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# ASYMPTOTIC BEHAVIOR OF ASSET MARKETS: ASYMPTOTIC INEFFICIENCY

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## 1. INTRODUCTION

Underlying the Walrasian (Arrow-Debreu) model of economic activity are two assumptions: that agents act as price takers, and that there is a market for every commodity. When there is uncertainty about the future, the latter assumption entails a complete set of contingent claims; i.e., claims to consumption streams dependent on the future state of the world.

Arrow (1953, 1964) presents a different model, with trading in futures markets for securities (assets) whose payoffs depend on the state of the world, and in spot markets for physical commodities.<sup>1</sup> Although the Walrasian model and the security market model are formally different, Arrow shows that, if security markets are complete (i.e., if every wealth pattern can be obtained from a portfolio of available securities), the two models are equivalent: they support the same equilibrium allocations (of physical commodities). In particular, if security markets are complete, equilibrium allocations are efficient (Pareto-optimal).

If security markets are incomplete, however, the situation is quite different: the Walrasian model and the security market model are not

<sup>1</sup>Arrow considers only nominal securities; i.e., securities denominated in units of account. Radner (1972) considers a model with real securities (i.e., securities denominated in physical commodities).

equivalent. In particular, equilibrium allocations of security markets need not be Pareto-optimal.<sup>2</sup> Indeed, equilibrium allocations need not even be optimal within the set of allocations that can be obtained through trades in the given securities. [See Hart (1975), Grossman (1977), Newberry and Stiglitz (1982), Stiglitz (1982), and especially Geanakoplos and Polemarchakis (1987).]

Security markets equilibria will be efficient if assets span all the uncertainty; they may be inefficient if assets fail to span all the uncertainty. Intuition might suggest (and mine did) that security markets equilibria will be "nearly" optimal if assets span "most" of the uncertainty. The results of this paper suggest that this intuition may be wrong—badly wrong. In general, security markets equilibria may be inefficient, and remain inefficient (i.e., bounded away from Pareto-optimal allocations) even when the set of assets expands to a set that resolves all the uncertainty. Moreover, this "asymptotic inefficiency" is robust; in fact, "asymptotic inefficiency" is a generic property of asset sequences.

Our results suggest that, although a complete securities market is a perfect substitute for a Walrasian (complete contingent claims) market, a large securities market need not be a good approximation for a Walrasian market.

To establish these results, we construct a model of a securities market with a countably infinite number of states of nature.<sup>3</sup> When the number of assets is finite, we are able to prove (under appropriate assumptions about the returns on assets, and the preferences, endowments, and feasible trading sets of consumers) that such a securities market always has an equilibrium (Theorem 1). To study the behavior of equilibrium allocations when the assets span "most" of the uncertainty, we fix an infinite sequence  $(\alpha_n)$  of assets, and consider, for each  $N$ , the corresponding securities market. We say that such a sequence is *asymptotically efficient* if, for all consumer preferences and endowments (in a well-behaved class), the equilibrium allocations of the security markets involving only the assets  $(\alpha_n; 1 \leq n \leq N)$  converge to Pareto-optimal allocations of the underlying Walrasian markets. We identify a condition on an infinite sequence of assets that is necessary and sufficient that it be asymptotically efficient. Modulo a small technical caveat (that preferences be uniformly proper), sequences of assets that are asymptotically efficient are also *asymptotically complete*, in the sense that the equilibrium allocations of the security markets involving only the assets  $(\alpha_n; 1 \leq n \leq N)$  converge to equilibrium

<sup>2</sup>Of course, they may be optimal in certain circumstances; the capital asset pricing model provides a notable example.

<sup>3</sup>It would be natural to allow for continuous uncertainty, but this would give rise to serious technical difficulties that we wish to avoid; see Section 7 for further discussion.

allocations of the underlying Walrasian market (Theorem 2).<sup>4</sup> The requisite condition for asymptotic efficiency (and asymptotic completeness) is that every Arrow security can be *uniformly* approximated by the returns from a *finite* portfolio. This condition is extremely strong: the sequences of assets that *fail* to satisfy this condition constitute a residual subset of the space of all asset sequences (Theorem 3).<sup>5</sup>

We find it convenient (for technical reasons) to work with *numeraire securities*; i.e., securities denominated in a single commodity. Numeraire securities constitute a convenient halfway station between the purely financial securities of Arrow and the general securities of Radner (1972). As shown by Geanakoplos and Polemarchakis (1987), the problems of existence identified by Hart (1975) for general security models do not arise in the case of numeraire securities.<sup>6,7</sup>

Our results do not depend on pathologies in securities structures, endowments, or preferences. We assume the existence of a riskless asset, and that the returns on other assets are bounded; we could, without loss, assume that all assets have strictly positive payoffs. We assume that endowments are bounded away from zero; we could also assume that endowments are bounded above. Finally, our negative results are obtained with preferences representable by separable, strictly concave utility functions (with bounded marginal rates of substitution); similar constructions could be carried out with preferences representable by homogeneous utility functions.

The crucial idea underlying our negative results is that the requirement that terminal consumption be nonnegative places severe constraints on the set of portfolios that can be traded. Terminal wealth constraints matter.

If consumption bundles are not required to be nonnegative (i.e., if we ignore terminal wealth constraints), the situation is quite different. On one hand, for a given set of assets an equilibrium need not exist; moreover, equilibrium allocations of security markets may become unbounded as the set of available assets expands. On the other hand, limits of equilibrium allocations of the finite securities markets will be equilibrium allocations

<sup>4</sup>This method of studying the assumption of complete markets seems analogous to the familiar method of studying the assumption of price-taking behavior by consumers; see Anderson (1986) for example.

<sup>5</sup>Recall that residual sets are large, and their complements are small, so "asymptotic inefficiency" is a generic property.

<sup>6</sup>This had already been established for models involving only purely financial securities by Cass (1984), Werner (1985) and Duffie (1985).

<sup>7</sup>There would be no particular difficulty in allowing for general real securities (i.e., securities denominated in arbitrary commodity bundles), provided we follow Radner (1972) and adopt appropriate lower bounds on asset trades.

of the underlying complete markets economy, *provided* that (i) the infinite sequence  $\{\alpha_n\}$  of assets spans all the uncertainty, (ii) equilibria of the finite security markets exists, and (iii) these equilibrium allocations converge (Theorem 4).

The first research of which I am aware on the asymptotic behavior of security markets is due to Green and Spear (1987, 1989), and their thought-provoking work has provided some of the impetus for the present paper. However, the conclusions of the present paper are somewhat different from those reached by Green and Spear. For further discussion, see Section 7.

The remainder of the paper is organized in the following way. We describe the model in Section 2. Section 3 provides the basic existence theorem and its proof. Section 4 discusses asymptotic efficiency and asymptotic completeness, Section 5 presents the generic analysis, and Section 6 discusses the case of unconstrained consumption. Finally, Section 7 concludes the paper.

## 2. THE MODEL

We use a variant of the model of Geanakoplos and Polemarchakis (1987), adapted to accommodate an infinite number of possible states of the world.

Transactions occur (at date 0) in assets (or securities) before the state of nature is known, and then (at date 1) in real commodities, after the state of nature is known. (There would be no difficulty in allowing for consumption before the state of nature is known.) The state of nature is described by an atomic probability space  $(S, \sigma)$ , where  $S = \{1, 2, 3, \dots\}$  is the set of *states* of nature, and  $\sigma$  is a probability measure on  $S$ .<sup>8</sup> We assume that  $\sigma(s) > 0$  for each  $s \in S$ . (This involves no loss of generality.)

At each state there are available for consumption  $l$  physical goods,  $1, \dots, l$ , of which the first is the *numeraire*. *Commodity bundles* (or consumption patterns) are elements of the *commodity space*  $L_1(S, \sigma)^l$ ; i.e., functions  $x : S \rightarrow \mathbb{R}^l$  for which the *norm*

$$\|x\| = \int |x(s)| d\sigma(s) = \int \sum_{i=1}^l |x_i(s)| d\sigma(s)$$

<sup>8</sup>The probability  $\sigma(s)$  may be interpreted as the objective probability that state  $s$  will occur, or as the unanimous assessments of consumers, but neither of these interpretations is necessary. All that is necessary for our purposes is that assessments of consumers be consistent, in the minimal sense of allowing for the same consumption patterns.

is finite. Since  $\sigma$  is a probability measure, the norm of  $x$  is just the expectation of  $|x| = \sum |x_i|$ . We seldom distinguish between functions  $x: S \rightarrow \mathbb{R}^l$  and  $l$ -tuples  $(x_1, \dots, x_l)$  of functions  $x_i: S \rightarrow \mathbb{R}$ . Since  $S$  is countable, a function  $x_i: S \rightarrow \mathbb{R}$  may be identified with a sequence of real numbers, but it is convenient to use functional notation; we shall usually write  $x(s, i)$  rather than  $x_i(s)$ . It is frequently convenient to identify a function  $w \in L_1(S, \sigma)$  with the  $l$ -tuple  $(w, 0, \dots, 0) \in L_1(S, \sigma)^l$ ;  $w$  is a *numeraire pattern*. Given bundles  $x, y \in L_1(S, \sigma)^l$ , we write  $x \leq y$  to mean  $x(s, i) \leq y(s, i)$  for each  $i, s$ ;  $x < y$  to mean  $x(s, i) < y(s, i)$  for each  $i, s$  and  $x \neq y$ ;  $x \ll y$  to mean  $x(s, i) < y(s, i)$  for each  $i, s$ . We write  $\chi_{si}$  for the consumption pattern that represents one unit of commodity  $i$  in state  $s$  and nothing in other states; i.e.,  $\chi_{si}(s, i) = 1, \chi_{si}(r, j) = 0$  if  $(r, j) \neq (s, i)$ .

An *asset* (or *security*) is a claim to a numeraire pattern at date 1. (Thus, securities are denominated in the numeraire commodity.) The *return* on asset  $\alpha$  in state  $s$  is  $\alpha(s)$ , which may be positive, negative, or zero. We frequently use the same notation for an asset and for its returns; it should always be clear from context what is intended. We assume that asset returns are *bounded*; i.e., for each  $\alpha$ , there is a constant  $c$  such that  $|\alpha(s)| \leq c$  for each state  $s$ .<sup>9</sup> We assume that the first asset  $\alpha_1$  is *riskless*; i.e.,  $\alpha_1(s) = 1$  for each  $s \in S$ .<sup>10</sup> [We usually write 1 for this element of  $L_1(S, \sigma)$ .]

If there are  $N$  assets  $\alpha_1, \dots, \alpha_N$ , a *portfolio* is a vector  $y = (y_1, \dots, y_N) \in \mathbb{R}^N$ ;  $y_k$  is the holding (number of shares) of the  $k$ th asset, and may be positive, negative, or zero. The *return* on the portfolio  $y = (y_1, \dots, y_N)$  is the numeraire pattern:

$$\text{return}(y) = \sum y_k \alpha_k \in L_1(S, \sigma).$$

We find it convenient to write  $\delta_k$  for the portfolio consisting of one share of the  $k$ th asset and nothing else.

*Asset prices* are vectors  $q \in \mathbb{R}^N$ , where  $q_k$  is the price of the  $k$ th asset; if  $y \in \mathbb{R}^N$  is a portfolio, then  $q \cdot y = \sum q_k y_k$  is the *value* of the portfolio  $y$  at the prices  $q$ . The asset prices  $q$  *admit arbitrage* if there is a consumer  $h$  and a portfolio  $y$  such that  $ty \in Y^h$  for every  $t > 0$ ,  $\text{return}(y) > 0$  and  $q \cdot y \leq 0$ .

*Commodity prices* are functions  $p: S \rightarrow (\mathbb{R}^l)^+$ ;  $p(s, i)$  is the price of commodity  $i$  in state  $s$ . We shall always normalize so that, for each state  $s$ ,  $\sum p(s, i) = 1$ . [This is a free normalization, because there will be a different budget constraint in each state; see Geanakoplos and Polemarchakis

<sup>9</sup>The requirement that asset returns be bounded is likely unnecessary, but it is very convenient.

<sup>10</sup>This requirement too is likely unnecessary but very convenient.

(1987).] The *income pattern*  $p \square x$  required to purchase a commodity bundle  $x$  at prices  $p$  is defined by

$$p \square x(s) = \sum p(s, i)x(s, i).$$

Consumers  $h \in \{1, \dots, H\}$  are defined by *consumption sets*, *endowments*  $e^h$ , *preferences*  $\preceq^h$ , and *feasible trading sets*  $Y^h \subset \mathbb{R}^N$ . Except in Section 6, where we consider unconstrained consumption, we shall always assume that *consumption sets* for each consumer are the positive cone  $[L_1(S, \sigma)^+]^+$ ; i.e., we require terminal consumption to be nonnegative. Henceforth, we suppress explicit reference to consumption sets. We assume that individual endowments are strictly positive in every state [i.e.,  $e^h(s) > 0$  for every  $h, s$ ] and that numeraire endowments are bounded away from 0 [i.e., there is a  $\delta > 0$  such that  $e^h(s, 1) \geq \delta$  for every  $h, s$ ]. *Preferences* are (norm) continuous, convex, and strictly monotone; i.e.,  $x \preceq^h y$  whenever  $x < y$ . [Such preferences are representable by continuous, quasiconcave, strictly monotonic utility functions  $[L_1(S, \sigma)^+]^+ \rightarrow [0, \infty)$ .] We assume that the numeraire is *desirable* in every state, in the sense that, for each commodity bundle  $x \in [L^1(S, \sigma)^+]^+$  and state  $s$ , there is a  $c > 0$  such that  $x \preceq^h c\chi_{s1}$ . (Note that the right-hand side is the consumption pattern that yields  $c$  units of the numeraire in state  $s$  and nothing in other states.) The *trading set*  $Y^h \subset \mathbb{R}^N$  defines the set of portfolios that consumer  $h$  is permitted to hold (and hence the set of asset trades that  $h$  is permitted to make). We assume throughout that  $Y^h$  is a closed, convex subset of  $\mathbb{R}^N$ , that 0 belongs to the interior of  $Y^h$ , and that  $t\delta_1 \in Y^h$  for every  $t \geq 0$ . Thus, we may impose limits on short sales and on purchases, but some trading is always permitted, and arbitrarily long positions in the riskless asset are permitted.

A *securities (or asset) market* is a pair  $\mathcal{E} = (\{\alpha_k\}, \{(e^h, \preceq^h, Y^h)\})$ , where  $\{\alpha_k: 1 \leq k \leq N\}$  is a finite set of assets and  $\{(e^h, \preceq^h, Y^h): 1 \leq h \leq H\}$  is a finite set of consumers. The assumptions above are understood to be in force at all times; in particular,  $\alpha_1$  is riskless.

Given securities prices  $q$ , commodity prices  $p$ , and endowments  $e^h$ , the *budget set*  $B^h(q, p, e^h)$  for consumer  $h$  is the set of pairs  $(x^h, y^h)$  consisting of a commodity bundle  $x^h \in [L_1(S, \sigma)^+]^+$  and a portfolio  $y^h \in Y^h$  with the following properties:

- (i)  $q \cdot y^h \leq 0$  ( $y^h$  is affordable);
- (ii)  $p \square [x^h - e^h - \text{returns}(y^h)] \leq 0$  ( $y^h$  finances  $x^h$ ).

[Note that (ii) is an infinite collection of budget constraints, one for each state, but that there is no overall budget constraint.]

A (*securities market*) *equilibrium* for the securities market  $\mathcal{E}$  is a 4-tuple  $(q, p, x, y)$ , where  $q \in \mathbb{R}^N$  are asset prices,  $p: S \rightarrow (\mathbb{R}^I)^+$  are

commodity prices,  $x = (x^1, \dots, x^H)$  is the *equilibrium allocation* (so that  $x^h$  is consumer  $h$ 's equilibrium consumption bundle), and  $y = (y^1, \dots, y^H)$  is the profile of portfolios, satisfying

- (i)  $\sum (x^h - e^h) = 0$  (commodity markets clear);
- (ii)  $\sum y^h = 0$  (assets are in zero net supply);
- (iii) for all  $h$ ,  $x^h$  is  $\preceq^h$ -maximal in  $B^h(q, p, e^h)$  (consumers optimize).

Notice that equilibrium asset prices cannot admit arbitrage.

Underlying every securities market  $\mathcal{E}$  is a Walrasian (Arrow-Debreu) economy  $\mathcal{E}^{\text{CM}}$ , with the same commodities and consumers, but with complete contingent claims to every state/commodity pair. *Commodity prices* for  $\mathcal{E}^{\text{CM}}$  are nonnegative linear functionals on the commodity space; i.e., elements of the space  $[L_\infty(S, \sigma)]^+$  of nonnegative, bounded functions,  $\pi: S \rightarrow (\mathbb{R}^I)^+$ . A *Walrasian (competitive) equilibrium* for  $\mathcal{E}^{\text{CM}}$  is, as usual, a pair  $(\pi, x)$ , where  $\pi$  are *commodity prices* and  $x = (x^1, \dots, x^H)$  is the *equilibrium allocation* (so that  $x^h$  is consumer  $h$ 's equilibrium consumption bundle), satisfying

- (i)  $\sum (x^h - e^h) = 0$ ;
- (ii) for each  $h$ ,  $\pi \cdot (x^h - e^h) \leq 0$ ;
- (iii) for each  $h$ , if  $x^h \prec^h y^h$ , then  $\pi \cdot (y^h - e^h) > 0$ .

### 3. EXISTENCE OF SECURITIES MARKET EQUILIBRIUM

In this section, we show that securities market equilibria exist, provided that assets and trading sets satisfy appropriate assumptions. Our assumption on assets is that their returns are linearly independent; this is a rather innocuous assumption, and could likely be avoided altogether.<sup>11</sup> Our assumption on trading sets guarantees that liabilities on asset holdings can exceed endowments in only a finite number of states; this assumption is

<sup>11</sup>Geanakoplos and Polemarchakis (1987), for instance, allow for redundant assets. However, they assume that trading sets are equal to  $\mathbb{R}^N$ , in which case redundant assets can be priced by arbitrage, and every equilibrium has the same allocation of consumption goods as an equilibrium in which redundant assets are not traded. When trading sets are restricted, however, redundant assets cannot always be priced by arbitrage, and trading in redundant assets cannot always be eliminated.

not so innocuous. However, Mas-Colell and Zame (1991b) have shown that, without this assumption or one like it, equilibrium need not exist.<sup>12</sup> We refer the reader to that paper for discussion and a specific counterexample.

**THEOREM 1.** *Let  $\mathcal{E} = (\{\alpha_k\}, \{(e^h, \preceq^h, Y^h)\})$  be a securities market. Assume that asset returns are linearly independent and that, for every  $h$  and every  $y^h \in Y^h$ , there is an index  $n$  such that  $[\text{ret}(y^h) + e^h](s) \geq 0$  for every  $s \geq n$ . Then  $\mathcal{E}$  has an equilibrium.*

*Proof.* We construct a securities market equilibrium for  $\mathcal{E}$  as the limit of securities market equilibria for finite-state securities markets that approximate  $\mathcal{E}$ .

The first step is to construct these finite-state markets and their equilibria. To this end, fix a positive integer  $n$ , and write  $S_n = \{1, \dots, n\}$ , the first  $n$  states. Let  $L_n$  be the space of functions from  $S_n$  to  $\mathbb{R}$ , and let  $L_n^l$  be the space of functions from  $S_n$  to  $\mathbb{R}^l$ . We identify  $L_n$  (respectively  $L_n^l$ ) with the subspace of  $L_1(S, \sigma)$  [respectively  $L_1(S, \sigma)^l$ ] consisting of functions that vanish off  $S_n$ . Let  $P_n: L_1(S, \sigma) \rightarrow L_n$  and  $Q_n: L_1(S, \sigma)^l \rightarrow L_n^l$  be the projections [so that  $P_n(w)$  is the restriction of  $w$  to  $S_n$ , etc.].

For each  $n$ , we let  $\mathcal{E}_n$  be the securities market with state space  $S_n$ , commodity space  $L_n^l$ , assets  ${}_n\alpha_k = P_n(\alpha_k)$ , endowments  ${}_ne^h = Q_n(e^h)$ , preferences the restrictions to  $L_n^l$  of the given preferences, and trading sets  $Y^h$ . By a result of Geanakoplos and Polemarchakis (1987),  $\mathcal{E}_n$  has a securities market equilibrium  $(q_n, p_n, {}_nx, {}_ny)$ .<sup>13</sup> Without loss of generality, we may assume that the riskless asset  $\alpha_1$  has price 1; i.e.,  $q_n \cdot \delta_1 = 1$ . This completes the first step.

The second step is to extract a convergent subsequence of the sequence  $\{(q_n, p_n, {}_nx, {}_ny)\}$  of securities market equilibria. Note first that the equilibrium bundles  ${}_nx^h$  are nonnegative and sum to the aggregate endowments  $\Sigma e^h$ . Passing to a subsequence if necessary, we may (because  $S$  is countable) assume that the equilibrium bundles converge; i.e., there are bundles  $x^h \in L_1(S, \sigma)^l$  such that  ${}_nx^h(s, i) \rightarrow x^h(s, i)$  for each  $s, i$ . Note that  ${}_nx^h \rightarrow x^h$  in the norm topology of  $L_1(S, \sigma)$  (again because  $S$  is countable and because all the bundles  ${}_nx^h$  are bounded by  $\Sigma e^h$ ). Write  $x = (x^1, \dots, x^H)$ .

Since we have normalized the state prices to sum to 1, we may (again passing to a subsequence if necessary) also assume that the state prices  $p_n(s, i)$  converge; say,  $p_n(s, i) \rightarrow p(s, i)$ . If  $p(s, i) = 0$  for any state  $s$  and

<sup>12</sup>This fact was not realized in earlier versions of the present paper.

<sup>13</sup>Geanakoplos and Polemarchakis assume that trading sets are equal to  $\mathbb{R}^N$ , and that asset returns are linearly independent. However, their argument depends only on the existence of a suitable cone of no-arbitrage prices; our assumptions are sufficient for this purpose.

commodity  $i$ , then, for  $n$  sufficiently large, the demands in state  $s$  would be unbounded. Hence,  $p(s, i) \neq 0$  for each  $s, i$ .

We assert that equilibrium security prices are bounded. To see this, fix an index  $k$ , and a consumer  $h$ . Since  $0$  belongs to the interior of  $Y^h$ ,  $t\delta_1 \in Y^h$  for each  $t > 0$ , and  $Y^h$  is closed, we can find a constant  $r > 0$  such that  $t\delta_1 - r\delta_k \in Y^h$  for each  $t > 0$ . Numeraire desirability implies that there is a constant  $c > 0$  such that  $h$  prefers  $c\chi_{11}$  to society's aggregate endowment. Since  $\alpha_k$  is bounded and  $\alpha_1$  is riskless, there is a constant  $c' > 0$  such that  $\alpha_1 + c'\alpha_k \geq 0$  and  $\alpha_1 - c'\alpha_k \geq 0$ ; hence  ${}_n\alpha_1 + c'({}_n\alpha_k) \geq 0$  and  $\alpha_1 - c'({}_n\alpha_k) \geq 0$  for each  $n$ . We have normalized so that  $q_n \cdot \delta_1 = 1$ ; if  $q_n \cdot \delta_k > c' + c/r$ , then the portfolio  $(c + c'r)\delta_1 - r\delta_k$  is affordable and yields a return of more than  $c$  units of the numeraire in state 1. Since  $h$  prefers  $c\chi_{11}$  to society's aggregate endowment, this is incompatible with equilibrium. We conclude that  $q_n \cdot \delta_k \leq c' + c/r$  for each  $k$  and  $n$ . Similarly,  $q_n \cdot \delta_k \geq -c' - c/r$ . In particular, the sequence  $\{q_n\}$  of securities prices lies in a bounded subset of  $\mathbb{R}^N$ . Passing to a subsequence if necessary, we may assume that the securities prices  $q_n$  also converge to a limit, say,  $q_n \rightarrow q$  for some  $q \in \mathbb{R}^N$ . Note that  $q \cdot \delta_1 = 1$ .

To show that the equilibrium portfolios  ${}_n y$  lie in a bounded subset of  $\mathbb{R}^N$ , we show first that the price system  $q$  does not admit arbitrage for the securities  $\{\alpha_k\}$ . To see this, let  $\gamma$  be a portfolio of the securities  $\{\alpha_k\}$ , with returns  $(\gamma) = g$ ; assume that  $t\gamma \in Y^h$  for all  $t > 0$  and that  $g(s) \geq 0$  for every  $s$ , and that  $g(r) > 0$  for some state  $r$ . Without loss, we may assume that  $g(r) = 1$ . For each  $n \geq r$ ,  $\gamma$  may be viewed as a portfolio of the securities  $\{{}_n\alpha_k\}$ , with returns  ${}_n g = P_n(g)$ . In particular,  ${}_n g(r) = g(r) = 1$ . By assumption,  $0 \in \text{int } Y^h$  and numeraire endowments are bounded away from 0, so there is a  $\delta > 0$  such that  $-\delta\alpha_1 \in Y^h$  and  $e^h(1, s) > \delta > 0$  for each state  $s$ . Since  $t\gamma \in Y^h$  for each  $t > 0$ , convexity implies that  $(\frac{1}{2})t\gamma - (\frac{1}{2})\delta\alpha_1 \in Y^h$ . If we choose  $t$  sufficiently large, numeraire desirability implies that the returns on this feasible portfolio, together with consumer  $h$ 's endowment, are preferred to any feasible consumption bundle. Hence, this portfolio cannot be affordable at equilibrium. Thus,  $q_n \cdot [(\frac{1}{2})t\gamma - (\frac{1}{2})\delta\alpha_1] > 0$  for each  $n$ , so that  $q_n \cdot \delta > (\delta/t)$ . Passing to the limit yields  $q \cdot \gamma > (\delta/t)$ ; in particular,  $q$  does not admit arbitrage.

We can now show that, for each  $h$ , the equilibrium portfolios  ${}_n y^h$  remain bounded (as  $n \rightarrow \infty$ ). For, suppose not. Passing to a subsequence if necessary, we may assume that the portfolio directions  ${}_n y^h / \|{}_n y^h\|$  have a limit  $\tau \in \mathbb{R}^N$ . Since  $\tau$  is the limit of portfolios of norm 1, it, too, has norm 1; in particular,  $\tau$  is not the 0 portfolio. Since  $q_n \cdot {}_n y^h = 0$ , it follows that  $q_n \cdot ({}_n y^h / \|{}_n y^h\|) = 0$  and hence that  $q \cdot \tau = 0$ . Since the portfolios  ${}_n y^h$  become unbounded (as  $n \rightarrow \infty$ ), it follows that  $t\tau \in Y^h$  for every  $t > 0$ . Since  $q$  does not admit arbitrage, it is impossible that  $\tau$  has returns that are nonnegative, and strictly positive in some state. We have assumed that the returns  $\alpha_1, \dots, \alpha_N$  are linearly independent, so that, for some  $m$ , the

returns  $P_m(\alpha_1), \dots, P_m(\alpha_N)$  are also linearly independent. This means that no nonzero portfolio of  $\alpha_1, \dots, \alpha_N$  can have 0 returns in the states  $1, \dots, m$ ; in particular,  $\tau$  cannot have 0 returns (because  $\tau \neq 0$ ). Hence  $\tau$  must have negative returns in some state  $s$ . The portfolios  ${}_n y^h$  are unbounded, and the state prices  $p_n(s, i)$  converge to nonzero limits; hence, for  $n$  sufficiently large, the returns on the portfolios  ${}_n y^h$  create liabilities in state  $s$  that cannot be satisfied. Since this contradicts the equilibrium conditions, we conclude that the portfolios  ${}_n y^h$  remain bounded, as asserted.

Passing once again to a subsequence if necessary, we may assume that the portfolios  ${}_n y^h$  also converge; say,  ${}_n y^h \rightarrow y^h$  for each  $h$ . Write  $y = (y^1, \dots, y^H)$ . This yields a limit 4-tuple  $(q, p, x, y)$ , completing the second step.

The third step is to show that  $(q, p, x, y)$  is a securities market equilibrium for  $\mathcal{E}$ . It is routine to verify that assets are in zero net supply and that commodity markets clear, that all portfolios are feasible and affordable, and that a consumption bundle is in each consumer's budget set. It remains only to verify optimality of portfolios and consumption bundles.

To this end, suppose that consumer  $h$ 's portfolio and consumption bundle are not optimal (in  $h$ 's budget set). Then there is a consumption plan  $x$  such that  $x \succ^h x^h$  and  $x$  is financed (at prices  $p$ ) by a portfolio  $y \in Y^h$  with  $q \cdot y \leq 0$ . We first construct a consumption plan  $x^*$  and a portfolio  $y^* \in Y^h$  having the following properties:

- (i) there is a  $k$  such  $x^*(s) = 0$  for all  $s \geq k$ ;
- (ii)  $x^* \succ^h {}_n x^h$  for all sufficiently large  $n$ ;
- (iii)  $q \cdot y^* < 0$ ;
- (iv)  $p \square [x^* - \text{returns}(y^*) - e^h](s) < 0$  for each  $s$ .

To achieve this, write  $\zeta = \text{returns}(y)$ , and let  $t$  be a real number (to be chosen later), with  $0 < t < 1$ . Since  $y$  finances  $x$ , we have  $p \square [x - \zeta - e^h] \leq 0$ , so

$$p \square [tx - t\zeta - te^h] = t\{p \square [x - \zeta - e^h]\} \leq 0.$$

Note that  $tx - t\zeta - te^h = tx + (1-t)e^h - t\zeta - e^h$ , so that the portfolio  $ty$  finances the consumption plan  $x_1 = tx + (1-t)e^h$ . Write  $\rho_1 = \text{returns}(\alpha_1)$ ;  $\rho_1$  is the consumption plan that yields 1 unit of the numeraire in each state. Since  $x \geq 0$  and numeraire endowments are bounded away from 0,  $x_2 = x_1 - 2\epsilon\rho_1 \geq 0$  for  $\epsilon > 0$  sufficiently small. Convexity of  $Y^h$  and the fact that  $0 \in \text{int } Y^h$  together imply that  $y_2 = ty - 2\epsilon\alpha_1$  belongs to  $Y^h$  (for  $2\epsilon < 1-t$ ); evidently,  $y_2$  finances  $x_2$ . If we choose  $t$  sufficiently close to 1 and  $\epsilon$  sufficiently small, continuity of preferences yields  $x_2 \succ^h x^h$ . For  $k$  an index to be chosen, define the consumption plan  $x^*$  by

$x^*(s) = x_2(s)$  for  $s \leq k$  and  $x^*(s) = 0$  for  $s > k$ . Continuity of preferences implies that, for  $k$  sufficiently large,  $x^* \succ^h x^h$ , and hence that  $x^* \succ_n^h x^h$  for  $n$  sufficiently large. Set  $y^* = ty - \varepsilon \alpha_1$ . The consumption plan  $x^*$  and the portfolio  $y^*$  satisfy all our requirements.

We claim that, for  $n \geq k$  sufficiently large,  $y^*$  finances  $x^*$  at the prices  $p_n$ . To see this, note that our assumption about trading sets implies that there is an index  $m$  such that  $[\text{returns}(y^*) + e^h](s) \geq 0$  for each  $s \geq m$ ; without loss, choose  $m \geq k$ . Since  $x^*(s) = 0$  for  $s \geq k$ , it follows that  $p_n \square [x^* - \text{returns}(y^*) - e^h](s) \leq 0$  for each  $s \geq m$  and every  $n$ . On the other hand, since  $p \square [x^* - \text{returns}(y^*) - e^h](s) < 0$  for each  $s$ , and  $p_n \rightarrow p$ , it follows that, for  $n$  sufficiently large,  $p_n \square [x^* - \text{returns}(y^*) - e^h](s) < 0$  for each  $s \leq m$ . Hence  $y^*$  finances  $x^*$  at prices  $p_n$ , for  $n$  sufficiently large. Since  $q_n \rightarrow q$  and  $q \cdot y^* < 0$ , it follows that  $q_n \cdot y^* < 0$  for  $n$  sufficiently large. If we keep in mind that  $x^*(s) = 0$  for  $s > n$ , we may interpret these statements in the securities market  $\mathcal{E}_n$  to conclude that  $y^*$  is a feasible and affordable (and security prices  $q_n$ ) portfolio that finances (at commodity prices  $p_n$ ) the consumption plan  $x^*$ , which is preferred to  $x^h$ . However, this contradicts the equilibrium conditions for the securities market  $\mathcal{E}_n$ . We conclude that portfolios and consumption bundles are optimal (given prices), so that  $(q, p, x, y)$  is an equilibrium for the securities market  $\mathcal{E}$ , as asserted. This completes the proof. ■

#### 4. ASYMPTOTICS

The result of the previous section guarantees that securities market equilibria exist; in this section we study the asymptotic behavior of such equilibria as the set of securities grows. We ask: When do equilibrium allocations of the securities markets converge to Pareto-optimal allocations or to Walrasian (competitive equilibrium) allocations of the underlying Walrasian economy?

Before formalizing these questions, we address a small but important point. As Mas-Colell (1986) has pointed out, the usual assumptions on preferences and endowments that suffice to guarantee the existence of Walrasian equilibrium in the finite-dimensional setting do not suffice in infinite-dimensional settings such as ours. The difficulty is that the consumption sets of consumers are assumed to be the positive cone, which has an empty interior. This leaves open the possibility that individual preferred sets may not be supportable by prices; in such a case, competitive equilibria need not exist. To avoid this difficulty, Mas-Colell introduced a restriction on preferences that he called *uniform properness*; in essence, uniform properness bounds marginal rates of substitution. In conjunction with the usual assumptions on preferences and endowments, uniform properness suffices to guarantee the existence of competitive equilibria. If

we hope to show that equilibrium allocations of securities markets converge to equilibrium allocations of the underlying Walrasian economy, we must surely make assumptions that are strong enough to guarantee the existence of Walrasian equilibria. The easiest way to do this is to assume the preferences are uniformly proper, and that is what we shall do.<sup>14</sup>

To formalize our questions about asymptotic behavior of securities market equilibria, fix consumer endowments and preferences  $(e^h, \preceq^h)$ ,  $h = 1, \dots, H$ , and an infinite sequence  $\{\alpha_k\}$  of assets (of which the first is riskless). For each  $N$ , we consider a security market  $\mathcal{E}_N = (\{\alpha_k: 1 \leq k \leq N\}, \{(e^h, \preceq^h, {}_N Y^h)\})$  in which the first  $N$  assets are available for trade. (Note that trading sets  ${}_N Y^h$  depend on the assets available for trade. For the purposes of definition, we make no restrictions on the way trading sets change with the number of available assets.) We say that this sequence  $\{\mathcal{E}_N\}$  of securities markets is *asymptotically efficient* if, for each  $\varepsilon > 0$ , there is an integer  $N_0$  such that: for  $N \geq N_0$ , every equilibrium allocation of  $\mathcal{E}_N$  is within  $\varepsilon$  (in norm) of a Pareto-optimal allocation of the underlying Walrasian economy  $\mathcal{E}^{\text{CM}}$ . [Since the space of states of nature is countable, all these allocations lie in a norm compact subset of the commodity space  $L_1(S, \sigma)$ , so this requirement is equivalent to the requirement that equilibrium allocations of  $\mathcal{E}_N$  converge, as  $N$  tends to  $\infty$ , to Pareto-optimal allocations of  $\mathcal{E}^{\text{CM}}$ . Norm continuity of preferences implies that the utilities of equilibrium allocations of  $\mathcal{E}_N$  also converge to the utilities of Pareto-optimal allocations.] We say that the sequence  $\{\mathcal{E}_N\}$  is *asymptotically complete* if, for each  $\varepsilon > 0$ , there is an integer  $N_0$  such that, for  $N \geq N_0$ , every equilibrium allocation of  $\mathcal{E}_N$  is within  $\varepsilon$  of an equilibrium allocation of the underlying Walrasian economy  $\mathcal{E}^{\text{CM}}$ . (Similarly, this is equivalent to the requirement that equilibrium allocations of  $\mathcal{E}_N$  converge, as  $N$  tends to  $\infty$ , to equilibrium allocations of  $\mathcal{E}^{\text{CM}}$ . Again, norm continuity of preferences implies that the utilities of equilibrium allocations of  $\mathcal{E}_N$  also converge to the utilities of competitive equilibrium allocations of  $\mathcal{E}^{\text{CM}}$ .) If the sequence  $\{\mathcal{E}_N\}$  of securities markets is not asymptotically efficient (respectively not asymptotically complete), we say it is *asymptotically inefficient* (respectively *asymptotically incomplete*).

Such sequences of securities markets are the appropriate objects of study if we view assets, endowments, preferences, and trading sets as given. Alternatively, we may view assets as given (known), but endowments, preferences, and trading sets as variable (unknown).<sup>15</sup> Taking the latter point of view, we shall say that the sequence  $\{\alpha_k\}$  of assets is *asymptotically efficient* (respectively *asymptotically complete*) if for all specifications  $\{(e^h, \preceq^h)\}$  of endowments and (uniformly proper) preferences, there are trading sets  ${}_N Y^h$  for which the sequence  $\{\mathcal{E}_N\}$  of securities

<sup>14</sup>For more on the meaning of uniform properness, see Richard and Zame (1986).

<sup>15</sup>Of course, many other points of view are also possible.

markets is asymptotically complete (respectively asymptotically efficient); otherwise  $\{\alpha_k\}$  is *asymptotically incomplete* (respectively *asymptotically inefficient*).<sup>16</sup>

From this point of view, the basic questions are: When are asset sequences asymptotically efficient? When are they asymptotically complete? Theorem 2 provides complete answers to these questions (and its proof yields insights into various other questions). Before giving the formal statement, we collect some notation and terminology.

For  $x, y \in L_1(S, \sigma)$ , we define  $d_x(x, y) = \sup_{s \in S} |x(s) - y(s)|$ . Of course, this supremum will be infinite if  $|x - y|$  is an unbounded function. Nevertheless, this "distance function" induces a well-defined (complete, metrizable) topology on  $L_1(S, \sigma)$ , which we call the *uniform topology*. If  $E$  is a subset of  $L_1(S, \sigma)$ , we denote its closure with respect to the uniform topology by  $\text{cl}_\infty(E)$ ;  $x \in \text{cl}_\infty(E)$  if and only if  $x$  can be *uniformly approximated* by elements of  $E$ . The distance from a point  $x$  to a set  $E$  is  $d_x(x, E) = \inf\{d_x(x, y) : y \in E\}$ . Note that  $d_x(x, E) = 0$  exactly when  $x \in \text{cl}_\infty(E)$ . By an *Arrow security* (for state  $s$ ) we mean the security  $\chi_{s,1}$  whose return is 1 in state  $s$  and 0 in every other state. (Note that this is in agreement with our previous usage.)

The following result completely characterizes asymptotic completeness and asymptotic efficiency for a sequence of assets.

**THEOREM 2.** *Let  $\{\alpha_k\}$  be a sequence of assets, of which the first is riskless. The following statements are equivalent:*

- (i) *the sequence  $\{\alpha_k\}$  is asymptotically efficient;*
- (ii) *the sequence  $\{\alpha_k\}$  is asymptotically complete;*
- (iii) *every Arrow security can be uniformly approximated by the returns on a finite portfolio of the securities  $\alpha_k$ .*

The last condition may be formulated equivalently as follows: For every state  $s$  and every  $\varepsilon > 0$ , there is a finite portfolio of the assets  $\{\alpha_k\}$  whose returns differ from  $\chi_{s,1}$  by at most  $\varepsilon$  in every state.

*Proof.* (iii)  $\Rightarrow$  (i) and (ii). Consider a sequence  $\{\alpha_k\}$  of assets (of which the first is riskless) having the property that every Arrow security can be uniformly approximated by the returns on a finite portfolio. Fix endowments and preferences  $\{(e^h, \preceq^h)\}$ . For each  $N$  and  $h$ , define the trading set  ${}_N Y^h$  as the set of portfolios  $y \in \mathbb{R}^N$  such that  $[\text{returns}(y) + e^h](s) \geq 0$  for  $s \geq N$ . Assume we are given a subsequence  $\mathcal{E}_{N(n)}$  of  $\{\mathcal{E}_N\}$ , and equilibrium allocations  $x_n = ({}_n x^1, {}_n x^2, \dots, {}_n x^H)$  for  $\mathcal{E}_{N(n)}$ , converging

<sup>16</sup>Because we only require convergence to Pareto-optimal allocations (or Walrasian allocations) for *some* sequence of trading sets, we have somewhat slanted these definitions in the direction of efficiency and completeness. This is consistent with our goal, which is to show that efficiency and completeness are difficult to obtain.

to  $x = (x^1, x^2, \dots, x^H)$ ; let  $q_n$  be the corresponding asset prices and let  $p_n$  be the corresponding state prices. We proceed by showing that  $x$  is in the core of the underlying Walrasian economy  $\mathcal{E}^{CM}$ .

Suppose this were not so. Then there would be a set of consumers, which we may assume to be the consumers  $M = \{1, 2, \dots, M\}$ , and an allocation  $x^* = (x^{*1}, \dots, x^{*M})$  that is a redistribution of the endowments  $(e^1, \dots, e^M)$  and is unanimously preferred to  $x$  by consumers in  $M$ . Continuity of preferences, together with the assumption that numeraire endowments are bounded away from zero, guarantees that we can find a state  $r$ , allocations  $z = (z^1, \dots, z^M)$  and  $\bar{z} = (\bar{z}^1, \dots, \bar{z}^M)$ , and a real number  $\delta > 0$  such that  $\bar{z}$  is unanimously preferred to  $x$  by consumers in  $M$  and

$$\begin{aligned} \bar{z}^m(s, i) &= z^m(s, i) = 0 && \text{for } s > r, 1 \leq i \leq l; \\ \bar{z}^m(s, i) &= z^m(s, i) && \text{for } s \leq r, 2 \leq i \leq l; \\ \bar{z}^m(s, 1) &= z^m(s, 1) - \delta && \text{for } s \leq r; \\ e^m(s, 1) &\geq \delta && \text{for all } s; \\ \sum_m z^m(s, i) &= \sum_m x^{*m}(s, i) = \sum_m e^m(s, i) && \text{for } s \leq r, 1 \leq i \leq l. \end{aligned}$$

Since we have normalized the state prices to sum to 1, we may (passing to a subsequence if necessary) assume that the state prices  $p_n(s, i)$  converge; say,  $p_n(s, i) \rightarrow p(s, i)$ . As noted in the proof of Theorem 1, if  $p(s, i) = 0$  for any state  $s$  and commodity  $i$ , then, for  $n$  sufficiently large, the state  $s$  demands in  $\mathcal{E}_{N(n)}$  would exceed total endowments. Hence,  $p(s, i) \neq 0$  for each  $s, i$ . (The limiting behavior of asset prices is irrelevant.)

For each price system  $p^*$  and each  $m \in M$ , define a numeraire pattern  $w^m(p^*)$  by

$$\begin{aligned} w^m(p^*, s) &= [1/p^*(s, 1)][p^* \square (z^m - e^m)](s) && \text{for } s \leq r; \\ w^m(p^*, s) &= 0 && \text{for } s > r. \end{aligned}$$

For  $s \leq r$ ,  $w^m(p^*, s)$  is the amount of numeraire that must be transferred into state  $s$  in order to make the net purchase  $(z^m - e^m)(s)$ , at prices  $p^*$ . Since  $z$  is a reallocation of endowments in states  $s \leq r$ , and  $w^m(p^*, s) = 0$  in states  $s > r$ , it follows that, for each price system  $p^*$  and each state  $s$ ,  $\sum w^m(p^*, s) = 0$  (summation over  $m \in M$ ).

For each  $m$ ,  $w^m(p)$  is a finite linear combination of Arrow securities, so (iii) enables us to find finite portfolios  $y^1, \dots, y^{M-1}$ , such that  $d_x(\text{returns}(y^m), w^m(p)) < \delta/M$  for  $1 \leq m \leq M - 1$ . The definition of the trad-

ing sets  ${}_N Y^h$  implies that these portfolios are feasible if  $N$  is sufficiently large. Set

$$y^M = - \sum_{m=1}^{M-1} y^m$$

so that  $y^M$  is a finite portfolio, and  $d_x(\text{returns}(y^M), w^M(p)) < \delta$ . Since the state prices  $p_n(s, i)$  converge to  $p(s, i)$  for each  $s, i$  and  $w^m(p^*, s) = 0$  for states  $s > r$ , we conclude that  $d_x(\text{returns}(y^m), w^m(p_n)) < \delta$ . For each  $m \in M$ , provided that  $n$  is sufficiently large.

Our construction guarantees that, at all prices  $p_n$  sufficiently close to  $p$ , the portfolio  $y^m$  is feasible for consumer  $m$  (i.e., it does not impose unsatisfiable liabilities). Since we have constructed  $y^M$  so that  $\sum y^m = 0$ , at least one of the portfolios  $y^m$  must have a nonpositive price (at asset prices  $q_n$ ). However, at prices  $p_n$  sufficiently close to  $p$ , the returns  $\zeta^m$  on the portfolio  $y^m$  will finance purchase of the commodity bundle  $\bar{z}^m$ . Since  ${}_n x^m \rightarrow x^m$  and  $x^m \prec^m \bar{z}^m$ , continuity of preferences implies that  ${}_n x^m \prec^m \bar{z}^m$  for  $n$  sufficiently large. Since  $\bar{z}^m$  belongs to the budget set of consumer  $m$ , this contradicts the equilibrium conditions in  $\mathcal{E}_{N(N)}$ . It follows that  $x$  is in the core of the underlying complete markets economy, as desired.

We have just proved that (iii) implies that every limit of equilibrium allocations of the finite securities markets is in the core of the underlying Walrasian economy. Since allocations in the core are Pareto-optimal, this certainly yields asymptotic efficiency, and we obtain the implication (iii)  $\Rightarrow$  (i).

If we replicate the economy, and note that a securities market equilibrium for the original economy is necessarily a securities market equilibrium for the replicated economy, we conclude that every limit of equilibrium allocations of the finite securities markets is in the core of every replication of the underlying complete markets economy. We can now apply a result of Aliprantis *et al.* (1987), which is the infinite-dimensional version of the Debreu and Scarf (1963) core convergence theorem; assuming that preferences are uniformly proper, equilibrium allocations are precisely those in the core of every replication. This yields asymptotic completeness, and we obtain the implication (iii)  $\Rightarrow$  (ii).

(i)  $\Rightarrow$  (iii) and (ii)  $\Rightarrow$  (iii). We establish the contrapositives. Suppose that some Arrow security, say  $\chi_{11}$  (without loss), cannot be uniformly approximated by returns on a finite portfolio. Note that the set of returns on finite portfolios constitutes a linear subspace of  $L_1(S, \sigma)$  coinciding with the linear span  $\text{span}\{\alpha_k\}$  of the securities. Set  $4\rho = d_x(\chi_{11}, \text{span}\{\alpha_k\}) > 0$ . We find two consumers so that the equilibrium allocations of the corresponding securities markets are bounded away from Pareto-optimal allocations of the underlying Walrasian economy, independently of the choice of trading sets. It is convenient to give the construction first for the

case of one commodity (the numeraire) in each state; the general case requires only a simple adaptation.

For the one commodity case, let  $u: [0, \infty) \rightarrow [0, \infty)$  be any continuously differentiable, strictly concave function such that  $u'(0) < \infty$  and  $u'(\infty) > 0$ . The two consumers will have identical utility functions  $U^1 = U^2 = U$ , where  $U(x) = \sum a_k u(x(k))\sigma(k)$ , and  $\{a_k\}$  is a bounded sequence of strictly positive numbers, to be chosen later.<sup>17</sup> Endowments are  $e^1 = (3, \rho, \rho, \rho, \dots)$ ,  $e^2 = (1, 3\rho, \rho, \rho, \dots)$ . Write  $e = e^1 + e^2$  for the aggregate endowment. Since consumers have identical, separable, strictly concave utility functions, it is easy to see that every Pareto-optimal allocation is of the form  $(\lambda e, (1 - \lambda)e)$  for some real number  $\lambda$  with  $0 \leq \lambda \leq 1$ . To obtain the Pareto-optimal allocation  $(\lambda e, (1 - \lambda)e)$  requires the net trade (for consumer 1):

$$\lambda e - e^1 = (4\lambda - 3, (4\lambda - 1)\rho, (2\lambda - 1)\rho, (2\lambda - 1)\rho, \dots).$$

Equilibrium allocations of securities markets (and hence their limits) are individually rational, so any Pareto-optimal allocation that is the limit of equilibrium allocations of the securities markets must also satisfy the individual rationality requirements  $U(\lambda e) \geq U(e^1)$  and  $U((1 - \lambda)e) \geq U(e^2)$ . By choosing the coefficient sequence  $\{a_k\}$  appropriately, we can guarantee that these inequalities are satisfiable only if  $\lambda$  is very close to  $\frac{1}{2}$ . (We leave the details to the reader.)

Now consider a securities market  $\mathcal{E}_N = (\{\alpha_k: 1 \leq k \leq N\}, \{(e^h, \leq^h)\})$ , and an equilibrium allocation  $(x^1, x^2)$  of  $\mathcal{E}_N$ ; suppose that  $(x^1, x^2)$  is close (in the  $L_1$  norm) to a Pareto-optimal allocation. If we choose  $\{a_k\}$  so that  $\lambda$  is close to  $\frac{1}{2}$ , then  $x^1$  must be close (in the  $L_1$  norm) to  $\lambda e$ , for some  $\lambda$  close to  $\frac{1}{2}$ , and the net trade of consumer 1, which is  $x^1 - e^1$ , must be close (in the  $L_1$  norm) to  $\lambda e - e^1$ . If  $\lambda$  is close to  $\frac{1}{2}$ , the net trade of consumer 1 must be close to 1 in the first state, be close to  $\rho$  in the second state, and be bounded by  $\rho$  in every other state (since neither consumer 1 nor consumer 2 can incur liabilities greater than endowments). Hence, we can obtain [by choosing  $\{a_k\}$  so that  $\lambda$  is sufficiently close to  $\frac{1}{2}$ ] that  $d_x(x^1 - e^1, \chi_{11}) \leq 2\rho$ . Since there is only one commodity in each state, equilibrium trades in the securities market  $\mathcal{E}_N$  must be effected entirely through transactions in available securities. In particular, we can find a finite portfolio whose returns are precisely  $x^1 - e^1$ , and hence differ from  $\chi_{11}$  by at most  $2\rho$ . This contradicts our supposition that  $d_x(\chi_{11}, \text{span}\{\alpha_k\}) = 4\rho$ . We conclude that (for appropriately chosen  $\{a_k\}$ ), no equilibrium allocation of the securities market

$$\mathcal{E}_N = (\{\alpha_k: 1 \leq k \leq N\}, \{(e^h, \leq^h)\})$$

<sup>17</sup>Any choice of the sequence  $\{a_k\}$  will lead to a well-defined utility function  $U$  on  $L_1(S, \sigma)$ ;  $U$  will be uniformly proper if we choose the sequence  $\{a_k\}$  bounded away from 0.

can be close to a Pareto-optimal allocation of the underlying Walrasian economy; since Walrasian allocations are Pareto-optimal, it follows a fortiori that no equilibrium allocation of the securities market  $\mathcal{E}_N = (\{\alpha_k: 1 \leq k \leq N\}, \{e^h, \preceq^h\})$  can be close to a Walrasian allocation of the underlying Walrasian economy. This completes the proofs of (iii)  $\Rightarrow$  (i) and (ii) for the case of one commodity in each state.

To obtain the case of  $l$  commodities from the case of one commodity, we simply choose utility functions so that, in each state, all  $l$  commodities are perfect substitutes. (Of course, this is incompatible with strict concavity of utility functions, but strict concavity can be restored by making a tiny perturbation. Again, we leave the details to the reader.) ■

The argument of Theorem 2 actually proves a bit more. The argument that equilibrium allocations of  $\mathcal{E}_N$  converge to a core allocation (and in particular, to a Pareto-optimal allocation) of the underlying Walrasian economy requires only that for every numeraire pattern  $x$  that is nonzero in only a finite number of states, every  $\varepsilon > 0$ , every state  $s_0$ , every consumer  $h$ , and all sufficiently large indices  $N_0$ , there is portfolio  $y \in_N Y^h$  whose returns differ from  $x$  by at most  $\varepsilon$  in states  $s \leq s_0$ , and are bounded (in absolute value) in every other state by numeraire endowments divided by the number of consumers. Informally: If the assets span a norm dense subspace of  $L_1(S, \sigma)$ , endowments are "large enough," and sufficiently large shorts sales are permissible, then securities markets equilibria will converge to equilibria of the underlying complete markets economy (and a fortiori to Pareto-optimal allocations). However, these are stringent requirements; see Example 4 below and Section 5, particularly the concluding discussion.

We now give a number of examples. [See also Green and Spear (1989) and Zame (1990).] It is convenient to describe an asset sequence by an infinite matrix, the  $j$ th column of which represents returns on the  $j$ th asset (so that the entry in the  $i$ th row and  $j$ th column is the return paid by the  $j$ th asset in the  $i$ th state, etc.).

EXAMPLE 1.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$A$  is asymptotically efficient; of course, it is simply a riskless security, together with a complete set of Arrow securities.

EXAMPLE 2.

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$\mathbf{B}$  is asymptotically efficient. This is merely to illustrate the point that asymptotic efficiency depends only on the space spanned by the assets; note that the space spanned by the first  $n$  columns of  $\mathbf{B}$  is precisely the same as that spanned by the first  $n$  columns of  $\mathbf{A}$ .

EXAMPLE 3.

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$\mathbf{C}$  is asymptotically inefficient. Indeed, the distance from  $\chi_{11}$  to the span of the columns of  $\mathbf{C}$  is  $\frac{1}{2}$ .

EXAMPLE 4.

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & -1 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 0 & -1 & 1 & 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & 0 & -1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$\mathbf{D}$  is asymptotically inefficient; as in Example 3, the distance from  $\chi_{11}$  to the span of the columns of  $\mathbf{D}$  is  $\frac{1}{2}$ . Note that any sequence in the span of the columns of  $\mathbf{D}$  that converges to  $\chi_{11}$  in the  $L_1(S, \sigma)$  norm is necessarily unbounded. (See the comments following the proof of Theorem 2 above.)

We have required that endowments be bounded away from 0; the final example shows the kind of difficulties that could arise if we did not make this restriction.

EXAMPLE 5. Consider the sequence of assets  $\{\alpha_k\}$  whose returns are  $\alpha_k(s) = k^{-s}$ . If endowments are  $e^h(s) = \exp(-s)$ , then *no finite nonzero portfolio* (except for the 0 portfolio) is dominated by endowments. Hence *no finite portfolio can be traded*. We surely cannot expect to say much about asymptotic efficiency for such sequences of assets.

## 5. GENERICITY

In Section 4 we presented a characterization of asymptotic efficiency and examples of sequences that are or are not asymptotically efficient. The purpose of this section is to show that it is the asymptotically inefficient examples that are typical: "most" asset sequences are asymptotically *inefficient*. To make this assertion precise, we need to identify the family of asset sequences under consideration and a topology on it.

Since all of our results are invariant to change of scale, there is no loss of generality in restricting our attention to assets satisfying  $\|\alpha\| \leq 1$ . (Recall that  $\|\alpha\|$  is the expectation with respect to  $\sigma$  of  $|\alpha(s)|$ .) Since we have required that the returns on assets be bounded, we should take account of this. The simplest way to do this is to restrict our attention to asset sequences for which the  $k$ th asset is bounded by a preassigned number. To this end, fix, once and for all, a sequence  $\{B_k\}$  of positive numbers; let  $\mathcal{A}$  be the set of asset sequences  $\alpha = \{\alpha_k\}$  such that  $\alpha_1 = 1$  is riskless, and  $\|\alpha_k\| \leq 1$  and  $|\alpha_k(s)| \leq B_k$  for each  $s, k$ .

$\mathcal{A}$  carries several natural topologies. The simplest corresponds to the distance function defined on pairs of asset sequences  $\alpha, \beta$  by

$$d^1(\alpha, \beta) = \sum \sigma(s) |\alpha_k(s) - \beta_k(s)|$$

the sum extending over all indices  $k$  and all states  $s$ . (Strictly speaking, this is not a metric, since it might be infinite; nonetheless, it induces a well-defined topology.)

An alternative topology may be defined on  $\mathcal{A}$  by noting that every element  $\alpha \in \mathcal{A}$  gives rise to a bounded operator (continuous linear transformation)  $R^\alpha$  from  $l_1$  (the space of all summable real sequences) into  $L_1(S, \sigma)$  defined by

$$R^\alpha(c) = \sum c_k \alpha_k$$

for  $c = (c_1, c_2, \dots) \in l_1$ . If we identify asset sequences with the operators they induce, it seems natural to define the *operator distance* between asset sequences  $\alpha, \beta$  as

$$d^{\text{op}}(\alpha, \beta) = \|R^\alpha - R^\beta\|_{\text{op}} = \sup \left\{ \sum \sigma(s) |R^\alpha(c)(s) - R^\beta(c)(s)| \right\},$$

where the sum extends over all states  $s$ , and the supremum extends over all sequences  $c \in l_1$  with  $\|c\|_{l_1} = \sum |c_s| \leq 1$ . In the topology induced by this distance function, two asset sequences  $\alpha, \beta$  are close together if the same portfolios yield nearly the same (expected) returns, uniformly over all portfolios representing total trades of at most 1 share [see Dunford and Schwartz (1958) for further discussion].<sup>18</sup>

It is not hard to see that the topology on  $\mathcal{A}$  induced by  $d^1$  is stronger than that induced by  $d^{op}$ , and that they are both complete, metric topologies.

Recall that a subset of a complete metric space is *residual* if it contains the countable intersection of dense, open sets. (Recall that residual sets are large and that their complements are small. In particular, the Baire category theorem implies that a residual subset of a complete metric space is dense.) Since we have identified two different complete metric topologies on  $\mathcal{A}$ , we have two possible interpretations of "residual" for subsets of  $\mathcal{A}$ . As it happens, the "bad set" is residual in *both* these topologies.

At this point, we need to address two issues. The first is that we have established the existence of equilibrium only when asset returns are linearly independent. It therefore seems reasonable to incorporate the requirement of linear independence into the definition of the "bad set." This poses no difficulties. The second issue is more substantive. As a consequence of the infinite dimensionality of  $L^1(S, \sigma)$ , there are linearly independent sequences of assets that do not span  $L^1(S, \sigma)$  or any dense subspace. Consider for example the sequence  $\alpha = \{\alpha_k\}$  for which  $\alpha_1$  is riskless, and  $\alpha_k = \chi_{1,2k}$  is the Arrow security yielding 1 unit of the numeraire in state  $2k$  and nothing in other states. This sequence is linearly independent and asymptotically inefficient. On the other hand, it fails—in every reasonable sense—to "span all the uncertainty." In particular, the Arrow security  $\chi_{11}$ , which yields 1 unit of the numeraire in state 1 and nothing in other states, does not lie in the closure of the space spanned by the sequence  $\alpha$ . It seems clear that  $\alpha$  is not a "complete sequence of assets" in any reasonable sense, but it is not entirely clear should constitute a "complete sequence of assets."

Two natural notions of completeness come to mind. In the finite-dimensional setting, completeness of a set of assets means that every prescribed numeraire pattern can be obtained as the returns of a suitable finite portfolio. Since the set of states and available assets is infinite, we

<sup>18</sup>If we identify asset sequences with the operators they induce, then  $d^1$  is the distance function corresponding to the *trace norm*; again, we refer to Dunford and Schwartz (1958) for further details. One point to note: if  $d^1(\alpha, \beta)$  is finite, then the difference  $R^\alpha - R^\beta$  is a compact operator from  $l_1$  to  $L_1(S, \sigma)$ ; i.e., it maps bounded subsets of  $l_1$  to relatively compact subsets of  $L_1(S, \sigma)$ .

should allow for approximations or for infinite portfolios. The identification of asset sequences as operators  $l_1 \rightarrow L_1(S, \sigma)$  provides an easy way to express these conditions. If  $R^\alpha$  has dense range, then every numeraire pattern can be approximated (in expectation) by the returns on a finite portfolio. If  $R^\alpha$  is onto, then every numeraire pattern can be obtained as the returns on a suitable (infinite) portfolio.<sup>19</sup> Because  $S$  is countable, the set of invertible operators  $l_1 \rightarrow L_1(S, \sigma)$  is not empty; moreover, this set is open in the topology induced by  $d^{\text{op}}$ , and a fortiori in the topology induced by  $d^1$  (since it is stronger). Hence, whichever definition we take for completeness of a set of asset sequences, we conclude that the set of complete asset sequences certainly contains a nonempty, open subset of  $\mathcal{A}$ . In particular, the set of complete asset sequences is quite large.

Having addressed these issues, we now turn to the main result of this section, which establishes that the set of linearly independent, asymptotically independent asset sequences is residual, and therefore large.

**THEOREM 3.** *The set of asymptotically inefficient, linearly independent asset sequences is a residual subset of  $\mathcal{A}$  in both the  $d^1$  and  $d^{\text{op}}$  topologies.*

To motivate the proof, write  $C_0(S)$  for the set of functions  $v \in L_1(S, \sigma)$  with the property that  $|v(s)| \rightarrow 0$  as  $s \rightarrow \infty$ . Note that  $C_0(S)$  is exactly the uniform closure of the subspace of  $L_1(S, \sigma)$  spanned by Arrow securities. Theorem 2 tells us that the asset sequence  $\{\alpha_k\}$  is asymptotically efficient exactly when every Arrow security, and hence every numeraire pattern in  $C_0(S)$ , lies in the  $d_\infty$  closure of  $\text{span}\{\alpha_k\}$ . Equivalently, for every vector  $v \in C_0(S)$ , we have  $d_\infty(v, \text{span}\{\alpha_k\}) = 0$ . Of course, 0 is the closest a numeraire pattern can be to  $\text{span}\{\alpha_k\}$ ; since  $\alpha_1 = 1$ , the furthest a numeraire pattern can be from  $\text{span}\{\alpha_k\}$  is its distance from the one-dimensional subspace spanned by the asset 1. We show that, for a residual set of asset sequences, all vectors in  $C_0(S)$  are at least half as far from  $\text{span}\{\alpha_k\}$  as they could possibly be.

Before beginning the proof proper, it is convenient to isolate the most technical part in a lemma.

**LEMMA.** *For each  $v \in L_1(S, \sigma)$ , the set  $\mathbf{Q}(v)$  of linearly independent asset sequences  $\{\alpha_k\}$  with the property*

$$d_\infty(v, \text{span}\{\alpha_k\}) \geq \left(\frac{1}{2}\right)d_\infty(v, \text{span}\{1\})$$

*is a residual subset of  $\mathcal{A}$  in both the  $d^1$  and  $d^\infty$  topologies.*

*Proof.* For each  $s \in S$ , and  $w \in L_1(S, \sigma)$ , write  $w^{(s)}$  for the numeraire pattern that coincides with  $w$  in states  $1, 2, \dots, s$  and is 0 else-

<sup>19</sup>It is not clear which infinite portfolios we should allow. However, if  $\theta \in l_1$ , then the series  $\sum \theta_j \alpha_j$  converges absolutely (in expectation), so we should certainly allow portfolios in  $l_1$ .

where. Fix positive integers  $m, n$ ; write  $\rho = (\frac{1}{2}) - 2^{-m}$ , and let  $\mathbf{Q}(v, m, n)$  be the set of asset sequences  $\{\alpha_k\}$  in  $\mathcal{A}$  such that

- (i)  $\alpha_1^{(n)}, \dots, \alpha_n^{(n)}$  are linearly independent;
- (ii)  $d_\infty(v, \text{span}\{\alpha_1, \dots, \alpha_n\}) > \rho d_\infty(v, \text{span}\{1\})$ .

It is easily seen that  $\mathbf{Q}(v)$  contains the intersection (taken over all integers  $n, m$ ) of the sets  $\mathbf{Q}(v, m, n)$ , so it suffices to show that each  $\mathbf{Q}(v, m, n)$  is an open subset of  $\mathcal{A}$  with respect to  $d^{\text{op}}$  and is dense with respect to  $d^1$ .

To see that  $\mathbf{Q}(v, m, n)$  is an open subset of  $\mathcal{A}$  with respect to  $d^{\text{op}}$ , note first that the linear independence of  $\alpha_1^{(n)}, \dots, \alpha_n^{(n)}$  is certainly preserved by any perturbation of the sequence  $\{\alpha_k\}$  that is small in the first  $n$  states and for the first  $n$  terms of the sequence. Hence the set of sequences  $\{\alpha_k\}$  satisfying the linear independence condition (i) is open. Note too that, if

$$d_\infty(v, \text{span}\{\alpha_1, \dots, \alpha_n\}) > \rho d_\infty(v, \text{span}\{1\}),$$

then there is a state  $s$  such that

$$d_\infty(v^{(s)}, \text{span}\{\alpha_1^{(s)}, \dots, \alpha_n^{(s)}\}) > \rho d_\infty(v, \text{span}\{1\}).$$

Since the vectors  $v^{(s)}, \alpha_1^{(s)}, \dots, \alpha_n^{(s)}$  all lie in the finite-dimensional space  $\mathbb{R}^s$ , a simple continuity argument shows that the last inequality is also satisfied by any perturbation of  $\{\alpha_k\}$  that is sufficiently small in the first  $s$  states and for the first  $n$  terms of the sequence. Hence, the set of sequences  $\{\alpha_k\}$  satisfying the span condition (ii) is open, as desired.

To see that  $\mathbf{Q}(v, m, n)$  is a dense subset of  $\mathcal{A}$  with respect to  $d^1$ , fix a sequence  $\beta = \{\beta_k\} \in \mathcal{A}$ ; we construct a perturbed sequence  $\tilde{\beta} = \{\tilde{\beta}_k\} \in \mathbf{Q}(v, m, n)$  such that  $\sum \sigma(r) |\tilde{\beta}_k(r) - \beta_k(r)|$  is arbitrarily small (the sum extending over all states  $r$ , and all indices  $k \leq n$ ). Since  $\|\beta_i - \beta_i^{(s)}\| \rightarrow 0$  as  $s$  tends to  $\infty$ , we may (choosing  $s$  sufficiently large), assume that  $\beta_i^{(s)} = \beta_i$  for  $i = 2, \dots, n$ . Moreover, since linear independence of  $\beta_1^{(n)}, \dots, \beta_n^{(n)}$  is equivalent to the vanishing of an  $n \times n$  determinant, this condition can be achieved by an arbitrarily small perturbation, which we assume to have been already carried out.

To achieve the span condition, it is convenient to work with the linear transformation  $R^\beta: \mathbb{R}^n \rightarrow L_1(S, \sigma)$ . If the span condition (ii) is not satisfied, let  $C$  be the set of vectors  $c \in \mathbb{R}^n$  such that

$$\|R^\beta(c) - v\|_\infty \leq \rho d_\infty(v, \text{span}\{1\}).$$

Since  $\{\beta_1^{(n)}, \dots, \beta_n^{(n)}\}$  is a linearly independent set, the linear transformation  $R^\beta$  is an isomorphism of  $\mathbb{R}^n$  with a finite-dimensional subspace of  $L_1(S, \sigma)$ , so that  $C$  is compact. For each  $c \in C$ , the above inequality, the

triangle inequality, and the facts that  $\beta_1 = 1$  and that  $\rho < \frac{1}{2}$  imply that

$$\sum_{i=2}^n |c_i| \|\beta_i\|_{\infty} \geq \|R^{\beta}(c) - c_1 1\|_{\infty} > \rho d_{\infty}(v, \text{span}\{1\}).$$

Choose any state  $r > s$ . For  $1 = 2, \dots, n$ , define  $\delta_i$  in the following way: if  $c_i = 0$ , then  $\delta_i = 0$ ; if  $c_i \neq 0$  but  $v(r) = 0$ , then  $\delta_i = \text{sign}(c_i)$ ; otherwise,  $\delta_i = \text{sign}(v(r)/c_i)$ . Define  $\tilde{\beta}_i$  and  $\beta_i^*$  by

$$\tilde{\beta}_i(t) = \beta_i(t) \quad \text{for } t \neq r,$$

$$\tilde{\beta}_i(r) = \delta_i \|\beta_i\|_{\infty},$$

$$\beta_i^* = \tilde{\beta}_i / \|\tilde{\beta}_i\|.$$

This yields a perturbation  $\{\beta_i^*\}$  of  $\{\beta_i\}$  such that

$$\|R^{\beta^*}(c) - v\|_{\infty} > \rho d_{\infty}(v, \text{span}\{1\}).$$

Since  $R^{\beta^*}$  is continuous, we conclude that

$$\|R^{\beta^*}(c') - v\|_{\infty} > \rho d_{\infty}(v, \text{span}\{1\})$$

for all  $c'$  in some neighborhood  $W_c$  of  $c$ . Since  $C$  is compact, we can cover  $C$  with a finite number of these neighborhoods. Since we can make these perturbations in different states  $r$ , we conclude that there is a single perturbation  $\{\tilde{\beta}_i\}$  such that  $\|R^{\tilde{\beta}}(c) - v\|_{\infty} > \rho d_{\infty}(v, \text{span}\{1\})$  for every  $c \in C$ . Since we have made these perturbations in states where the  $\beta_i$  vanished, we conclude that  $\|R^{\tilde{\beta}}(c'') - v\|_{\infty} > \rho d_{\infty}(v, \text{span}\{1\})$  for every  $c'' \in \mathbb{R}^n \setminus C$ . Finally, since we can make these perturbations only in states of arbitrarily low probability, we can guarantee that  $d^1(\beta, \tilde{\beta})$  is as small as we like. Hence the perturbation  $\tilde{\beta}$  has all the required properties. We conclude that  $\mathbf{Q}(v, m, n)$  is dense with respect to  $d^1$ , as desired. This completes the proof. ■

With this technical result in hand, we turn to the proof of Theorem 3.

*Proof of Theorem 3.* Write  $\mathbf{Q}$  for the set of linearly independent asset sequences  $\{\alpha_k\} \in \mathcal{A}$  having the property that, for each  $v \in C_0(S)$ ,

$$d_{\infty}(v, \text{span}\{\alpha_k\}) \geq \left(\frac{1}{2}\right) d_{\infty}(v, \text{span}\{1\}).$$

We claim that  $\mathbf{Q}$  is a residual subset of  $\mathcal{A}$  in both the topologies  $d^1$  and  $d^{\infty}$ . To see this, note that for each  $v \in C_0(S, \sigma)$ , the lemma provides a

residual set  $\mathbf{Q}(v)$  of asset sequences  $\{\alpha_k\}$  such that

$$d_x(v, \text{span}\{\alpha_k\}) \geq \left(\frac{1}{2}\right)d_x(v, \text{span}\{1\}).$$

If  $\{\alpha_k\} \in \mathbf{Q}(v)$  and  $d_x(v, v') < \left(\frac{1}{4}\right)d_x(v, \text{span}\{1\})$ , the triangle inequality implies that

$$d_x(v', \text{span}\{\alpha_k\}) \geq \left(\frac{1}{4}\right)d_x(v', \text{span}\{1\}).$$

Since  $C_0(S, \sigma)$  is  $d_x$ -separable, we may choose a countable dense subset  $\{v_i\}$ . Set  $\mathbf{Q} = \bigcap \mathbf{Q}(v_i)$ ; as the countable intersection of residual sets,  $\mathbf{Q}$  is also a residual set. For each nonzero vector  $w \in C_0(S)$ , we can find a  $v_i$  such that

$$d_x(w, v_i) < \left(\frac{1}{2}\right)d_x(w, \text{span}\{1\}).$$

It follows that  $d_x(w, \text{span}\{\alpha_k\}) > \left(\frac{1}{2}\right)d_x(w, \text{span}\{1\})$  for all  $\{\alpha_k\} \in \mathbf{Q}(v_i)$ , and a fortiori for all  $\{\alpha_k\} \in \mathbf{Q}$ . Thus,  $\mathbf{Q}$  has the properties required.

In view of Theorem 2, each asset sequence in  $\mathbf{Q}$  is asymptotically inefficient; this completes the proof of Theorem 3. ■

An observation about  $\mathbf{Q}$  may serve to explicate further the sense in which "large trades" are important. Consider any two consumers  $(e^1, \preceq^1)$ ,  $(e^2, \preceq^2)$ , for whom

- (i) there is an  $s^*$  such that  $e^1(s) + e^2(s) \leq 1$  for  $s \geq s^*$ ;
- (ii) if  $z$  is an individually rational, Pareto-optimal net trade (for consumer 1), then  $z(1) \geq +4$  and  $z(2) \leq -4$ .

Fix an asset sequence  $\alpha \in \mathbf{Q}$ . For any  $N$ , consider a securities market  $\mathcal{E} = (\{\alpha_k: 1 \leq k \leq N\}, \{(e^h, \preceq^h, Y^h)\})$ , and let  $w$  be an equilibrium net trade in  $\mathcal{E}$ . Feasibility of  $w$  implies that  $|w(s)| \leq 1$  for  $s \geq s^*$ . On the other hand, if  $z$  is an individually rational, Pareto-optimal net trade then (ii) entails  $d_x(z^{(s^*)}, \text{span}\{1\}) > 4$ , and the definition of  $\mathbf{Q}$  therefore entails  $d_x(z^{(s^*)}, \text{span}\{\alpha_k\}) > 2$ . In particular,  $d_x(z^{(s^*)}, w) > 2$ . Since  $|w(s)| \leq 1$  for  $s \geq s^*$ , we conclude that  $d_x(z^{(s^*)}, w^{(s^*)}) > 1$ , and hence that

$$\|z - w\| > \sum_{s \leq s^*} \sigma(s).$$

That is, no security market equilibrium net trade is close to any Pareto-optimal net trade. In particular, no security market equilibrium allocation is close to a Pareto-optimal allocation.

We can obtain similar genericity results if we restrict ourselves to sequences of assets with positive returns; all that is required is a small change in the proof of the Lemma. For details, and for discussion of genericity in assets and endowments, see Zame (1990).

## 6. UNCONSTRAINED CONSUMPTION

As we noted earlier, the asymptotic inefficiency we have demonstrated may be traced directly to the requirement that consumption bundles be nonnegative in each state. In this section, we ask what happens when we drop this requirement; i.e., we assume in what follows that the consumption set of each consumer is the entire commodity space  $L_1(S, \sigma)$ . (Allowing for negative consumption may be natural when the commodities are themselves assets, and the discussion that follows might be viewed in the light of asset pricing models.) The other assumptions of Section 2 are understood to remain in force. In particular, preferences are monotone, norm continuous, and convex, and endowments are positive.

The first comment that needs to be made is that, in the absence of consumption constraints, the set of feasible consumption bundles is not compact; as a consequence, equilibria need not exist. This can, of course, be the case even for complete market economies with one commodity and two states of nature. [See Werner (1987) and Nielsen (1986) for elegant treatments of the existence problem in the finite-dimensional case.] Unboundedness of feasible consumption bundles also opens the possibility that the securities markets corresponding to each finite set of assets might have equilibria, but that equilibrium allocations might not converge (or have a convergent subsequence) as the set of assets expands. In short, without consumption constraints, existence and convergence of equilibria are problematic; as we shall see, however, *efficiency* of limits of equilibria is not problematic. If assets span a dense subspace of the set of all wealth streams (a condition that seems like a natural formalization of the idea that assets span all the uncertainty), if trading sets are sufficiently large, if the securities markets corresponding to a sequence of assets have equilibria, and if the equilibrium allocations converge, then the limit is an equilibrium allocation of the underlying complete markets economy (and in particular is Pareto-optimal).

A small point should be addressed here. Positivity constraints imply that all feasible allocations (and thus all equilibrium allocations of finite securities markets) lie in a norm compact set, so norm convergence of equilibrium allocations of finite securities markets is the relevant notion. Moreover, since preferences are norm continuous, norm convergence of allocations implies convergence of the corresponding utilities. The absence of positivity constraints raises the possibility that equilibrium allocations lie in a set that is weakly compact but not norm compact, so it seems useful to consider *weak* convergence of equilibrium allocations. However, since utility functions need not be weakly continuous, weak convergence of arbitrary allocations does not imply convergence of the corresponding utilities. As we show, however, weak convergence of *equilibrium* allocations does imply convergence of the corresponding utilities.

A final issue concerns trading sets. It is intuitively clear that equilibrium allocations cannot be nearly efficient if trading sets are very small. To rule out this possibility, consider a sequence  $\{N Y\}$  of trading sets,  $N Y \subset \mathbb{R}^N$ . For  $N < N'$ , we regard  $\mathbb{R}^N$  as the subset of  $\mathbb{R}^{N'}$  consisting of vectors whose last  $N' - N$  coordinates are 0. Say that  $\{N Y\}$  expands unboundedly if for each  $N$  and each compact set  $K \subset \mathbb{R}^N$  there is an  $N_0$  such that, if  $N < N'$ , then  $K \subset_{N'} Y$ .

**THEOREM 4.** Fix consumers  $\{(e^h, \preceq^h)\}$ , and norm continuous, quasi-concave utility functions  $u^h: L_1(S, \sigma)^l \rightarrow \mathbb{R}$  representing the preference relations  $\preceq^h$ . Let  $\{\alpha_k\}$  be a linearly independent sequence of assets that spans a weakly dense subspace of  $L_1(S, \sigma)$ ; assume that  $\alpha_1$  is riskless. For each  $N$ , consider the securities market  $\mathcal{E}_N = (\{\alpha_k: 1 \leq k \leq N\}, \{(e^h, \preceq^h, N Y^h)\})$  in which the first  $N$  assets are available for trade, and let  $(q_N, p_{N,N} x, N y)$  be an equilibrium for  $\mathcal{E}_N$ . Assume that the equilibrium allocations  $\{N x\}$  converge weakly to an allocation  $x$ , and that the trading sets  $\{N Y^h\}$  expand unboundedly. Then  $x$  is an equilibrium allocation for the underlying complete markets economy, and  $u^h(N x^h) \rightarrow u^h(x^h)$  for every consumer  $h$ .<sup>20</sup>

*Proof.* The argument is very similar to the argument of Theorem 2. We first establish the following:

*Claim.* There does not exist a set  $C$  of consumers and an allocation  $X$  that is feasible for consumers in  $C$  (i.e., the restriction of  $X$  to  $C$  is a reallocation of the endowments of consumers in  $C$ ), such that

$$u^c(X^c) > \limsup u^c(N x^c) \quad \text{for every } c \in C.$$

If this is not so, then there is a set of consumers, which we may assume to be the consumers  $M = (1, 2, \dots, M)$ , and an allocation  $X = (X^1, \dots, X^M)$  that is a redistribution of the endowments  $(e^1, \dots, e^M)$  and has the property that  $u^m(X^m) > \limsup u^m(N x^m)$  for  $m \in M$ . Continuity of preferences, together with the assumption that numeraire endowments are bounded away from zero, guarantees that we can find a state  $r$ , allocations  $z = (z^1, \dots, z^M)$  and  $\bar{z} = (\bar{z}^1, \dots, \bar{z}^M)$ , and a real number  $\delta > 0$  such that  $u^m(\bar{z}^m) > \limsup u^m(N x^m)$  for  $m \in M$  and

$$\begin{aligned} \bar{z}^m(s, i) &= z^m(s, i) = 0 && \text{for } s > r, 1 \leq i \leq l; \\ \bar{z}^m(s, i) &= z^m(s, i) && \text{for } s \leq r, 2 \leq i \leq l; \\ \bar{z}^m(s, 1) &= z^m(s, 1) - \delta e^m(s, 1) && \text{for } s \leq r; \\ \sum_m z^m(s, i) &= \sum_m X^m(s, i) = \sum_m e^m(s, i) && \text{for } s \leq r, 1 \leq i \leq l. \end{aligned}$$

<sup>20</sup>Note that the latter property is independent of the choice of utility functions representing the given preferences.

Since we have normalized the state prices to sum to 1, we may (passing to a subsequence if necessary) assume that the state prices  $p_n(s, i)$  converge; say,  $p_n(s, i) \rightarrow p(s, i)$ . If  $p(s, i) = 0$  for any state  $s$  and commodity  $i$ , then, for  $n$  sufficiently large, the demands in state  $s$  would be unbounded. Hence,  $p(s, i) \neq 0$  for each  $s, i$ . (The limiting behavior of asset prices is irrelevant.)

For each price system  $p^*$  and each  $m \in M$ , define a numeraire pattern  $w^m(p^*)$  by

$$\begin{aligned} w^m(p^*, s) &= [1/p^*(s, 1)][p^* \square (z^m - e^m)](s) & \text{for } s \leq r; \\ w^m(p^*, s) &= 0 & \text{for } s > r. \end{aligned}$$

For  $s \leq r$ ,  $w^m(p^*, s)$  is the amount of numeraire that must be transferred into state  $s$  in order to make the net purchase  $(z^m - e^m)(s)$ , at prices  $p^*$ . Since  $z$  is a reallocation of endowments in states  $s \leq r$ , and  $w^m(p^*, s) = 0$  in states  $s > r$ , it follows that, for each price system  $p^*$  and each state  $s$ ,  $\sum w^m(p^*, s) = 0$  (summation over  $m \in M$ ).

We now use the density of  $\text{span}\{\alpha_k\}$  in  $L_1(S, \sigma)$  to find portfolios  $\theta^1, \dots, \theta^{M-1}$  whose returns  $\zeta^1, \dots, \zeta^{M-1}$  are within  $\delta/M$  of  $w^m(p)$  in the  $L_1$  norm; i.e.,  $\|\zeta^m - w^m(p)\| < \delta/M$  for  $1 \leq m \leq M-1$ . If we set

$$\theta^M = - \sum_{m=1}^{M-1} \theta^m$$

we obtain a portfolio whose returns  $\zeta^M$  are also within  $\delta$  of  $w^M(p)$  in the  $L_1$  norm; i.e.,  $\|\zeta^M - w^M(p)\| < \delta$ . Since the state prices  $p_n(s, i)$  converge to  $p(s, i)$  for each  $s, i$  and  $w^m(p^*, s) = 0$  for states  $s > r$ , we conclude that  $\|\zeta^m - w^m(p_n)\| < \delta$  for each  $m \in M$ , provided that  $n$  is sufficiently large.

Since the trading sets expand unboundedly, our construction guarantees that, if  $n$  is sufficiently large then the portfolio  $\theta^m \in {}_n Y^m$ . Since we have constructed  $\theta^M$  so that  $\sum \theta^m = 0$ , at least one of the portfolios  $\theta^m$  must have a nonpositive price (at asset prices  $q_n$ ). However, if  $n$  is sufficiently large, so that prices  $p_n$  are sufficiently close to  $p$ , then the returns  $\zeta^m$  on the portfolio  $\theta^m$  will finance purchase of the commodity bundle  $\bar{z}^m$ . By construction,  $u^m(\bar{z}^m) > \limsup u^m({}_n x^m)$  for each  $m$ . Since  $\bar{z}^m$  belongs to the budget set of consumer  $m$ , this contradicts the equilibrium conditions in  $\mathcal{E}_{N(n)}$ . This establishes the claim.

If  $x$  is not in the core of the underlying complete markets economy, there is a set  $C$  of consumers and an allocation  $y$  that is feasible for consumers in  $C$  such that  $u^c(y^c) > u^c(x^c)$  for  $c \in C$ . However, norm continuity and quasiconcavity of utility functions imply that  $\limsup u^h({}_n x^h) \leq u^h(x^h)$  for every  $h$  (i.e., utility functions are weakly upper semicontinuous), so this would contradict the claim. We conclude

that  $x$  is in the core of the underlying complete markets economy; that it is a competitive equilibrium allocation follows, as before, by replication.

It remains only to establish that utilities converge. If not there would be a consumer, say, consumer 1, for whom  $\{u^n(x^1)\}$  does not converge to  $u^1(x^1)$ . Passing to a subsequence if necessary, and keeping in mind that utility functions are weakly upper semicontinuous, we obtain  $\limsup u^n(x^1) < u^1(x^1)$ . Continuity allows us to choose a number  $r < 1$ , sufficiently close to 1 so that  $\limsup u^n(x^1) < u^1(rx^1)$ . Write  $y^1 = rx^1$  and  $y^h = x^h + (1/H)(1-r)x^1$  for  $h \neq 1$ . Then  $y = (y^1, \dots, y^H)$  is a redistribution of endowments, and  $u^j(y^j) > \limsup u^n(x^j)$  for every  $j$ , so this again contradicts the claim. This completes the proof. ■

## 7. CONCLUDING REMARKS

There is an old intuition that opening new markets should lead to a Pareto improvement (of the equilibrium allocation). Hart (1975) has shown that this intuition is not always correct: opening new markets may, in fact, lead to a Pareto worsening (of the equilibrium allocation). Despite Hart's examples, the intuition may persist that opening more and more markets should eventually be an improvement, and indeed that, as markets come closer and closer to being complete, equilibrium allocations should become closer and closer to Pareto-optimal allocations.

In this paper, we have examined this intuition by studying the asymptotic behavior of securities markets as the number of securities grows. Our results show that equilibrium allocations of the securities markets need not converge to Pareto-optimal allocations, and that such behavior is not at all pathological. Thus a large—but incomplete—security market may not be a good approximation to a Walrasian market.

Green and Spear (1987, 1989) consider similar questions, and isolate a condition on asset returns and endowments that guarantees asymptotic efficiency. Roughly speaking, their requirement (Assumption A') is that endowments are eventually large in comparison to asset returns. This condition is very strong; indeed, if endowments are fixed, the set of asset sequences *failing* this condition is a residual set. Their result is thus not at variance with the results here (although they interpret their results slightly differently than I interpret the results in the present paper).

An alternative way to examine the intuition that, as markets come closer and closer to being complete, equilibrium allocations should become closer and closer to Pareto-optimal allocations, is to consider a framework in which a complete set of securities is available for trade, but participation is restricted, so that some fraction of consumers are constrained from trading in some markets. Letting this fraction converge to zero is another way of letting the market converge to completeness. Such

an analysis has been carried out by Cass (1990) in a model with a finite number of states. Cass shows that, as the fraction of consumers whose participation is restricted converges to zero, equilibrium allocations converge to Pareto-optimal allocations (indeed, to Walrasian allocations of the complete markets economy).<sup>21</sup>

We have found it convenient to work in the commodity space  $L_1(S, \sigma)$ , but other choices could be made. For instance, so long as the set of states of nature is countable, all of our work could be carried out in any of the spaces  $L_p(S, \sigma)$ ,  $1 \leq p \leq \infty$ , with suitable adjustments.<sup>22</sup>

As noted earlier, it would be natural to model the set of states of nature by a continuum (rather than by a countable set, as we have done). However, the existence proof given here does not generalize to this setting; the difficulty is that the wealth mapping  $(p, x) \rightarrow p \square x$  need not be jointly continuous in the relevant topologies. In the complete markets setting, Bewley (1972) cleverly finesses a similar issue [for a general discussion of this point, see Mas-Colell and Zame (1991a)], but it seems that Bewley's method does not work when markets are incomplete, and the existence of equilibrium in this setting is problematical. For the case of separable preferences, see Hellwig (1991), Mas-Colell and Monteiro (1991), Mas-Colell and Zame (1991b), and Monteiro (1991).

The methods employed here could easily be adapted to the context of multiple trading dates, but they do not seem easily adaptable to continuous time models of trading (for precisely the reason above); see Duffie and Huang (1985), Duffie and Zame (1989) for example.

As we have shown, asymptotic inefficiency arises because some portfolios create liabilities that are unsatisfiable, given endowments and the constraints that terminal consumption be positive. Relaxing these terminal consumption constraints to allow for unboundedly negative terminal consumption restores asymptotic efficiency. However, when consumption goods are physical commodities (rather than assets), consumption of negative quantities does not seem to have a clear interpretation. A more attractive method of addressing this problem would be to insist that terminal consumption be nonnegative, but allow for the possibility that some liabilities are not met; i.e., to allow for the possibility of default. For such a model, see Zame (1990).

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<sup>21</sup>I am grateful to a referee for pointing out this work.

<sup>22</sup>For  $1 \leq p < \infty$ , virtually no changes are required, save for altering the definitions in the appropriate way. The case  $p = \infty$  is a little more complicated; see Zame (1990).

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