss in § 4.1, for many, but not all, distributions of precisions and costs, the restriction to non-decreasing contracts is without loss. Also, if a contract has regions of decrease, we cannot implement it in revealing strategies. Finally, the set of non-decreasing contracts is often compact and the payoffs continuous.

Assumption 1. *The non-atomic part of* Q *has a derivative with respect to Lebesgue measure.*

If $Q((0, 0)) > 0$, we have a positive mass of what we think of as "noise traders," people with no useful information.

Proposition 2 (Existence)**.** *If Assumption* 1 *holds, then the problems* $\max_{c \in \mathcal{C}} \ \Pi^{open}(S(c))$ *and* $\max_{c \in \mathcal{C}} \ \Pi^{emp}(S(c))$ *have solutions.*

In outline, there is no loss in restriction attention to those recruitment contracts $c \in \mathcal{C}(p)$ that are bounded above by p because the maximal benefit in reducing uncertainty to 0 is bounded above by p. Indicators of the subgraphs of such c's form an L^1-compact set, and dominated convergence shows that the expected payoffs are continuous.[10]

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[10] Details are in the appendix.

Knowing that an optimal recruitment contract exists, we turn to the derivative conditions that it must satisfy. This kind of analysis requires that the objective functions Π^{open} and Π^{emp} move infinitesimally when c moves infinitesimally. A more primitive sufficient condition for the following is that q be strictly positive on $(\mathbb{R}_+ \times [0,p])^N$.

Assumption 2 (Smooth support condition)**.** *Q has a continuous density, q, that is strictly positive on a set with interior including the graph of c^*.*

The expressions that arise involve several aspects of the distribution Q evaluated at $(\tau_i, c_i) = (\tau^\circ, c^\circ)$: the cdf of i's costs conditional on i's precision, $H(c^\circ \big| \tau^\circ)$; the marginal density of i's precision and costs, $q(\tau^\circ, c^\circ)$; and the marginal density of i's precision, $\varphi(\tau^\circ)$. The expressions also involve evaluating the improvements in precision from including a new source with precision τ°. Letting $C^* = \{(\tau_i, c_i) \in \mathbb{R}^2_+ : c_i \leq c^*(\tau_i)\}$ denote the subgraph of $c^*(\cdot)$, this is p times

$$\mathsf{m}((\tau_{-i}, c_{-i}) \big| \tau^\circ) := \frac{1}{\tau_0 + \sum_{j\neq i} \tau_j 1_{C^*}(\tau_j, c_j)} - \frac{1}{\tau_0 + \tau^\circ + \sum_{j\neq i} \tau_j 1_{C^*}(\tau_j, c_j)}. \tag{6}$$

The expected benefit to adding a source of precision τ° is p times

$$\mathsf{M}(\tau^\circ) = \int_{(\mathbb{R}^2_+)^{N-1}} \mathsf{m}((\tau_{-i}, c_{-i}) \big| \tau^\circ)\, dQ_{-i}(\tau_{-i}, c_{-i}). \tag{7}$$

For any non-zero c^*, M takes values in the interval $(0,1)$, $\mathsf{M}(\tau^\circ) > 0$, $\partial \mathsf{M}(\tau^\circ)/\partial\tau > 0$, $\partial^2 \mathsf{M}(\tau^\circ)/\partial\tau^2 < 0$, and M is strictly decreasing in N.

Proposition 3. *If Q satisfies Assumption 2 and c^* is a solution to $\max_{c\in\mathcal{C}(p)} \Pi^{open}(c)$ with a non-zero one-sided derivative at the point (τ°, c°), then*

$$p \cdot \mathsf{M}(\tau^\circ) = \left[c^\circ + H(c^\circ \big| \tau^\circ) \frac{\varphi(\tau^\circ)}{q(\tau^\circ, c^\circ)} \right]. \tag{8}$$

If we substitute the maximization problem $\max_{c\in\mathcal{C}(p)} \Pi^{emp}(c)$, then

$$p \cdot \mathsf{M}(\tau^\circ) = \left[(1+\gamma)c^\circ + H(c^\circ \big| \tau^\circ) \frac{\varphi(\tau^\circ)}{q(\tau^\circ, c^\circ)} \right]. \tag{9}$$

The left-hand sides of (8) and (9) are the marginal benefit of an increase in precision that arises from a tiny increase in the recruitment contract c^* at the point τ°. The right-hand sides are the marginal effects on the costs of the optimal recruitment contract, and contain two terms. The first term is the cost of the newly recruited source. The second term reflects the the increased payments to every source with costs below the curve c^* at τ°.

The derivative conditions (8) and (9) contain functions of the optimal c^* on both sides of the equations. As a result, the optimal c^* depends on subtle aspects of the distribution Q, and closed form solutions are not likely, even in highly structured special cases.

3. A Special Case

One relatively easy special case has precision and costs independent, and uniformly distributed on $[0, \overline{\tau}]$ and $[0, \overline{c}]$. Equivalently, Q is the uniform distribution on $([0, \overline{\tau}] \times [0, \overline{c}])^N$. In this case, the FOC are uniquely satisfied by a recruitment contract that satisfies the curvature conditions needed for implementability (p. 10). The uniqueness arises from a contraction mapping, and the lattice structure of the set of non-decreasing functions makes the analysis of the optimal contract transparent.

The density of each i's precisions and costs is $q(\tau^\circ, c^\circ) = \frac{1}{\overline{\tau}\overline{c}}$, the conditional cdf of costs given precision is $H(c^\circ \big| \tau^\circ) = \frac{c^\circ}{\overline{c}}$, and the marginal of i's precision is $\varphi(\tau^\circ) = \frac{1}{\overline{\tau}}$. Here, the FOC (8) and (9) in Proposition 3 reduce to

$$p \cdot \mathsf{M}(\tau) = \left[c^*(\tau) + \tfrac{c^*(\tau)}{\overline{c}} \tfrac{1/\overline{\tau}}{1/(\overline{\tau}\cdot\overline{c})}\right] = 2c^*(\tau) \text{ and } p \cdot \mathsf{M}(\tau) = (2+\gamma)c^*(\tau). \tag{10}$$

If M_c denotes the dependence of M on c, then the FOC in (10) can be rewritten as $c = \kappa p \cdot \mathsf{M}_c$ with κ being $\frac{1}{2}$ or $\frac{1}{2+\gamma}$.

3.1. **Competition faced by sources.**

The basic result about competition faced by the sources is

Proposition 4. *If c is a fixed point of the mapping $c \mapsto \kappa p \cdot \mathsf{M}_c$, $0 < \kappa < 1$, then* (1) *an increase in τ_0 or a decrease in p,* (2) *an increase in N,* (3) *an increase in $\overline{\tau}$, or* (4) *a decrease in $\overline{c}$ lead to a decrease in the optimal recruitment contract at all values of τ.*

A decrease in the recruitment contract for all τ is uniformly bad for the sources since $V(\tau)$, their indirect utility at τ, is equal to $c(\tau)$. The sources are competing with each other to improve the initial precision at a cost the principal is willing to pay. One expects that increasing the initial precision and/or making the amount or quality of the competition higher is bad for the sources. Proposition 4 shows that these guesses are accurate because it considers:

(1) an increase in the initial precision from τ_0 to τ_0' or a decrease in the penalty term p both make the principal less willing to pay for any increase in precision, or
(2) an increase in the number of sources from N to N', or
(3) a first order stochastic increase in the sources' precisions, here modeled as an increase from $\overline{\tau}$ to $\overline{\tau}'$, or
(4) a first order stochastic decrease in the sources' costs, here modeled as an decrease from $\overline{c}$ to $\overline{c}'$.

The proof of Proposition 4 relies on the following, which has some independent interest. Recall that $\mathcal{C}(p)$ is the set of non-decreasing functions bounded above by p.

Lemma 2. *For $0 < \kappa < 1$, the mapping $c \mapsto \kappa p \cdot \mathsf{M}_c$ is an increasing contraction mapping from the compact, convex lattice $\mathcal{C}(p)$ to itself.*

By direct investigation of equations (6) and (7), the changes considered in Proposition 4 lower $\mathsf{M}_c(\tau)$ for all τ. Combined with contraction and the lattice structure, this yields the proof. Since each M_c is smooth, Lemma 2 also means that iterative numerical techniques can yield arbitrarily close approximations to the optimal recruitment contracts.

Being a fixed point of the mapping $c \mapsto \kappa p \cdot \mathsf{M}_c$ also implies that implementability is guaranteed. Recall from (p. 10) that implementability of a recruitment contract required a curvature condition, specifically, $\left|\frac{c''(\tau)}{c'(\tau)}\right| < \frac{2}{(\tau_0+\tau)}$.

Lemma 3. *If c is a fixed point of the mapping $c \mapsto \kappa p \cdot \mathsf{M}_c$, $\kappa > 0$, then $\left|\frac{c''(\tau)}{c'(\tau)}\right| \leq \frac{1}{(\tau_0+\tau)}$.*

3.2. **Correlation between precision and costs.** In general, one expects that precision and costs are positively related, that, on average, higher precision sources are more costly, especially if the skills that give them higher precision are broadly valuable. The higher the positive correlation, the more we expect the principal to have to pay higher precision sources since high precision sources are now more costly.

Negative correlation could arise with an influx of low cost, high precision sources. In this case, we would expect the principal to engage in a kind of outsourcing — paying less to all high precision sources to take advantage of the newly available lower cost sources of high quality.

In both of these cases, there can be a secondary effect of the opposite sign on the recruitment contract's offering to below average sources. To get at these effects, we consider changing the uniform density, $q(\tau, c) = \frac{1}{\overline{\tau}\overline{c}}$ to

$$q_r(\tau, c) = \tfrac{1}{\overline{\tau}\overline{c}} + r(\tau - \tfrac{\overline{\tau}}{2})(c - \tfrac{\overline{c}}{2}) \tag{11}$$

defined on the same area, $[0, \overline{\tau}] \times [0, \overline{c}]$. For small[11] $r > 0$ and above-average τ, this has the effect of moving costs from below to above average, with the reverse effect for below-average τ, leading to positive correlation between precision and costs. For $r < 0$, these effects are reversed, and we have negative correlation. Increases in $r > 0$ increase the positive correlation between precision and costs, while decreases in $r < 0$ decrease the negative correlation. In either case, the overall distribution of precisions is unchanged, $\varphi_s(\tau^\circ) = \frac{1}{\overline{\tau}}$.

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[11] For $|r| < 4/(\overline{\tau}\overline{c})$, $q_r(\cdot, \cdot)$ is a strictly positive density.

Straight-forward calculation yields

$$\begin{aligned} h_r(c^\circ \big| \tau^\circ) &= \tfrac{1}{\overline{c}} + \overline{\tau} r(\tau^\circ - \tfrac{\overline{\tau}}{2})(c^\circ - \tfrac{\overline{c}}{2}), \\ H_r(c^\circ \big| \tau^\circ) &= c^\circ \cdot \left[\tfrac{1}{\overline{c}} + \overline{\tau} r(\tau^\circ - \tfrac{\overline{\tau}}{2})(\tfrac{c^\circ - \overline{c}}{2})\right], \text{ and} \\ H_r(c^\circ \big| \tau^\circ) \tfrac{\varphi \tau^\circ}{q(\tau^\circ, c^\circ)} &= c^\circ \left[1 - \tfrac{m}{d}\right] = c^\circ \left[1 - \tfrac{m(\tau^\circ, c^\circ)}{h_r(c^\circ \big| \tau^\circ)}\right] \text{ where} \\ m(\tau^\circ, c^\circ) &= \overline{\tau} r(\tau^\circ - \tfrac{\overline{\tau}}{2})\tfrac{c^\circ}{2}. \end{aligned}$$

The FOC (8) for Π^{open} in Proposition 3 reduces to $p \cdot \mathsf{M}_c(\tau) = c(\tau)\left[2 - \frac{m}{d}\right]$, or $c = \kappa p \cdot \mathsf{M}_c$ where $\kappa = \frac{1}{\left[2 - \frac{m}{d}\right]}$ (instead of $\kappa = \frac{1}{2}$ as we had before). The FOC (9) for Π^{emp} in Proposition 3 reduces to $c = \kappa p \cdot \mathsf{M}_c$ where $\kappa = \frac{1}{\left[(2+\gamma) - \frac{m}{d}\right]}$ (instead of $\kappa = \frac{1}{2+\gamma}$ as we had before).

Since $d > 0$, for positive r, $-\frac{m}{d} > 0$ for all $\tau \in [0, \frac{\overline{\tau}}{2})$, and $-\frac{m}{d} < 0$ for all $\tau \in (\frac{\overline{\tau}}{2}, \overline{\tau}]$. This has the effect of reducing $\mathsf{M}_c(\tau)$ for below average sources and raising it for above average sources. We reverse these signs when $r < 0$, which yields

Lemma 4. *Increases in r raise the recruitment contract for sources with above-average precision and reduce it for sources with below-average precision.*

The essential intuition is that if there there is a change to having more sources below the recruitment contract at a given precision τ, decreasing the contract at τ may lose very few sources, but saves money on all the sources that are below the contract.

4. SOME GENERALITIES

We now turn to some of the general results about the monotonicity of the recruitment contracts, the effects on optimal precision and costs of changes in the initial precision, the value of the prediction, the penalty for inaccuracy, and first order stochastically dominating shifts in the distribution of the costs and precisions of the sources.

For ease of reference, the explicit integrals giving the expected profits are

$$\Pi^{open}(c) = v - \int_{(\mathbb{R}^2_+)^N} \left[\tfrac{p}{\tau_0 + \sum_i \tau_i 1_C(\tau_i, c_i)} + \textstyle\sum_i c(\tau_i) 1_C(\tau_i, c_i)\right] dQ(\vec{\tau}, \vec{c}) \text{ and} \tag{12}$$

$$\Pi^{emp}(c) = v - \int_{(\mathbb{R}^2_+)^N} \left[\tfrac{p}{\tau_0 + \sum_i \tau_i 1_C(\tau_i, c_i)} + \textstyle\sum_i (c_i + c(\tau_i)) \cdot 1_C(\tau_i, c_i)\right] dQ(\vec{\tau}, \vec{c}). \tag{13}$$

4.1. **Monotonicity.** We use the monotonicity of the recruitment functions both for examining the existence of an optimal recruitment contract for sources and for implementing an optimal recruitment plan. In the argument for existence, the set of non-decreasing recruitment functions is compact in a metric that makes the payoffs in (12) and (13) continuous.

The monotonicity of a recruitment function and its implementability are interlocked. If implemented and revealing, a monotone recruitment function gives higher rewards to higher precision sources by motivating them to bet more on better evidence about the true value being estimated. As noted above, monotonicity is required for implementation of this property.

To summarize, for the mechanism we design here, monotonicity is crucial. If we could design a yet better mechanism, one that could implement revealing non-monotonic recruitment functions, there is still the question of whether we would want to. The answer is, sometimes, yes, and the first order conditions in Proposition 3 contain some information about this.

Suppose that for some interval of source precisions, $(a, b]$, most of the costs are high, but that in the next interval, $(b, c]$, most of the costs are low. For intermediate values of c, this corresponds to $H(c\big|\tau)$ being low over $(a, b]$ then becoming (much) higher over $(b, c]$. In order to satisfy the FOC, which have an increasing function on the left-hand side, we might expect that the recruitment function must decrease from $(a, b]$ to $(b, c]$ to compensate for the behavior of $H(\cdot\big|\tau)$.

Intuitively, lowering payments to sources with precisions in the $(b, c]$ range selects essentially none of them out of the principal's pool. However, if c must be monotone, this requires selecting out many sources with precisions in the interval $(a, b]$. There is a strong parallel with a monopolist's pricing problem without perfect discrimination where lowering the price charged to one group requires lowering it to all.

4.2. **Expensive Errors and Better Sources.** For any recruitment contract c, increases in the penalty, p, unambiguously reduce both $\Pi^{open}(c)$ and $\Pi^{emp}(c)$. This directly implies that principal's optimal profit decreases in p. There are two kinds of sources, the one with the initial precision τ_0, and the ones that the principal may wish to recruit. The same logic implies that the principal's optimal profit is increasing as the sources become better, and increasing as the sources become less expensive.

If there is an increase in the initial precision, τ_0, the principal's profits unambiguously increase. If there is a first order (stochastically dominating) shift in the sources' precisions that is independent of costs, a more involved argument shows that the principal's profits unambiguously increase.

From Strassen's theorem, we can replace each (τ_i, c_i) pair by $(r(\tau_i), c_i)$ where $r(x) \geq x$. Define a new monotonic recruitment function by $c_{new}(\tau) = c(r^{-1}(\tau))$ and let C_{new} denote its subgraph. This has the effect of changing the precision term in the profit functions from $p/(\tau_0 + \sum_i \tau_i 1_C(\tau_i, c_i))$ to $p/(\tau_0 + \sum_i r(\tau_i) 1_{C_{new}}(r(\tau_i), c_i))$, an unambiguous benefit to the principal. At the same time, a stochastic increase in τ has no effect on the cost terms. A completely parallel analysis shows that stochastically decreasing costs independent of precisions is unambiguously good for the principal.

Less clear are the effects of these shifts on the optimal recruitment contract itself, though it is fairly easy to tell how the optimal precision and the optimal recruitment function respond to changes in p.

Define $x_1(c) = -\int 1/(\tau_0 + \sum_i \tau_i 1_C(\tau_i, c_i))\, dQ(\vec{\tau}, \vec{c})$, the negative inverse precision, and $x_2(c) = \int \sum_i c(\tau_i) 1_C(\tau_i, c_i)\, dQ(\vec{\tau}, \vec{c})$, the open structure recruitment costs. With these,

$$\Pi^{open}(c) = v + p x_1(c) - x_2(c).$$

Define $x_2^*(x) = \min_{c \in \mathcal{C}} x_2(c)$ subject to $x_1(c) \geq x$. It is easy to show that x_2^* is non-decreasing in the standard lattice ordering on functions. In terms of precision and costs, maximizing Π^{open} is equivalent to the problem

$$\max_{x_1, x_2} v + p x_1 - x_2 \text{ subject to } x_2 \geq x_2^*(x_1). \tag{14}$$

Since x_2^* is non-decreasing, we immediately have

Proposition 5. *If $p > p'$ and c and c' are corresponding optimal recruitment contracts, then $x_1(c) \geq x_1(c')$ and $x_2(c) \geq x_2(c')$, that is, the principal's optimal precision and costs are higher.*

This kind of geometric analysis is more difficult for changes in τ_0 and first order stochastic shifts in the costs and the precisions — these shifts have the unambiguous effect of shifting the $x_2^*(\cdot)$ curve inwards or outwards, but have ambiguous or difficult-to-discover effects on the slope of the curve. As we saw in the previous section, the FOC from Proposition 3 can sometimes be used to answer these kinds of questions.

4.3. **Competition Between Sources.**

We now give more detail about the result that recruitment contracts decrease with the number of sources. This property has significant implications on re-structuring of the organizational form. If the firm restricts the participation to only employees in the company, the number of agents is not likely to be large. One simple way to extend the set of sources is to utilize the outsiders to collect information. The philosophy of outsourcing, open source software production, and openly traded prediction markets reflects a similar idea.

The essential intuition is that the competition between the N sources drive each other's marginal product, that is, their value to the principal, down, and this effect becomes larger as N increases.

Note that the value of any gain in precision in equations (12) and (13) is bounded above by p/τ_0. Therefore, lowering $c(\cdot)$ below p on any set having positive probability increases expected profits. This yields

Lemma 5. *If c^* is optimal for Π^{open} or Π^{emp}, then $Q\{(\tau, c) : c^*(\tau) \leq p/\tau_0\} = 1$.*

The first terms in the profit functions are the same, $v-\int p/(\tau_0+\sum_i \tau_i 1_C(\tau_i, c_i))\, dQ(\vec{\tau}, \vec{c})$, and take values in the range $[v - p/\tau_0, v)$. The cost terms involve N identical pieces, the average per source costs. These are equal to either $AC^{open} = \int c(\tau) \cdot 1_C(\tau, c)\, dQ(\tau, c)$ or $AC^{emp} = \int (c + c(\tau)) \cdot 1_C(\tau, c)\, dQ(\tau, c)$. Since $v - p/\tau_0$ is feasible, we know that the cost terms are bounded above by p/τ_0, so that, at their respective optima, $AC^{open} < \frac{p/\tau_0}{N}$ and $AC^{emp} < \frac{p/\tau_0}{N}$.

Using the conditional cdf at τ, $H(x\big|\tau) = Q(c \in [0, x]\big|\tau)$, the average costs are

$$AC^{open} = \textstyle\int_{\mathbb{R}_+} c(\tau) \cdot H(c(\tau)\big|\tau)\, d\mathrm{marg}\, Q(\tau),$$

where $\mathrm{marg}\, Q$ is the marginal distribution of Q, and

$$AC^{emp} = \textstyle\int_{\mathbb{R}_+} (c(\tau) + E\left(c\big|c \leq c(\tau), \tau\right)) \cdot H(c(\tau)\big|\tau)\, d\mathrm{marg}\, Q(\tau).$$

These considerations yield

Lemma 6. *If c_N^* is optimal for Π^{open} when there are N potential sources, then for all $\epsilon > 0$, $Q\{c_N^*(\tau) \cdot H(c_N^*(\tau)\big|\tau) > \epsilon\} \to 0$ as $N \to \infty$, and if c_N^* is optimal for Π^{emp} when there are N potential sources, then for all $\epsilon > 0$, $Q\{(c_N^*(\tau) + E(c\big|c \leq c_N^*(\tau), t)) \cdot H(c_N^*(\tau)\big|\tau) > \epsilon\} \to 0$.*

This means that for large N, the only way that $c_N^* = c_N^*(\tau)$, the principal's willingness to pay for a source of precision τ, can be large is if $H(c_N^*\big|\tau)$ is small. At face value, this seems to be an odd conclusion. However, combined with $c_N^*(\cdot)$ being non-decreasing, Lemma 6 tells us that $c_N^*(\cdot)$ can only be large at τ if it is very unlikely that there is any source having precision τ or larger.

5. Extensions and Discussion

We now turn to a discussion of the sensitivity of our results to violations of the various assumptions, and to what may, or may not be possible by way of solutions.

5.1. **Agents' preferences.** If agents are not risk neutral, we can alter or betting mechanism as suggested in Chen *et. al.* (2001). In our context, their suggestion would involve using a preliminary round of betting to estimate risk preferences, and then using the estimated curvature to adjust the precision infered from the size of the bets.

A separate, more difficult issue arises if the agents have preferences over the decisions that the principal will make on the basis of the prediction, e.g. if the decision will affect large parts of the workforce within a firm. At one level, this is Kadane and Winkler's (1988) cautionary work showing that it is very difficult to elicit probabilities if there is a violation of what they call the "no-stakes" condition, that is, if the agents have a stake in the decision. Beyond this, we are back to the Lucas critique we discussed above — if the forecasts are known to be accurate and used for decision making, this can raise the incentives to manipulate the forecasts.

5.2. **Agents' anonymity.** Another violation of the "no-stakes" conditions arises if the sources' reports are not anonymous. The major benefit of anonymity is to eliminate the agents' strategic report due to their reputation concerns. Sharfstein and Stein (1990) have listed several different kinds of strategic behavior caused by concerns about reputations. For example, agents may herd (e.g. report similarly or conservatively) when they are worrying about looking "stupid," and tend to deviate from consensus when they are seeking a "ranking effect." Prendergast (1993) describes how pressure to conform to the opinion/prior of supervisors arises in incentive contracts designed to induce effort from employess. In an anonymous system, the incentives to engage in these behaviors will diminish.

One downside of anonymity is difficulty in detecting agents' collusion. For example, if two agents talk to each other, they both improve their precision without incurring costs. However, the principal, observing their bets, need not know that the errors in the two sources are no longer independent. The importance of both of these types of problems diminishes quickly as the number of agents increases.

5.3. **Auditable costs.** If it were available, costless auditing would be the best system for the principal. Suppose that the agents' costs of information acquisition are aduitable by the principal, the principal can adjust the reward function so as to get as much precision as possible at any cost $\epsilon > 0$. The idea is to adjust the reward function so as to make the agents' expected reward ϵ above the acquisition cost c_i.

If accurate auditing is too expensive, the principal may well be better off with the system proposed here. That is, letting the sources earn the information rents, $c(\tau_i) - c_i$, may be the better solution.

5.4. **Some statistical issues.** If one weights observations with weights that are themselves noisy estimates, the implied loss of precision can outweigh the benefits (e.g. Kendall and Stuart 1973, Ch. 9). We assumed that each agent knows their precision τ_i. If the agents' precisions are themselves estimates, the overall quality of the aggregate prediction will be reduced.

Further, we assume that each individual observes independent signals. If the errors that the agents make are very highly correlated but are treated as independent, the principal will incorrectly conclude that the estimate has very high precision. More generally, when the signals are dependent, the entire variance-covariance matrix is needed for the optimal heteroskedasticity corrections. In forecasting, Clemen (1989) conclude that, mostly, the noise introduced by use of estimated correlations in heteroskedasticity corrections outweighs the gain in statistical efficiency.

5.5. **Predicted precision and noise traders.** Our mechanism puts up a sign saying "Noise traders not welcome." By contrast, introducing noise traders into a market is often thought of as a way to subsidize the informed traders, though this clearly introduces noise into the predictions. We directly pay participants, seeking a prediction for private use rather than a publicly observable prediction. One implied merit of this is that we also have a predicted precision.

In their review of the experience with such markets, Wolfers and Zitzewitz (2004) discuss the general considerations for a prediction market to be reliable.

> These insights suggest that some prediction markets will work better when they concern events that are widely discussed, since trading on such events will have higher entertainment value and there will be more information on whose interpretation traders can disagree. (*ibid.* p. 121)

However, there is no guidance as to how reliable a market that is working will be. Further, since many of the most successful prediction markets offer bragging rights or prizes instead of real money, we are not likely to gain a good theoretical understanding of this. By contrast, in our proposal, we understand where the precision comes from and how much there is.

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6. PROOFS

Proof of Proposition 1: Suppose that $V(\tau) = c(\tau)$. There is no loss in writing $g(b) = c(b) + b + m(b)$ for some function $m(b)$, which yields, after substitution, $\eta(b) = c(b)+m(b)-\frac{h(b)}{(\tau_0+\tau)}$. Direct revelation of τ requires that for all τ, $\eta(\tau) = c(\tau)$, i.e. that for all τ, $m(\tau) = \frac{h(\tau)}{(\tau_0+\tau)}$. This yields $h(b) = (\tau_0 + b)m(b)$, so that $\eta(b) = c(b) + m(b)\left[1 - \frac{(\tau_0+b)}{(\tau_0+\tau)}\right]$. The first order condition requires that $\eta'(b)$ should equal 0 when $b = \tau$. Now, $\eta'(b) = c'(b) + m'(b)\left[1 - \frac{(\tau_0+b)}{(\tau_0+\tau)}\right] - m(b)\frac{1}{(\tau_0+\tau)}$. Evaluating at $b = \tau$ yields, $\frac{m(b)}{(\tau_0+b)} = c'(b)$, that is, $m(b) = (\tau_0 + b)c'(b)$. ∎

The next proof shows that the objective functions $\Pi^{open}(\cdot)$ and $\Pi^{emp}(\cdot)$ are continuous functionals on a compact class of monotonic functions. The compactness proof is moderately involved, and simple examples can be constructed showing that compactness may fail if the non-atomic part of the distribution of Q fails to have a derivative.

Proof of Proposition 2: As noted in the text, it is sufficient to consider only $c \in \mathcal{C}(p)$, the subset of the non-decreasing functions that are bounded above by p. For any such c, let $C \subset \mathbb{R}^2_+$ denotes its subgraph, $C = \{(\tau, y) : 0 \leq y \leq c(\tau)\} \subset \mathbb{R}^2_+$. Define the distance $d(C, C')$ as the $L^1(Q)$-distance between the indicator functions of C and C (in any of the axes) so that $d(C, C') = Q(C \Delta C')$. For $(\vec{\tau}, \vec{c}) \in (\mathbb{R}^2_+)^N$, we will use $1_C(\tau_i, c_i)$ to denote $1_C(\text{proj}_i(\vec{\tau}, \vec{c}))$ where "proj_i" is the canonical projection mapping. With this notation, we have (repeating equations (12) and (13) for ease of reference),

$$(15)\quad \Pi^{open}(c) = v - \int_{(\mathbb{R}^2_+)^N} \left\{ \left[\tfrac{p}{1+\sum_i \tau_i 1_C(\tau_i, c_i)}\right] + \left[\textstyle\sum_i c(\tau_i) 1_C(\tau_i, c_i)\right] \right\} dQ(\vec{\tau}, \vec{c}), \text{ and}$$

$$(16) \Pi^{emp}(c) = v - \int_{(\mathbb{R}^2_+)^N} \left\{ \left[\tfrac{p}{1+\sum_i \tau_i 1_C(\tau_i, c_i)}\right] + \left[\textstyle\sum_i (\gamma c_i + c(\tau_i)) 1_C(\tau_i, c_i)\right] \right\} dQ(\vec{\tau}, \vec{c}).$$

Routine applications of Lebesgue's dominated convergence theorem shows that these two objective functions are continuous.

Compactness: To complete the proof, it is sufficient to show that $\mathcal{C}(p)$ is compact. Since $L^1(Q)$ is complete, it is sufficient to show that $\mathcal{C}(p)$ is *closed* and *totally bounded.*[12]

(1) Closedness: To show closedness, we prove that any convergent sequence $C^n \in \mathcal{C}(p)$ converges to a $C \in \mathcal{C}(p)$.

(a) We show by contradiction that $C \subset (R_+ \times (p, +\infty))$: Let C^n be a sequence in $\mathcal{C}(p)$ converging to C. It is sufficient to look at a single axis, which we do without changing notation. If $Q(C \cap (\mathbb{R}_+ \times (p, \infty)) > 0$, then for large n, $Q(C^n \cap (\mathbb{R}_+ \times (p, \infty)) > 0$, contradicting $C^n \in \mathcal{C}(p)$.

(b) We show by contradiction that C has to be non-decreasing: C fails to be non-decreasing Q-a.e. iff there exist sets $A < B \subset \mathbb{R}_+$, $0 \leq r < s \leq p$, such that

(1) $Q(A \times [0, r]) > 0$ with $(A \times [0, r]) \subset C$ Q-a.e., and
(2) $Q(B \times (r, s]) > 0$ with $(B \times (r, s]) \subset C^c$ Q-a.e.

However, if these are true, then $Q((A \times [0, r]) \cap C^n) > 0$ and $Q((B \times (r, s]) \cap (C^n)^c) > 0$ for all large n as well, contradicting $C^n \in \mathcal{C}(p)$.

Summarizing (a) and (b), we obtain *closedness.*

(2) Total boundedness: Pick $\epsilon > 0$, we must show there exists a finite ϵ-net for $\mathcal{C}(p)$. The idea for constructing the finite ϵ-net is to look at finite sets of non-decreasing step functions with fine enough steps that they come ϵ-close to any point in $\mathcal{C}(p)$. The details of choosing the step sizes and their locations are a slightly messy as they depend on Q. The density of the non-atomic part of Q is denoted q.

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[12] Compactness criteria for subsets of L^1 spaces given in terms of conditional expectations with respect to finite sub-σ-fields can be found in Dunford and Schwartz (Theorem IV.4.18, p. 297). Verifying those conditions involves essentially the same steps as we present here.

Pick $\epsilon > 0$.

There is a possibly empty, finite set of atoms, (τ_α, c_α), $\alpha = 1, \ldots, At$, with the property that the Q-mass of all other atoms is less than $\epsilon/4$.

Because $(\mathbb{R}_+ \times [\tau, \infty)) \downarrow \emptyset$ as $\tau \uparrow \infty$, we can pick $\overline{\tau}$ such that $Q(\mathbb{R}_+ \times [\overline{\tau}, \infty)) < \epsilon/4$.

Because $\{(c, \tau) : q(c, \tau) \geq B\} \downarrow \emptyset$ as $B \uparrow \infty$, we can pick $\overline{B}$ such that $Q(q(c, \tau) \geq \overline{B}\}) < \epsilon/4$.

Pick M large enough that $2(M + At) \cdot \frac{\overline{B}p\overline{\tau}}{M^2} < \epsilon/4$. Partition $[0, \overline{\tau}) \times [0, p)$ into the M^2 rectangles $[a_m, a_{m+1}) \times [b_n, b_{n+1})$ where each $a_m = \frac{mp}{M}$, $m = 0, \ldots, M-1$ and each $b_n = \frac{n\overline{\tau}}{M}$, $n = 0, \ldots, M-1$. The Euclidean area of each rectangle is $\frac{p\overline{\tau}}{M^2}$. Ignoring any contribution from either the atoms or the set $\{q(c, \tau) \geq \overline{B}\}$, each rectangle has probability less than or equal to $\frac{\overline{B}p\overline{\tau}}{M^2}$.

In the domain of the functions in $\mathcal{C}(p)$, let $T_{domain} \subset [0, \overline{\tau}]$ be the set $T_{domain} = \{\tau_\alpha : \alpha = 1, \ldots, At\} \cup \{a_m : m = 0, \ldots, M-1\}$. In the range of the functions in $\mathcal{C}(p)$, let $T_{range} \subset [0, p]$ be the set $T_{range} = \{c_\alpha : \alpha = 1, \ldots, At\} \cup \{b_m : m = 0, \ldots, M-1\}$. Let $N(T_{domain}, T_{range})$ be the finite set of non-decreasing functions constant between adjacent points in T_{domain} and taking on only values in T_{range}. To complete the proof, it is sufficient to show that $N(T_{domain}, T_{range})$ is an ϵ-net for $\mathcal{C}(p)$.

Pick arbitrary $c \in \mathcal{C}(p)$. Let $c' \in N(T_{domain}, T_{range})$ be the largest function everywhere less than or equal to c. The symmetric difference $C \Delta C'$ has four components:

(a) the atoms of Q not in the set (τ_α, c_α), $\alpha = 1, \ldots, At$, a set which contributes at most $\epsilon/4$ to its mass;
(b) a subset of $\mathbb{R}_+ \times [\overline{\tau}, \infty)$ which contributes at most $\epsilon/4$ to its mass;
(c) a subset of $\{q(\tau, c) > \overline{B}\}$, which contributes at most $\epsilon/4$ to its mass; and
(d) at most $2(M + At)$ rectangles (where 0 width rectangles are allowed) up and to the left of the graph of c', which contribute at most $2(M + At) \cdot \frac{\overline{B}p\overline{\tau}}{M^2} < \epsilon/4$ to its mass.

Since the mass of these four sets adds to less than ϵ, $d(C, C') < \epsilon$. ∎

Proof of Proposition 3: Let (τ°, c°) be a point on the graph of $c^*(\cdot)$ at which there is a non-zero derivative, s. For $\Delta > 0$, define $h_\Delta(t) = s(\tau^\circ - t)1_{(\tau^\circ - \Delta, \tau^\circ]}(t)$, for $\Delta < 0$, define $h_\Delta(t) = [c^\circ - c^*(t)] \cdot 1_{(\tau^\circ - \Delta, \tau^\circ]}(t)$ so that $(c^* + h_\Delta)(t) = c^\circ$ if $\tau^\circ - \Delta < t \leq \tau^\circ$ and $(c^* + h_\Delta)(t) = c^*(t)$ otherwise. We arrive at the FOC, (8) and (9), by setting

$$\lim_{\Delta \to 0} \frac{\Pi^{open}(c^* + h_\Delta) - \Pi^{open}(c^*)}{\Delta^2} = 0, \quad \text{and } \lim_{\Delta \to 0} \frac{\Pi^{emp}(c^* + h_\Delta) - \Pi^{emp}(c^*)}{\Delta^2} = 0.$$

Let $C^* + H_\Delta$ denote the subgraph of $c^* + h_\Delta$ and C^* the subgraph of c^*. Because q is smooth, $Q((\tau_i, c_i) \in (C^* + H_\Delta) \setminus C^*) = q(\tau^\circ, c^\circ) \cdot \frac{s \cdot \Delta^2}{2} + o(\Delta^2)$. The penalty terms in

Π^{open} and Π^{emp} are $\Psi(c^*) := \frac{p}{1+\sum_i \tau_i 1_{C^*}(\vec{\tau},\vec{c})}$, so

$$[\Psi(c^* + h_\Delta) - \Psi(c^*)] = p\left[\int_{(\mathbb{R}^2_+)^N} \left(\frac{1}{1+\sum_j \tau_j 1_{C^*}(\tau_j, c_j)} - \frac{1}{1+\sum_j \tau_j 1_{(C^*+H_\Delta)}(\tau_j, c_j)}\right) dQ(\vec{\tau},\vec{c})\right].$$

The difference in the integrand is equal to 0 except when a (τ_i, c_i) falls into the triangle; the probability that (τ_i, c_i) and (τ_j, c_j), $i \neq j$, both fall into the triangle is $o(\Delta^4)$, so disappears in the limit. This implies that there are N terms to consider, one for each i, and each term involves an integral over the $j \neq i$ axes. Since the (τ_i, c_i) vectors are iid, we need only evaluate one of these integrals. For (τ_i, c_i) in the triangle, the difference in the integrand converges to $-\mathsf{m}((\tau_{-i}, c_{-i})\big|\tau^\circ)$. Finally, since the area of the triangle is $q(\tau^\circ, c^\circ) \cdot \frac{s\Delta^2}{2}$,

$$\lim_{\Delta\to 0} \tfrac{1}{\Delta^2}[\Psi(c^* + h_\Delta) - \Psi(c^*)] = -q(\tau^\circ, c^\circ)\tfrac{s}{2}Np \cdot \mathsf{M}(\tau^\circ). \tag{17}$$

The recruitment costs in (15) are $\xi(c^*) := \sum_i \int_{\mathbb{R}^2_+} c^*(\tau_i) 1_{C^*}(\tau_i, c_i)\, dQ(\vec{\tau},\vec{c})$, so that

$$\xi(c^* + h_\Delta) - \xi(c^*) = N \cdot \int_{\mathbb{R}^2_+} \left[(c^* + h_\Delta)(\tau) 1_{(C^*+H_\Delta)}(\tau, c) - c^*(\tau) 1_C(\tau, c)\right]\, dQ_i(\tau, c), \tag{18}$$

Q_i being the marginal. The integrands are equal expect in the triangle and in the quadrilateral below the triangle, $C^* \cap \{(\tau^\circ - \Delta, \tau^\circ] \times \mathbb{R}_+\}$. To within higher order terms in Δ, triangle contributes $c^\circ \cdot q(\tau^\circ, c^\circ)\frac{s\Delta^2}{2}$ to the difference, and the quadrilateral contributes $s\frac{\Delta}{2} \cdot H(c^\circ\big|\tau^\circ)\varphi(\tau^\circ)\Delta$, so that

$$\lim_{\Delta\to 0} \tfrac{\xi(c^*+h_\Delta)-\xi(c^*)}{\Delta^2} = \tfrac{s}{2}N\left[H(c^\circ\big|\tau^\circ)\varphi(\tau^\circ) + c^\circ q(\tau^\circ, c^\circ)\right]. \tag{19}$$

From (17) and (19) and cancellation/rearrangement,

$$p\mathsf{M}(\tau^\circ) = \left[H(c^\circ\big|\tau^\circ)\tfrac{\varphi(\tau^\circ)}{q(\tau^\circ, c^\circ)} + c^\circ\right], \tag{20}$$

which is the FOC, (8). The extra opportunity cost term in Π^{emp} leads, by the same logic, to

$$p\mathsf{M}(\tau^\circ) = \left[H(c^\circ\big|\tau^\circ)\tfrac{\varphi(\tau^\circ)}{q(\tau^\circ, c^\circ)} + (1+\gamma)c^\circ\right], \tag{21}$$

which is the FOC, (9). ▮

Proof of Lemma 2: Suppose that $c_1(\tau) \geq c_2(\tau)$ for all τ and $Q(C_2 \setminus C_1) > 0$. For any τ°, $\mathsf{M}_{c_1}(\tau^\circ) - \mathsf{M}_{c_2}(\tau^\circ)$ is equal to

$$E\left(\tfrac{1}{\tau_0+\sum_j \tau_j 1_{C_1}(\tau_j, c_j)} - \tfrac{1}{\tau_0+\tau^\circ+\sum_j \tau_j 1_{C_1}(\tau_j, c_j)}\right) - \left(\tfrac{1}{\tau_0+\sum_j \tau_j 1_{C_2}(\tau_j, c_j)} - \tfrac{1}{\tau_0+\tau^\circ+\sum_j \tau_j 1_{C_2}(\tau_j, c_j)}\right).$$

Since $x \mapsto 1/x$ is strictly convex and $Q(C_1 \setminus C_2) > 0$, this is strictly positive for all $\tau > 0$. Thus, $c_1 > c_2$ implies that $\mathsf{M}_{c_1} > \mathsf{M}_{c_2}$.

Define $T : \mathcal{C} \to \mathcal{C}$ by $T(c) = \kappa p \cdot \mathsf{M}_c$. Since T is a continuous, increasing map from a compact convex lattice to itself, we know it has a non-empty sublattice of fixed points, and that the set of fixed points varies in the usual fashion when we move T monotonically. Such analyses are even easier if T has a unique fixed point. To show that it does, we find a metric, $\rho(C_1, C_2)$, equivalent to $d(C_1, C_2) = Q(C_1 \Delta C_1)$, in which T is a contraction.

We first normalize the precision units so that initial precision, τ_0, is equal to 1. With the appropriate changes in the recruitment function, changing p to p/τ_0 and the upper bound for precision to $\overline{\tau}/\tau_0$ leaves the form of the principal's objective functions unchanged. Now we multiply the principal's utility function by a small positive constant to change the units of cost so that $\overline{c} < (N-1)\overline{\tau}$ and $p < 1$. Again, this leaves the form of the objective functions unchanged.

For recruitment functions c_α with associated subgraphs C_α, $\alpha = 1, 2$, define the random variable $X_{C_\alpha} = 1 + \sum_{j \neq i} \tau_j 1_{C_\alpha}(\tau_j, c_j)$. We define $\rho(C_1, C_2) = \|X_{C_1} - X_{C_2}\|_1 = E\,|X_{C_1} - X_{C_2}|$. Dominated convergence yields $[d(C_n, C) \to 0] \Rightarrow [\rho(C_n, C) \to 0]$, the τ_i being uniformly distributed on $[0, \overline{\tau}]$ yields $[\rho(C_n, C) \to 0] \Rightarrow [d(C_n, C) \to 0]$, so that ρ is equivalent to d.

There are two intermediate steps to showing that T is a contraction mapping.

Step 1: If $\sup_\tau |c_1(\tau) - c_2(\tau)| = \delta$, then $d(C_1, C_2) = E\,|1_{C_1 \Delta C_2}| \leq \delta/\overline{c}$. Now,

$$|X_{C_1} - X_{C_2}| = |\textstyle\sum_{j \neq i} \tau_j (1_{C_1} - 1_{C_2})| \leq \sum_{j \neq i} \tau_j |1_{C_1} - 1_{C_2}| < \overline{\tau} \sum_{j \neq i} |1_{C_1 \Delta C_2}|.$$

Taking expectations, this yields

$$\rho(C_1, C_2) = E\,|X_{C_1} - X_{C_2}| \leq (N-1)\overline{\tau} E\,|1_{C_1 \Delta C_2}| \leq (N-1)\overline{\tau}\tfrac{\delta}{\overline{c}} < \delta$$

where the last inequality follows from $\overline{c} < (N-1)\overline{\tau}$.

Step 2: From the definition of m and M in (6) and (7), we see that for all $\tau \geq 0$,

$$|\mathsf{M}_{c_1}(\tau) - \mathsf{M}_{c_2}(\tau)| \leq E\,\left|\left(\tfrac{1}{X_{C_1}} - \tfrac{1}{\tau + X_{C_1}}\right) - \left(\tfrac{1}{X_{C_2}} - \tfrac{1}{\tau + X_{C_2}}\right)\right|.$$

For any $x, y \geq 1$ and $\tau \geq 0$,

$$\left|\left(\tfrac{1}{x} - \tfrac{1}{\tau + x}\right) - \left(\tfrac{1}{y} - \tfrac{1}{\tau + y}\right)\right| \leq |x - y|,$$

so that

$$E\,\left|\left(\tfrac{1}{X_{C_1}} - \tfrac{1}{\tau + X_{C_1}}\right) - \left(\tfrac{1}{X_{C_2}} - \tfrac{1}{\tau + X_{C_2}}\right)\right| \leq E\,|X_{C_1} - X_{C_2}| = \rho(C_1, C_2).$$

Step 2 shows that the sup norm distance between M_{c_1} and M_{c_2} is less than or equal to $\rho(C_1, C_2)$. Let M_α be the subgraph of M_{c_α}, $\alpha = 1, 2$. Step 1 shows that this implies that $\rho(\mathsf{M}_1, \mathsf{M}_2)$ is less than this sup norm distance. Combining, $\rho(\mathsf{M}_1, \mathsf{M}_2) < \rho(C_1, C_2)$. Since $p < 1$ and $\kappa < 1$, T is a contraction. ∎

Proof of Lemma 3: If $c = \kappa \cdot \mathsf{M}_c$, then $\left|\frac{c''(\tau)}{c'(\tau)}\right| = \left|\frac{\mathsf{M}''_c(\tau)}{\mathsf{M}'_c(\tau)}\right|$. Since the integrand is smooth in τ and bounded, we can interchange the order of integration and differentiation. This yields $\mathsf{M}'_c(\tau) = \int \frac{dQ}{(\tau_0+\tau+\sum_{j\neq i}\tau_j 1_C(\tau_j,c_j))^2}$ and $-\mathsf{M}''_c(\tau) = \int \frac{dQ}{(\tau_0+\tau+\sum_{j\neq i}\tau_j 1_C(\tau_j,c_j))^3}$. Therefore, $\frac{1}{(\tau_0+\tau)} \geq \left|\frac{\mathsf{M}''_c(\tau)}{\mathsf{M}'_c(\tau)}\right|$ iff $\int \frac{dQ}{(\tau_0+\tau+\sum_{j\neq i}\tau_j 1_C(\tau_j,c_j))^2} \geq \int \frac{(\tau_0+\tau)dQ}{(\tau_0+\tau+\sum_{j\neq i}\tau_j 1_C(\tau_j,c_j))^3}$. Since $1 \geq \frac{(\tau_0+\tau)}{(\tau_0+\tau+\sum_{j\neq i}\tau_j 1_C(\tau_j,c_j))}$, the inequality holds. ∎

Proof of Proposition 4: Give $\mathcal{C}$ the usual pointwise lattice order. Let $T : \mathcal{C} \to \mathcal{C}$ be the mapping $T(c) = \kappa \mathsf{M}_c$ at initial precision τ_0 (respectively at number of sources N). After an increase in τ_0 (respectively N), we have a new mapping $S : \mathcal{C} \to \mathcal{C}$ such that $T(c) \geq S(c)$ for all c. Let c_T and c_S be the respective fixed points. For any c, we have the inequality $T^{(n)}(c) \geq S^{(n)}(c)$. Passing to the limit, $c_T \geq c_S$. The other changes in T are handled exactly the same way. ∎

Proof of Proposition 4: Partially order $\mathcal{C}$ by $c_1 \succ c_2$ if $c_1 > c_2$ on $(0, \frac{\overline{\tau}}{2})$ and $c_1 < c_2$ on $(\frac{\overline{\tau}}{2}, \overline{\tau})$. With this partial order, $\mathcal{C}$ is a lattice. Let $\mathsf{M}_s(c)$ denote $\kappa p \cdot \mathsf{M}_s$ when the density is q_s. For all c and all $s > s'$, $\mathsf{M}_s(c) \succ \mathsf{M}_{s'}(c)$ and $s \mapsto \mathsf{M}_s$ is lattice-increasing in s. ∎