The Algebraic Geometry of Competitive Equilibrium

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1 Introduction

The classical tools of general equilibrium theory are convexity and general topology. One of Debreu's lasting contributions has been to show how the tools of differential topology may serve to yield more refined information about equilibrium. In particular, Debreu (1970) showed that differential topology could provide a rigorous formalization of "counting equations and unknowns" to provide a satisfactory result on the determinacy of equilibrium.

Debreu required that preferences be representable by C^2 utility functions with non-vanishing gradients and that indifference surfaces have non-vanishing curvature. Following Debreu, a natural question to ask is: Are there other interesting classes of preferences (or demands) exhibiting regularity of behavior sufficient to guarantee the generic local determinacy of equilibrium prices? The first results in this direction were obtained by Rader (1972, 1973), who showed that generic finiteness of the equilibrium price set was a consequence of demand being differentiable almost everywhere and satisfying condition N: that the image of a null set is null. Rader also gave conditions on preferences sufficient to generate demand meeting these hypotheses; a further set of sufficient conditions was developed in a later paper (1979). The concave-utility requirement of Rader (1973) has only recently been relaxed to local concavifiability of the preference relation, by Pascoa and Werlang (1989).

The work reported here extends the work of Debreu in a different direction. Its motivation comes from two sources. The first is a remark made by Rader at an NBER Conference in the 1970's about the nature of demand when preferences are analytic. The second is the work of Blume and Zame (1989) and Schanuel, Simon and Zame (forthcoming) on the applications of algebraic geometry in non-cooperative game theory. When asked about the potential applications of the methods of those papers to general equilibrium theory, our response was to be skeptical because the required hypotheses seemed to be so strong. However, we later realized that these algebraic-geometric methods could in fact be extended to a much wider class of

preferences than we had previously thought (roughly, the piecewise analytic preferences), and that the strength of these assumptions bought many compensations. Our central result is that, for preferences in this class, generically in endowments, the equilibrium set is finite and depends nicely on endowments. We allow for flats and kinks of indifference surfaces, we make no curvature requirements, and we allow for equilibria on the boundary of the price simplex. Moreover, our conclusions are somewhat stronger than those in the smooth case in that they yield more structure on the exceptional set of endowments.

The classes of preferences we consider are those with utility representations that are semi-algebraic (roughly, piecewise algebraic) and finitely sub-analytic (roughly, piecewise analytic); precise definitions are given in Section 2. Most of the familiar preferences used in applications have representations that are finitely sub-analytic on the relevant portions of their domain: Cobb-Douglas, logarithmic, exponential, CES, HARA, piecewise-linear. However our methods, which are those of real algebraic geometry and mathematical logic, apply potentially to other classes of preferences. For our purposes, the crucial property of preferences is that their graphs belong to an O-minimal Tarski system containing the graphs of addition and multiplication (again, Section 2 provides a precise definition); the semi-algebraic sets and the finitely sub-analytic sets comprise the known examples. Once we know that preferences belong to such a system, the key properties of demand and the local finiteness of equilibrium follow from a few basic properties shared by all such systems. It seems preferable, therefore, to present the arguments in the somewhat more abstract setting of O-minimal Tarski systems rather than to carry around the excess baggage that comes from assuming that preferences are specifically semi-algebraic or finitely sub-analytic.

Section 2 contains the relevant mathematical background, including definitions and a summary of results about O-minimal Tarski systems, including the family of semi-algebraic sets and the family of finitely sub-analytic sets. Section 3 discusses preferences and demands. The local finiteness result and its proof are found in Section 4.

2   Mathematical Background

In this Section we set out, following van den Dries (1986), the basic properties of O-minimal Tarski systems. Our goal is to delimit a class of functions; we proceed, however, by delimiting a class of sets. This may seem less strange if we keep in mind that every property of a function $f : \mathbf{R} \to \mathbf{R}$ can be expressed in terms of its graph: f is measurable if and only if its graph is a measurable set, f is smooth if and only if its graph is a smooth manifold, etc.

A *Tarski system* is a family $\mathcal{S} = \{\mathcal{S}_n\}$ such that:

1. For each n, $\mathcal{S}_n$ is a Boolean algebra[1] of subsets of $\mathbf{R}^n$.

2. If $X \in \mathcal{S}_n$, then $\mathbf{R} \times X \in \mathcal{S}_{n+1}$ and $X \times \mathbf{R} \in \mathcal{S}_{n+1}$.

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[1]That is, $\mathcal{S}_n$ contains $\emptyset$ and $\mathbf{R}^n$ and is closed under formation of complements, finite unions, and finite intersections.

3. For each n, $D_n = \{(x_1, \ldots, x_n) \; : \; x_1 = x_n\} \in \mathcal{S}_n$.

4. If $X \in \mathcal{S}_n$ and $\pi : \mathbf{R}^n \to \mathbf{R}^{n-1}$ is the projection onto the first $n-1$ coordinates, then $\pi(X) \in \mathcal{S}_{n-1}$.

A Tarski system $\mathcal{S}$ is *O-minimal* if, in addition:

5. It contains the graph of the less than relation: $L = \{(x, y) \; : \; x < y\} \in \mathcal{S}_2$.

6. For each $r \in \mathbf{R}$ the singleton $\{r\}$ belongs to $\mathcal{S}_1$.

7. Every set in $\mathcal{S}_1$ is a finite union of intervals and points.

It should be emphasized that a Tarski system is a *system*—that its properties achieve much of their bite in conjunction with each other. A simple example may make this point. Let $B \in \mathcal{S}_2$ and let B^s be the symmetrically reversed set: $B^s = \{(x, y) \; : \; (y, x) \in B\}$. Property 2 guarantees that $\mathbf{R} \times B \in \mathcal{S}_3$. Since properties 3 and 1 guarantee that $D_3 \in \mathcal{S}_3$ and $\mathcal{S}_3$ is a Boolean algebra, it follows that $(\mathbf{R} \times B) \cap D_3 \in \mathcal{S}_3$. Since B^s is the projection of $(\mathbf{R} \times B) \cap D_3$ into the first two coordinates, we conclude from property 4 that $B^s \in \mathcal{S}_2$.

To see the additional bite of O-minimality, observe that we may build a Tarski system by beginning with an arbitrary family $\{T_n\}$, adjoining the diagonal subsets, and closing under Boolean operations, products, and projections. However, such a process may already lead to a family which violates property 7. For instance, if we begin with the singletons in $\mathbf{R}$ and the single additional set $A = \{(x, y) \; : \; y = \sin x\} \subset \mathbf{R}^2$ (the graph of the sine function), we must include $\mathbf{R} \times \{0\} \subset \mathbf{R}^2$, and $A \cap (\mathbf{R} \times \{0\})$, and the image of $A \cap (\mathbf{R} \times \{0\})$ under the first coordinate projection. But this last set is precisely the set of zeroes of the sine function, which is not a finite union of points and intervals.

If $\mathcal{S}$ is an O-minimal Tarski system, it is convenient to say that a set X *belongs* to $\mathcal{S}$ or X is an $\mathcal{S}$-set if $X \in \mathcal{S}_n$ for some n. We say that a function (or correspondence) $f : A \to B$ *belongs* to $\mathcal{S}$ or f is an $\mathcal{S}$-*function* if the graph of f belongs to $\mathcal{S}$.[2] It is useful to note that the inverse correspondence of every function belonging to $\mathcal{S}$ again belongs to $\mathcal{S}$. The simple proof follows along the same lines as the coordinate-reversal example above.

For our purposes, we shall mainly be interested in O-minimal Tarski systems which contain (the graphs of) addition and multiplication; i.e.,

8. $A = \{(x, y, z) \; : \; z = x + y\} \in \mathcal{S}_3$.

9. $M = \{(x, y, z) \; : \; z = xy\} \in \mathcal{S}_3$.

Two O-minimal Tarski systems are well known: The polyhedral (or piecewise linear) sets satisfy all the above properties but property 9, while the semi-algebraic sets satisfy all nine properties.

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[2]Note that the domain of f is the projection of the graph of f, and so belongs automatically to $\mathcal{S}$ whenever f does.

A subset of $\mathbf{R}^n$ is *polyhedral* if it is a finite union of sets defined by linear equalities and inequalities; i.e., if it is a finite union of sets of the form $\{x \ : \ \lambda_i(x) = \alpha_i, \ \mu_j(x) < \beta_j, \ 1 \leq i \leq N, \ 1 \leq j \leq M\}$, where the α_i and β_j are real numbers, and the λ_i and μ_j are linear functionals.

A subset of $\mathbf{R}^n$ is *semi-algebraic* if it is a finite union of sets defined by polynomial equalities and inequalities; i.e., if it is a finite union of sets of the form $\{x \ : \ p_i(x) = \alpha_i, \ \ q_j(x) < \beta_j, \ \ 1 \leq i \leq M, \ 1 \leq j \leq N\}$, where p_i and q_j are polynomials.

It is easy to see that the polyhedral sets form an O-minimal Tarski system, and that they are contained in every O-minimal Tarski system satisfying property 8. (Of course they themselves do not satisfy property 9.) The polyhedral functions are precisely the familiar piecewise linear functions.

The semi-algebraic sets form the smallest O-minimal Tarski system satisfying properties 8 and 9. All of the defining properties of O-minimal Tarski systems except for property 4 are easily verified for the family of semi-algebraic sets. The fourth property is a consequence of a deep theorem of logic, the Tarski-Seidenberg Theorem. The semi-algebraic functions include the piecewise linear functions, all the familiar algebraic functions (polynomials, rational functions, roots, etc.), and their compositions, algebraic combinations, and their derivatives; but not the transcendental functions such as the exponential function, the logarithm and the trigonometric functions, and indefinite integrals of semi-algebraic functions.

A larger O-minimal Tarski system was discovered by van den Dries (1986), relying on the work of Lojasiewicz (1965) and Gabrielov (1968): the finitely sub-analytic sets. To define this family, we must first define semi-analytic sets and sub-analytic sets.

A subset $X \subset \mathbf{R}^n$ is *semi-analytic* if for each $y \in \mathbf{R}^n$ (not just $y \in X$) there is an open neighborhood U of y such that $U \cap X$ is the finite union of sets defined by real analytic inequalities and inequalities; i.e., $U \cap X$ is a finite union of sets of the form $\{x \ : \ f_i(x) = \alpha_i, \ \ g_j(x) < \beta_j, \ \ 1 \leq i \leq M, \ 1 \leq j \leq N\}$, where f_i and g_j are real analytic functions.

A subset $X \subset \mathbf{R}^n$ is *sub-analytic* if for each $y \in \mathbf{R}^n$ (not just $y \in X$) there is an open neighborhood V of y and a bounded semi-analytic set $Y \subset \mathbf{R}^{n+m}$ such that $V \cap X$ is the image of Y under the projection onto the first n coordinates.

A subset $X \subset \mathbf{R}^n$ is *finitely sub-analytic* if it is the image under the map

$$(x_1, \ldots, x_n) \mapsto \left(\frac{x_1}{\sqrt{1 + x_1^2}}, \ldots, \frac{x_n}{\sqrt{1 + x_n^2}} \right)$$

of a sub-analytic subset of $\mathbf{R}^n$.

Of course, semi-algebraic sets and functions are finitely sub-analytic, but many transcendental functions are finitely sub-analytic but not semi-algebraic, including

the restrictions of the exponential function, the logarithm and the trigonometric functions to compact subsets of their domains. Compositions, algebraic combinations, and derivatives of finitely sub-analytic functions are finitely sub-analytic, but indefinite integrals are not. Neither are the exponential function, the logarithm and the trigonometric functions on their entire domains.

For our purposes, the crucial properties of O-minimal Tarski systems are the various finiteness properties (discussed below), and the fact that O-minimal Tarski systems are *closed under definability*. To see what this means, consider an O-minimal Tarski system $\mathcal{S}$ and a set $X \in \mathcal{S}_n$. The set

$$Y = \{(x_1, \ldots, x_{n-1}) \ : \ \exists x_n \ (x_1, \ldots, x_{n-1}, x_n) \in X\}$$

is the projection of X onto its first $n - 1$ components, and thus belongs to $\mathcal{S}_{n-1}$. Similarly, the set

$$Z = \{(x_1, \ldots, x_{n-1}) \ : \ \forall x_n \ (x_1, \ldots, x_{n-1}, x_n) \in X\}$$

is the complement of the projection of the complement of X, and thus also belongs to $\mathcal{S}_{n-1}$. In like manner, if F is a first order formula involving the free variables x_1 through x_{n-1}, any finite number of quantified variables, and sets in $\mathcal{S}$, then it is a finite string of conjunctions and disjunctions of expressions such as those in the definitions of Y and Z and their negations. Negation corresponds to set complementation, conjunction corresponds to intersection and disjunction corresponds to union, so the fact that $\mathcal{S}$ is a Boolean Algebra (property 1) implies that the set

$$\{(x_1, \ldots, x_{n-1}) \ : \ F(x_1, \ldots, x_{n-1}, x_n) \quad \text{is true}\}$$

is an element of $\mathcal{S}$.

At this point, closure under definability might seem to be an obscure property, but the fact that the O-minimal Tarski systems we are interested in contain (the graphs of) familiar relations and functions, including $=$, $<$, $+$ and $-$, make it quite powerful.

To give a simple example of the implications of closure under definability, let $\mathcal{S}$ be any O-minimal Tarski system that contains addition and scalar multiplication, and let X be any set belonging to $\mathcal{S}$; we show that the closure $\mathrm{cl}\, X$ of X also belongs to $\mathcal{S}$. To see this, let us write $\|w\| = \sum_n |w_n|$. Then the closure of X is:

$$\mathrm{cl}\, X = \{y \in \mathbf{R}^n \ : \ \forall \epsilon > 0, \ \exists x \in X, \ \|x - y\| < \epsilon\}$$

On the face of it, the expression that defines $\mathrm{cl}\, X$ is not a formula in the sense discussed above; however, it may easily be expanded into such a formula. Let π denote the projection from $\mathbf{R}^2$ onto the first coordinate. For any $X \in \mathcal{S}_2$, let X^s denote the symmetric reversal of X, and recall that L is the graph of the less-than relation. Note first that the sets of non-negative real numbers, non-positive real numbers, and positive real numbers are all $\mathcal{S}$-sets. This is trivially true for the systems of semi-algebraic and finitely sub-analytic sets. However, it can be shown to be true for all O-minimal Tarski systems satisfying properties 1 through 6. For instance:

$$\mathbf{R}_+ = \{x \ : \ 0 \leq x\} = \pi\Big((L \cap (\{0\} \times \mathbf{R}))^s\Big) \cup \{0\}$$

Let $\mathbf{R}_0$ denote the set $\{0\} \times \mathbf{R}$. Property 6 implies that the set $\{0\}$ is an $\mathcal{S}$-set. Property 2 then implies that $\mathbf{R}_0$ is an $\mathcal{S}$-set. It follows from properties 5 and 2 that $L \cap \mathbf{R}_0$ is an $\mathcal{S}$-set, from properties 1 through 4 that $(L \cap \mathbf{R}_0)^s$ is an $\mathcal{S}$-set, from property 5 that $\pi((L \cap \mathbf{R}_0)^s)$ is an $\mathcal{S}$- set, and finally from property 1 that $\pi((L \cap \mathbf{R}_0)^s) \cup \{0\}$ is an $\mathcal{S}$-set.

Next, observe that the graph of the absolute value function on $\mathbf{R}$ is:

$$\text{Graph}(|\cdot|) = \{(r,s) \;:\; r \in \mathbf{R}_+ \quad \text{and} \quad s = r\}$$
$$\cup \{(r,s) \;:\; r \in \mathbf{R}_- \quad \text{and} \quad s = -r\} \cup \{(0,0)\}$$

The first set is $(\mathbf{R} \times \mathbf{R}_+) \cap D_2$, which is clearly an $\mathcal{S}$-set. The third set is $(\{0\} \times \mathbf{R}) \cap D_2$, which is clearly an $\mathcal{S}$ set. The second set is $\{(r,s) \;:\; r \in \mathbf{R}_-\} \cap \{(r,s) \;:\; r + s = 0\}$. The set on the left is clearly an $\mathcal{S}$-set. The set on the right is $A \cap (\mathbf{R} \times \mathbf{R} \times \{0\})$ (where A is the graph of addition). It follows from properties 8, 6 and 2 that this set is an $\mathcal{S}$ set. Thus the second set is an $\mathcal{S}$ set, and so $\text{Graph}(|\cdot|)$ is an $\mathcal{S}$-set. In other words, $|\cdot|$ is an $\mathcal{S}$-function for any O-minimal Tarski system satisfying properties 1 through 6 and 8.

Similarly we can use property 8 to conclude that $(x,y) \mapsto \|x - y\|$ is also an $\mathcal{S}$-function, and so $N = \{(x,y,\epsilon) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \;:\; \|x - y\| < \epsilon\}$ is an $\mathcal{S}$-set. Now the closure of X may be expressed as:

$$\text{cl}\, X = \left\{ y \in \mathbf{R}^n \;:\; \forall \epsilon \left((\epsilon \in \mathbf{R}_-) \text{ or } (\epsilon \in \mathbf{R}_{++} \text{ and } (\exists \mathbf{x} \in \mathbf{X}, \; (\mathbf{x}, \mathbf{y}, \epsilon) \in \mathbf{N})) \right) \right\}$$

which is an $\mathcal{S}$-set, as asserted.

Closure under definability has been exploited by Blume and Zame (1989) and Schanuel, Simon and Zame (forthcoming) to show that, for finite games, the graph of the Nash equilibrium correspondence, and the graphs of many refinements of the Nash equilibrium correspondence, are semi-algebraic correspondences. We refer to these papers for more examples of the sort of $\epsilon - \delta$ manipulations used above.

We now turn to the basic finiteness properties. In what follows, we fix an O-minimal Tarski system $\mathcal{S}$ that contains addition and multiplication, and list some important consequences of properties 1 through 9. For further discussion, see van den Dries (1986) and Bochnak, Coste and Roy (1987).

GLOBAL FINITENESS: Every set X belonging to $\mathcal{S}$ has a finite number of connected components, and each of them belongs to $\mathcal{S}$. If $f : X \to Y$ belongs to $\mathcal{S}$, then each point inverse $f^{-1}(y)$ belongs to $\mathcal{S}$, and there is a positive integer $N(f)$ such that each point inverse $f^{-1}(y)$ has at most $N(f)$ connected components.

TRIANGULABILITY: If $X \in \mathcal{S}_n$, then there is a finite simplicial complex $K \subset \mathbf{R}^n$ and a homeomorphism h belonging to $\mathcal{S}$, mapping X onto K.

In view of this property, we may unambiguously define the dimension of X, $\dim X$, to be the largest dimension of any subsimplex of K.

DIMENSION: If $X \in \mathcal{S}_n$ then $\text{cl}\, X, \partial X \in \mathcal{S}_n$, $\dim \text{cl}\, X = \dim X$ and $\dim \partial X < \dim X$. If $f : X \to \mathbf{R}^n$ belongs to $\mathcal{S}$, then $\dim f(X) \leq \dim X$.

CONTINUITY: If $F : X \to Y$ is a closed-valued correspondence belonging to $\mathcal{S}$, then there is a subset $X' \subset X$ belonging to $\mathcal{S}$ such that $X \backslash X'$ is closed,

$\dim X \backslash X' < \dim X$, and the restriction of F to X' is continuous. In particular, if $f : X \to Y$ is an $\mathcal{S}$-function, then it is continuous on the complement of a closed, lower dimensional $\mathcal{S}$-set $X' \subset X$.

GENERIC TRIVIALITY: Let $f : X \to Y$ be a continuous function belonging to $\mathcal{S}$. Then there is a subset $Y' \subset Y$ belonging to Y such that $Y \backslash Y'$ is closed, Y' has a finite number of connected components, and for each such connected component Y_i of Y', there is a set Z_i belonging to $\mathcal{S}$ and a homeomorphism h_i belonging to $\mathcal{S}$ which maps $Y_i \times Z_i$ onto $f^{-1}(Y_i)$ and has the property that $f(h_i(y, z)) = y$ for all y, z.

As we shall see, Generic Triviality serves both as a version of Sard's Theorem and as a version of the Implicit Function Theorem, and we shall make extensive use of it. It is useful to understand the differences, however. Given smooth manifolds X, Y, and a smooth function $f : X \to Y$, Sard's Theorem guarantees that most points $y \in Y$ are regular values of f (i.e., that the differential df has rank equal to the dimension of Y at every point of $f^{-1}(y)$). The set Y' of exceptional (irregular) values is a set of measure zero.[3] If X is compact, the set of regular values is open and for every regular value $y \in Y \backslash Y'$ there is a neighborhood W of Y, a manifold Z, and a smooth homeomorphism $h : W \times Z \to f^{-1}(W)$ such that $f(h(w, z)) = w$. Generic triviality allows us to obtain the same product structure, in the complement of an exceptional set Y', when X and Y are $\mathcal{S}$-sets and $f : X \to Y$ is an $\mathcal{S}$-function, but there are some important differences. The first of these is that we do not need to insist that X be compact. (In the smooth case, compactness of X is required to guarantee that the set of regular values be open. If X is not compact the exceptional set Y' could be dense, in which case no open subset of Y will admit a product structure.) A bit more subtly, note that in the case of $\mathcal{S}$-sets and $\mathcal{S}$-functions, there is a *finite* covering of the complement of the exceptional set by neighborhoods whose inverse images are products; this need not be possible in the smooth case. Most importantly, in the case of $\mathcal{S}$-sets and $\mathcal{S}$-functions it will be possible to say a great deal about the structure of the exceptional set Y' and its inverse image $f^{-1}(Y')$. Indeed, Y' is an $\mathcal{S}$-set and of lower dimension than Y, $f^{-1}(Y')$ is also an $\mathcal{S}$-set, the restriction of f to $f^{-1}(Y')$ is an $\mathcal{S}$-function, and the entire apparatus can then be applied to the mapping $f : f^{-1}(Y') \to Y'$. In the smooth case, by contrast, Y' will be a closed set of measure zero, but otherwise may be entirely arbitrary. In particular, both Y' and $f^{-1}(Y')$ may fail to be manifolds.

3   Preferences and Demand

For the rest of the paper, we fix an O-minimal Tarski system $\mathcal{S}$ that contains addition and multiplication. (Recall that the two known examples of such systems are the semi-algebraic sets and the finitely sub-analytic sets.) We consider a consumer, characterized by a consumption set $X \subset \mathbf{R}^{\mathbf{L}}$ and a preference order $\succeq$. We assume that X is closed and convex, and that the preference order is complete, transitive and continuous. We show first that, if the consumption set and (the graph of) the

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[3]The precise degree of smoothness required depends upon the dimensions of X and Y.

preference order belongs to $\mathcal{S}$, then the preference order is representable by a utility function belonging to $\mathcal{S}$.

Theorem 1 *If the consumption set X and (the graph of) the preference order $\succeq$ belong to $\mathcal{S}$, then $\succeq$ is representable by a continuous utility function belonging to $\mathcal{S}$.*

Proof: Fix an arbitrary reference point $x \in X$, and write $X' = \{y \ : \ y \succeq x\}$ and $X'' = X \backslash X'$. Both of these sets belong to $\mathcal{S}$. Now define a utility function $u(\,\cdot\,)$ by

$$u(y) = \begin{cases} +\inf\{\|z - x\| \ : \ z \succeq y\} & \text{for } y \in X', \\ -\inf\{\|z - x\| \ : \ y \succeq z\} & \text{for } y \in X''. \end{cases}$$

A standard and completely straightforward argument shows that u is lower semicontinuous and represents the preference ordering $\succeq$. The graph of u is:

$$\text{Graph}\, u = \{(y, r) : y \in X' \text{ and } r > 0 \text{ and not}(\exists z (z, y) \in \text{Graph} \succeq \text{ and } \|z - x\| < r)$$
$$\text{and } (\forall \epsilon > 0 \ \exists z' \ (z', y) \in \text{Graph} \succeq \text{ and } \|z - x\| < r + \epsilon)\}$$
$$\cup \{(y, r) \ : \ y \in X'' \text{ and } r < 0 \text{ and not } (\exists z \ (z, y) \in \text{Graph} \succeq \text{ and } \|z - x\| < -r)$$
$$\text{and } (\forall \epsilon > 0 \ \exists z' \ (z', y) \in \text{Graph} \succeq \text{ and } \|z' - x\| < -r + \epsilon)\}$$

Since O-minimal Tarski systems are closed under definability, we conclude that Graph u belongs to $\mathcal{S}$.

We now have an $\mathcal{S}$-function u which represents $\succeq$, but it may not be continuous. Next we modify it to make it continuous. Since u is an $\mathcal{S}$-function, its range must be a finite union of intervals and single points (Property 7). Using standard arguments it is easily seen that the range contains no isolated points, and is the union of a finite number of intervals such that each bounded interval is open from below and closed from above, and is such that the points of discontinuity are precisely those in the inverse image of the upper bound for each bounded interval. In other words, the range is

$$(a_1, b_1] \cup (a_2, b_2] \cdots \cup (a_{n-1}, b_{n-1}] \cup (a_n, \infty)$$

where each $b_i < a_{i+1}$ and $a_n < \infty$ if and only if $\succeq$ is globally non-satiated. Define

$$v(x) = \begin{cases} u(x) & \text{if } x \in u^{-1}(a_1, b_1], \\ u(x) - a_i & \text{if } x \in u^{-1}(a_i, b_i] \text{ for } i = 2, \ldots, n. \end{cases}$$

Each of the sets $u^{-1}(a_i, b_i]$ is an $\mathcal{S}$-set, and each of the functions $u(x) - a_i$ is an $\mathcal{S}$-function, so the set

$$\text{Graph}\, v = \{\text{Graph}\, u \cap (u^{-1}(a_1, b_1] \times \mathbf{R})\} \bigcup$$
$$\cup_{i=2}^{n} \Big(\{(x, r) \ : \ \exists s (r = s - a_i) \text{ and}((x, s) \in \text{Graph}\, u)\} \cap \{u^{-1}(a_i, b_i] \times \mathbf{R}\} \Big)$$

is in $\mathcal{S}$. The function v is a continuous utility representation for the preference order $\succeq$. $\qquad\Box$

It is convenient to collect here the following result, which assures us that the demand correspondence also belongs to $\mathcal{S}$ and has the "correct" dimension. The

reader will recognize this result as the analog in our setting of a well-known result of Debreu for smooth preferences. Its proof employs a duality idea that seems novel. View demand as depending on price and income. For each price/income pair, we identify the utility level the consumer achieves, and then make use of some facts about the relationship between prices and those consumption bundles which minimize expenditure at the given price among all bundles achieving that level of utility. We make no assumptions about curvature or smoothness. In particular, we allow for the possibility that indifference surfaces have flats and kinks. (To prove the corresponding result for the case of smooth preferences, one would check for regularity of the value 0 for the mapping which describes the first order conditions characterizing demand. Of course, there would be no way to verify the regularity condition unless the mapping is smooth, ruling out flats and kinks.)

Write Δ for the simplex of non-negative prices in $\mathbf{R}^\mathbf{L}$ which sum to 1. Demand depends on income y and prices p. We allow for some prices to equal 0, even though demand may be undefined at such prices. Let $d : \Delta \times \mathbf{R}_+ \to X$ denote the demand correspondence with arguments price and income, and define the demand correspondence $D : \Delta \times \mathbf{R}_+^\mathbf{L} \to X$, with arguments price and endowment, by the equation $D(p,e) = d(p, p \cdot e)$. The conclusion of the Theorem states that the dimension of the graph of d is less than or equal to L, and that of the graph of D is less than or equal to $2L - 1$. The smooth-preference version of this Theorem says that the dimensions are *equal* to L and $2L - 1$, respectively. We only have inequalities because our hypotheses are insufficient to guarantee the existence of demand for all price-income pairs.

Theorem 2 *If the consumption set X and preference order $\succeq$ belong to $\mathcal{S}$, then the graph of the demand correspondences $d : \mathbf{R}_+ \times \Delta \to X$ and $D : \Delta \times \mathbf{R}_+^\mathbf{L} \to X$ generated by $\succeq$ belongs to $\mathcal{S}$, $\dim \operatorname{Graph} d \leq L$ and $\dim \operatorname{Graph} D \leq (L-1) + L$.*

The duality argument is facilitated by the following result on convex sets and their supports. Let $C \subset \mathbf{R}^\mathbf{n}$ be a closed, convex set, and let $\rho : \partial C \to \mathbf{R}^\mathbf{n}$ denote correspondence that assigns to each $x \in \partial C$ its supporting hyperplanes:

$$\rho(x) = \{p \; : \; p \cdot x \leq p \cdot z \quad \text{for all} \quad z \in C\}.$$

Lemma 1 *The map $(x,p) \mapsto x - p$ is one-to-one on the domain $\operatorname{Graph} \rho$.*

Proof of Lemma 1: Suppose that (x,p) and (y,q) are two points in $\operatorname{Graph} \rho$, and that $x - p = y - q$. Then

$$p \cdot x - p \cdot p = p \cdot y - p \cdot q,$$
$$q \cdot y - q \cdot q = q \cdot x - q \cdot p.$$

Rearranging,

$$-p \cdot p + p \cdot q = p \cdot (y - x) \geq 0,$$
$$-q \cdot q + q \cdot p = q \cdot (x - y) \geq 0.$$

Adding these two inequalities gives $(p-q)\cdot(p-q)\leq 0$, so $p=q$ and therefore $x=y$. $\square$

Proof of Theorem 2: The first assertion, that Graph d is an $\mathcal{S}$-set, is again a simple exercise. The graph of the demand correspondence d is:

$$\text{Graph}\,d = \{(p,y,x) \; : \; y\in\mathbf{R}_+ \text{ and } x\in X \text{ and } p\cdot x\leq y$$
$$\text{and not }(\exists z \; z\in Z \text{ and } p\cdot z\leq y \text{ and } z\succ x)\}$$

Since Graph d is definable from $\mathcal{S}$, it is an $\mathcal{S}$-set.[4]

The calculation of the dimension of the graph of demand relies upon a duality argument. We will apply Lemma 1 to "at least as good as" sets. Let $\succeq x = \{y : (y,x)\in\succeq\}$.

First take prices to be in all of $\mathbf{R}^{\mathbf{L}}_+\backslash\{0\}$, and call the demand correspondence $\hat{d}$. The demand correspondence d is the restriction of $\hat{d}$ to the domain $\Delta\times\mathbf{R}_+$. Let $u:\mathbf{R}^{\mathbf{L}}\to\mathbf{R}$ be an $\mathcal{S}$-utility representation of the consumer's preferences $\succeq$.

It is a consequence of Lemma 1 that the $\mathcal{S}$-function

$$\phi:(p,y,d)\mapsto(d-p,u(d))$$

is one-to-one from Graph $\hat{d}$ to $\mathbf{R}^{\mathbf{L+1}}$. To see this, suppose that $\phi(p,y,d)=\phi(p',y',d')$. Then $u(d)=u(d')$. Since preferences are continuous and locally non-satiated, it follows that if z is demanded at any price q, then z is on the boundary of the closed set $\succeq x$ and q supports $\succeq x$ at z. Since $\succeq$ is locally non-satiated, d and d' are both on the boundary of $\succeq d$. Since p and p' support $\succeq d$ at d and d', respectively, it follows from the equality of $d-p$ and $d'-p'$ and Lemma 1 that $p=p'$ and $d=d'$.

Local non-satiation of $\succeq$ implies that $y=p\cdot d$. Since Graph $\hat{d}$ is the image of the $\mathcal{S}$-function ϕ^{-1}, and since $\mathcal{S}$-functions do not increase dimension, $\dim\text{Graph}\,\hat{d} = \dim\phi(\text{Graph}\,\hat{d})\leq L+1$.

Let $\|p\|$ denote the sum $\sum_l p_l$, and observe that the map $\pi:\mathbf{R}^{\mathbf{L}}_+\times\mathbf{R}\times X\to\Delta\times\mathbf{R}\times X$ defined by $\pi(p,y,x)=((1/\|p\|)p,(1/\|p\|)y,x,\|p\|)$ is an $\mathcal{S}$-isomorphism from Graph $\hat{d}$ to Graph $d\times\mathbf{R}_{++}$. Thus $\dim\text{Graph}\,d+1\leq L+1$, so Graph d has dimension at most L.

Finally, let π_L denote the projection of vectors in $\mathbf{R}^{\mathbf{L}}$ onto their first $L-1$ coordinates. The map $\phi:\Delta\times\mathbf{R}^{\mathbf{L}}_+\times X\to\Delta\times\mathbf{R}_+\times X$ given by $\psi(p,e,x)=(p,p\cdot e,x,\pi_L(e_1))$ defines an $\mathcal{S}$-isomorphism between Graph D and Graph $d\times\mathbf{R}^{\mathbf{L-1}}_+$. Thus Graph D has dimension no greater than $L+L-1$. $\square$

Having identified new classes of preferences, we ask which of the common specifications of preferences are included and which are not. Of course, preferences represented by piecewise linear utility functions are semi-algebraic. More generally, preferences represented by piecewise polynomial (spline) utility functions are semi-algebraic, as are preferences represented by Cobb-Douglas and CES utility functions with rational exponents (eg., $x^{1/4}y^{3/4}$, $x^{1/4}+y^{3/4}$). For irrational exponents, logarithmic, or exponential utilities the situation is slightly more complicated, because

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[4]Note that, even if preferences are piecewise linear, the definition of demand involves quadratic terms. Hence demand may fail to be piecewise linear even when preferences are.

the logarithm and exponential function are not finitely sub-analytic on the domain $(0, \infty)$. However, a slight twist will frequently land us back in the finitely sub-analytic class. Consider for example an economy in which consumption sets are the positive orthant and utility functions are Cobb-Douglas. Let $K \subset \mathbf{R}_{++}^{NL}$ be a compact set, and restrict attention to endowments lying in K. There is a compact set $K^* \subset \mathbf{R}_{++}^{NL}$ that is a product of intervals (and hence polyhedral), and contains all feasible, individually rational allocations. The economy we obtain by restricting consumption sets to K^* is then finitely sub-analytic, and its competitive equilibria coincide with the competitive equilibria of our original economy.

We are also compelled to ask whether the requirement that preferences belong to some O-minimal Tarski system containing additional and multiplication (eg. the finitely sub-analytic sets or the semi-algebraic sets) places any restrictions on observable data; the answer is that it does not. Indeed, the now classic constructions of Afriat (1967) and Diewert (1973) construct piecewise linear utility functions that rationalize any finite number of demand observations satisfying the Strong Axiom of Revealed Preferences. Matzkin and Richter (1991) construct piecewise polynomial (hence semi-algebraic) utility functions that rationalize (in a stronger sense) any finite number of demand observations satisfying SARP.

4 Local Determinacy of Equilibrium

In this Section we present our main result, the local finiteness and determinacy of equilibrium prices when consumption sets and preferences belong to a given O-minimal Tarski system S containing the graphs of addition and multiplication. We shall state and prove our result first for strictly convex preferences (which generate demand functions), in order to compare our technical apparatus to that used for smooth economies, and then for general prefrences (which generate demand correspondences).

We consider exchange economies with L commodities and N consumers, having consumption sets X_n and preferences $\succeq_n$. Throughout, we assume that consumption sets are closed and convex, that preferences are complete, transitive, continuous, monotone and locally non-satiated, and that consumption sets and preferences belong to a fixed O-minimal Tarski system S containing the semi-algebraic sets. We view consumption sets and preferences as fixed an endowments as variable. Write e_n for the endowment of consumer n, and e_{-1} for the vector of endowments of consumers other than consumer 1. Write $\mathcal{E} = \prod X_n$ for the space of endowments, and $\mathcal{E}_n$ for the space of endowment vectors for all consumers other than consumer 1. The space of prices is the non-negative unit simplex $\Delta \subset \mathbf{R}_+^L$.[5]

Theorem 3 *If preferences are strictly convex, then there is a closed, lower-dimensional S-set $E_0 \subset \mathcal{E}$ such that, if $e \in \mathcal{E} \backslash E_0$, then the set of equilibrium prices is finite. Moreover, the restriction to $e \in \mathcal{E} \backslash E_0$ of the equilibrium price correspondence is continuous.*

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[5]Since preferences are not required to be strictly monotone, we allow for the possibility that some equilibrium prices are 0. As in Section 3, we allow for the possibility that demand is undefined at some prices. Of course, such prices cannot be equilibrium prices, so this causes no difficulty.

Proof: The proof follows Debreu (1970). Define the map $F : \Delta \times \mathbf{R}_+ \times \mathcal{E}_n \to \mathbf{R}^{LN}$ by:

$$F(p, y, e_{-1}) = \begin{pmatrix} d_1(p, y) + \sum_{n=2}^{N} d_n(p, p \cdot e_n) - e_n \\ e_2 \\ \vdots \\ e_N \end{pmatrix}.$$

Given our assumptions, for all $e \in \mathcal{E}$, the set $F^{-1}(e)$ is non-empty, and its projection onto Δ is the set of equilibrium prices when endowments are e. According to generic triviality, there is a closed, lower-dimensional $\mathcal{S}$-set $E_0 \subset E$ such that, for each of the finite number of connected components E_i of $\mathcal{E} \backslash E_0$, there is an $\mathcal{S}$-set A_i and an $\mathcal{S}$-homeomorphism $h : E_i \times A_i \to F^{-1}(A_i)$ such that $F(h(e, a)) = e$. Thus, for all $e \in E_i$, $\dim F^{-1}(e) = \dim A_i$. Now we count dimensions. Each E_i has dimension LN, and $E_i \times A_i$ is homeomorphic to a set of dimension LN, so $\dim A_i = 0$. Hence $F^{-1}(e)$ is 0-dimensional. Since it is an $\mathcal{S}$-set, it is finite. Since $F^{-1}(A_i)$ is a product, the restriction of the equilibrium price correspondence to each A_i (and hence to $\mathcal{E} \backslash E_0$) is continuous. $\qquad\square$

Note how Generic Triviality simultaneously plays the roles of Sard's Theorem and of the Implicit Function Theorem: it tells us that almost all values are "regular" and gives us a device for counting dimensions. Notice, though, that unlike the Implicit Function Theorem, we have no criterion for identifying regular values.

Rader's (1972, 1973) result on the generic determinateness of equilibrium for absolutely continuous demand applies to semi-algebraic and finitely sub-analytic exchange economies. Semi-algebraic and finitely sub-analytic functions are continuously differentiable almost everywhere and satisfy his Condition N, that the image of a null set is null. According to Rader, equilibrium is therefore locally determined for almost all endowment allocations. Our result differs in two respects from that obtained through the application of Rader's Theorem. Our additional hypotheses allow us to say more about the exceptional set of endowments (closed and lower dimensional) and more about the equilibrium correspondence (generic continuity).

In the absence of strict convexity of preferences, demand is a correspondence, rather than a function. In this case, F will also be a correspondence, rather than a function, and we work with inverse images of appropriate projections of its graphs. The proof uses the same map and counts dimensions in much the same way the proof of the preceding Theorem does, but the argument requires that the graph of the map be cut up in such a way that the dimension of the demand sets and their supporting price sets can be controlled.

Theorem 4 *If preferences are convex, then there is a closed, lower-dimensional $\mathcal{S}$-set $E_0 \subset \mathcal{E}$ such that, if $e \in \mathcal{E} \backslash E_0$, then the set of equilibrium prices is finite. Moreover, the restriction to $\mathcal{E} \backslash E_0$ of the equilibrium price correspondence is continuous.*

Proof: Define the $\mathcal{S}$-set $H \subset \Delta \times \mathbf{R}_{++} \times \mathbf{R}^{L(N-1)} \times \mathbf{R}^{LN}$ such that:

$$H = \{(p, y, e_{-1}, d_1, \ldots, d_N) \ : \ d_1 \in D_1(p, y), d_n \in D_n(p, p \cdot e_n) \quad \text{for} \quad n = 2, \ldots, N\}$$

where D_n is consumer n's demand (correspondence) with arguments price and income. Let ϕ denote the map from H to $\mathbf{R}^{\mathbf{L}}$ such that $\phi(h) = (d_1 + \sum_{n=2}^{N} d_n - e_n, e_{-1})$. The function ϕ on H is the analog to F in the previous Theorem: $(p, y, e_{-1}, d_1, \ldots, d_N)$ is in the inverse image of (z, e_{-1}) if and only if p is an equilibrium price for the economy with endowment allocation (z, e_{-1}). For a given (p, y, e_{-1}), let $y_1 = y$ and $y_n = p \cdot e_n$ for $n = 2, \ldots, N$. Let $k = (k_1, \ldots, k_N)$ be a vector of integers, and define $H_k \subset H$ to be the subset of H where consumer 1's demand has dimension k_1, and so forth:

$$H_k = \{h \in H \ : \ \dim D_1(p, y) = k_1, \ldots, \dim D_N(p, y_n) = k_N\}.$$

This set will be empty unless all $k_i \leq L - 1$. Each H_k can be defined by linear inequalities, and so each H_k is an $\mathcal{S}$-set. The sets H_k partition H.

Next, let $\succeq d_n$ denote the consumption bundles consumer n finds at least as good as d_n. Define

$$H_{k,m} = \{h \in H_k \ : \ \dim\{p \ : \ p \text{ supports each } \succeq d_n\} = m\},$$

where $m \leq L - 1$. All the $H_{k,m}$ partition H.

If the projection of the set of endowments of a particular $H_{k,m}$ has dimension less than L, the set of economies having some equilibria described by H_k is lower dimensional, and H_k can be ignored. Assume, then, that for each consumer $n = 2, \ldots, N$ the projection of H_k onto his endowments has full dimension L. Then

$$\dim H_{k,m} = m + a + L(N - 1) + \sum_{n=1}^{N} k_n,$$

where a, the dimension of the projection of $H_{k,m}$ onto consumer 1's income, is either 0 or 1.

Define ϕ on $H_{k,m}$ such that

$$\phi(h) = \left(\sum_{n=1}^{N} d_n - \sum_{n=2}^{N} e_n, e_{-1}\right).$$

The price vector p is an equilibrium price for $e = (e_1, \ldots, e_N)$ if and only if there is a y and $d = (d_1, \ldots, d_N)$ such that (p, y, e_{-1}, d) is in the inverse image of e. We can suppose that the range of ϕ has dimension NL. Applying generic triviality to ϕ, there is a closed, lower dimensional set E_0 of endowment allocations such that, on each of the finite number of connected components E_i of $\mathcal{E} \backslash E_0$, $\phi^{-1}(e)$ is an $\mathcal{S}$-set of dimension $m + a - L + \sum_{n=1}^{N} k_n$.

Now h is in $\phi^{-1}(e)$ if and only if endowments of consumers 2 through N are correct and there are $d_n \in D_n(p, y_n)$ such that $\sum_{n=1}^{N} d_n = \sum_{n=1}^{N} e_n$. The set $\sum_{n=1}^{N} D_n(p, y_n)$ has dimension no more than $L - m$, since if $d \in \sum_{n=1}^{N} D_n(p, y_n)$, then $p \cdot d - \sum_{n=1}^{N} y_n = 0$ for an m-dimensional set of prices p. Thus the set of demands summing to d has codimension $L - m$, in other words, dimension $m + a - L + \sum_{n=1}^{N} k_n$. It follows from a routine generic triviality argument that, since the dimension of $H_{k,m}$ is the same as the dimension of each "fiber" over a price p in the projection of $H_{k,m}$ onto Δ, the dimension of this projection must be 0. Continuity follows as in the proof of the previous Theorem. $\square$

The foregoing results extend the local finiteness results of Debreu to an important class of preferences which allows for kinks and flats. We have chosen to parameterize economies by endowments, rather than by endowment distributions, and for good reason: the corresponding results are not generally true in that setting. Consider, for example, an Edgeworth box economy. If consumers have identical Leontief utility functions $u_i(x_i, y_i) = \min(x_i, y_i)$, and the box is square (i.e., the aggregate endowments of the two goods are equal) then any individually rational diagonal allocation, which is to say any individually rational Pareto optimal allocation, can be supported as an equilibrium (and different allocations will be supported by different prices). Hence equilibrium prices will be indeterminate for every endowment distribution. Of course, for most endowments the box will not be square, and determinacy will be restored.

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