

# Clubs and the Market: Large Finite Economies\*

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April 1997  
Revised April 13, 2000

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\*We thank Robert Anderson for tutelage, and Kenneth Arrow, Peter Hammond, Joe Ostroy, and many other seminar participants for comments. We thank the National Science Foundation, the Fulbright Foundation, the Social Science Research Council of Denmark and UC Berkeley and UC Los Angeles for financial support, and the University of Copenhagen Institute of Economics and the UC Los Angeles and UC Berkeley Departments of Economics for gracious hospitality during preparation of this paper.

## Abstract

This paper builds a general equilibrium model of finite economies with exchange and club formation. Agents trade multiple private goods widely in the market, can belong to several clubs, and care about the characteristics of other club members. Because club memberships are indivisible and choices of club memberships must be coordinated across the population, the core of such an economy may be empty and equilibrium may not exist. However, for large finite club economies, an approximate core is not empty and states in this approximate core can be approximately decentralized by prices for private goods and for club memberships. The arguments use convexification tools familiar from the literature on private goods economies, and a new tool, of some independent interest, that addresses the special problems created by the nature of club memberships. *Journal of Economic Literature* Classification Numbers: D2, D5, H4

**Keywords:** clubs,  $\varepsilon$ -cores, approximate cores, approximate equilibrium, approximate decentralization.

# 1 Introduction

Club theory deals with group formation, while general equilibrium theory deals with markets and prices. But groups and markets are interdependent: the company we keep affects our demand for private goods, and our consumption of private goods affects the company we seek. This paper and the companion Ellickson, Grodal, Scotchmer and Zame (1999) (henceforward EGSZ) integrate club theory and general equilibrium theory. Our theory builds on the intuition that club memberships and private goods should be regarded as parallel objects of choice and objects to be priced. However, the parallel only goes so far: clearing the markets for club memberships requires a level of coordination of individual choices considerably more demanding than that ordinarily associated with private goods.

Our companion paper builds a model of a club economy with a continuum of agents, and shows that such club economies share many of the pleasant features of exchange economies with a continuum of agents. In particular, the core is not empty, and core states can be decentralized by prices. But does the continuum model represent a good idealization of a finite club economy with a large number of agents? In the present paper, we answer this question in the affirmative by showing that, if the number of agents is large enough, an appropriate approximate core (for reasons discussed below, we use the club analog of Anderson's (1985) *fat  $\varepsilon$ -core*) is not empty, and every state in this approximate core can be approximately decentralized by prices. Moreover, the "rate of convergence" in each of these approximations is inversely proportional to the number of agents in the economy, just as for exchange economies. Roughly speaking, therefore, the present paper bears the same relationship to EGSZ as the work of Kannai (1970), Hildenbrand, Schmeidler and Zamir (1973), Grodal, Trockel, and Weber (1984) and Anderson (1985) bear to Aumann's (1964, 1966) analysis of exchange economies with a continuum of agents.

We describe a *club type* by the external characteristics of its members (those characteristics that matter to others) and the activity in which the

club is engaged. A *club membership* is an opening in a particular type of club available to an agent with a specified external characteristic. We allow agents to belong to many clubs, so if the economy has  $n$  divisible private goods and  $m$  types of club memberships then individual choice sets lie in  $\mathbb{R}^{n+m}$ . Because club memberships are indivisible, such a club economy resembles a private goods exchange economy with  $n$  divisible goods and  $m$  indivisible goods. Indivisibility presents a complication, but it is a complication familiar from the exchange case, and not particularly troublesome. (It is not troublesome at all with a continuum of agents.) Because indivisibility implies that the core of a club economy may be empty, we of necessity employ a notion of the core that is approximate. A much more subtle — and troublesome — complication is that agents' choices of club memberships must be consistent across the population.<sup>1</sup> For example, if we think of a (heterosexual, monogamous) marriage as a club, then the same numbers of males as of females must choose marriage.

To understand why this consistency requirement is troublesome, it is useful to outline a construction, in the spirit of Hildenbrand, Schmeidler and Zamir (1973), of a state in an approximate core of a finite exchange economy with  $n$  divisible goods and  $m$  indivisible goods. Begin with a continuous representation of  $\mathcal{E}$  — an economy with a continuum of agents representing each agent in  $\mathcal{E}$ . Use Aumann's (1966) existence theorem to find an equilibrium for this continuous representation. Average the choices for each type, obtaining choices  $x$  for the original finite economy that lie in the convex hull of individual demand sets at the given prices. Use the Shapley–Folkman theorem to construct new choices  $\hat{x}$  that keep the sum unchanged, keep the choices in the convex hulls of the demand sets, and have the property that that the choices of all but at most  $n + m$  exceptional agents lie in the demand sets themselves. Set the choices of the exceptional agents equal to 0, obtaining choices  $\bar{x}$ . Because the sum of all endowments might exceed the sum of all consumptions,  $\bar{x}$  need not be exactly feasible; distribute the excess arbitrarily to obtain a choice vector  $x^*$  in the fat  $\varepsilon$ -core.

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<sup>1</sup>Consistency, it seems, is the hobgoblin of (small) clubs.

Consider now a club economy  $\mathcal{E}$  with  $n$  divisible private goods and  $m$  types of club membership, and try to mimic the argument above. Begin with a continuous representation of  $\mathcal{E}$ . Use the existence theorem in EGSZ to find an equilibrium for this continuous representation. Average the choices for each type, obtaining private good choices  $x$  and club membership choices  $\mu$  for the original finite economy such that for every agent  $(x, \mu)$  is in the convex hull of his demand set at the given prices. Use the Shapley–Folkman theorem to construct new choices  $(\hat{x}, \hat{\mu})$  that keep the sum unchanged, keep the choice of every agent in the convex hull of his demand set, and have the property that the choices of all but  $n + m$  exceptional agents lie in the demand sets themselves. Set the choices of these  $n + m$  exceptional agents equal to 0, obtaining a choice vector  $(\bar{x}, \bar{\mu})$ . However,  $(\bar{x}, \bar{\mu})$  may not be a feasible state of the economy. The difficulty is not with private goods consumption, but rather with club membership choices. In the continuous representation of the economy, the representatives of the  $n + m$  exceptional agents and the representatives of some of the non-exceptional agents might belong to the same clubs. (Put differently, the non-exceptional agents might find that some of the clubs they wish to join have not filled their membership quotas.) If this is the case, the club membership choices represented in  $\bar{\mu}$  need not be consistent. This difficulty is not easily handled, because no obvious perturbation of  $\bar{\mu}$  will make these club membership choices consistent.

Lemma 3.1 — our main technical innovation — provides a way around this difficulty. Although the club membership choices represented by  $\bar{\mu}$  are not consistent, the construction of  $\bar{\mu}$  guarantees that they are nearly so. Lemma 3.1 guarantees that we can find a large subset  $A'$  of the set of non-exceptional agents (those whose choices are in their demand sets) so that the club memberships of agents in  $A'$  are consistent. Setting the club memberships of agents not in  $A'$  equal to zero (and adjusting private good consumption so that the sum of endowments equals the sum of consumption), we obtain a state in the fat  $\varepsilon$ -core.

Like the existence theorem, our decentralization theorem parallels an argument from the exchange case, and again relies on Lemma 3.1. Given an

allocation in the fat  $\varepsilon$ -core of an economy  $\mathcal{E}$ , let  $Z$  denote the set of sums (over all coalitions) of net preferred sets. As in Anderson (1978,1985), a decentralizing price is constructed by separating the convex hull of  $Z$  from a translate of an appropriate cone. There are three subtleties. First, in Anderson (1978), a net trade is a consumption vector net of endowment. In the club context, we need to account for the cost of forming clubs; we do so by defining a net trade to be a consumption vector net of endowment and the share of the input requirements of the clubs chosen. Second, in Anderson (1978) the feasible cone from which  $Z$  must be separated is a translate of the negative orthant. In the club context, the analogous cone would be the product of a translate of the negative orthant and the subspace representing consistent membership choices. However, separating  $Z$  from that cone might not guarantee that private goods prices are non-zero. To ensure that private good prices are non-zero, we must separate from a bigger cone. These two subtleties also arise in EGSZ. The third subtlety is special to our finite club economy and echoes the consistency problem described above. To show that  $Z$  is disjoint from the appropriate translate of the negative orthant, Anderson (1978) uses the Shapley-Folkman theorem to show that the contrary supposition leads to a blocking coalition formed by excluding the exceptional agents from the population. However, in the club context, excluding the exceptional agents need not lead to an allocation that is feasible. An additional, and more subtle, argument based on Lemma 3.1 allows us to prove that by excluding even more agents we can construct a blocking coalition.

As mentioned above, our notion of the approximate core is the fat  $\varepsilon$ -core, in which the resources used to form a blocking coalition are proportional to the total number of agents in the economy. Kannai (1970), Hildenbrand, Schmeidler and Zamir (1973) and Grodal (1976) use the weak  $\varepsilon$ -core, in which the resources used to form a blocking coalition are proportional to the size of the coalition. The reason we use the fat  $\varepsilon$ -core is that the weak  $\varepsilon$ -core of a club economy might easily be empty, no matter how large the economy. (See Example 5.2.) Moreover, as seen from Anderson (1985) and our decentralization theorem, the fat  $\varepsilon$ -core is the natural concept to use when decentralizing approximate core allocations. Since the weak  $\varepsilon$ -core is

contained in the fat  $\varepsilon$ -core, our results imply that allocations in the former can also be decentralized, but the rate of convergence for decentralization of the weak  $\varepsilon$ -core is no different from the rate for the fat  $\varepsilon$ -core (inversely proportional to the number of agents in the economy).

This paper is related to a vast literature on group formation that follows Buchanan (1965) and Tiebout (1956). The papers of that literature differ widely in the way in which they conceive and model entry, geographic space, political processes, price mechanisms, and the meaning of equilibrium; even a brief summary would be beyond the scope of this Introduction. However, the concerns of that literature also have much in common with those of the present paper, particularly non-existence of equilibrium, emptiness of the core, and the need to appeal to approximation in order to justify a competitive solution.<sup>2</sup> Where this paper and its companion differ most sharply from much of that literature is in our focus on the model with a continuum of agents as the proper idealization of perfect competition and as the proper idealization of a “real” economy with a large finite number of agents.

Several features of our model are worth reiterating. First, the set of possible types of clubs is given exogenously, but the number of clubs that actually form is determined endogenously, and many clubs of each type may form. Our clubs are not defined by pre-specified boundaries, nor are they “jurisdictions” in the political sense. Second, we allow each agent to belong to many clubs, as is natural if a club is interpreted as a marriage, a firm, a school, or a social group. (It might be natural to constrain agents to belong to a single club if a club were interpreted as a nation or a city or a political jurisdiction.) Third, we price all memberships in all types of clubs — even those which are not chosen. Fourth, we allow agents to trade with the market, and not just within the clubs they join.

Following this Introduction, Section 2 presents the formal description of

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<sup>2</sup>For a representative, but hardly exhaustive, list see Pauly (1970), Ellickson (1973, 1979), Bewley (1981), Scotchmer (1985, 1997), Scotchmer and Wooders (1987), Wooders (1988), Engl and Scotchmer (1996), Cole and Prescott (1997), Conley and Wooders (1997), Gilles and Scotchmer (1997).

a club economy. Section 3 discusses our main technical innovation and the integer programming problem on which it rests. Section 4 presents our main results: approximate decentralization of states in the fat  $\varepsilon$ -core and non-emptiness of the fat  $\varepsilon$ -core. Section 5 presents some illustrative examples. Most of the proofs are collected in Section 6.



## 2 Club Economies

### 2.1 Private goods

There are  $N \geq 1$  perfectly divisible, publicly traded private goods. Thus, the space of private goods is  $\mathbb{R}^N$  and the space of private good prices is also  $\mathbb{R}^N$ . Write  $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^N$  for the bundle of 1 unit of each private good, and write

$$\Delta = \{p \in \mathbb{R}_{++}^N : |p| = 1\}$$

for the open simplex of normalized private good prices.

For  $x, x' \in \mathbb{R}^J$  we write  $x \geq x'$  to mean  $x_j \geq x'_j$  for each  $j$ ,  $x > x'$  to mean that  $x \geq x'$  but  $x \neq x'$ , and  $x \gg x'$  to mean that  $x_j > x'_j$  for each  $j$ . We write  $|x| = \sum_{j=1}^J |x_j|$ . If  $B$  is a finite set we write  $|B|$  for its cardinality. If  $f : B \rightarrow \mathbb{R}^J$  is a function we write

$$f_B = \sum_{b \in B} f(b)$$

### 2.2 Clubs

As in EGSZ, we describe a *club type* by the number and characteristics of its members and the activity in which the club is engaged.

Let  $\Omega$  be a finite set of *external characteristics* of potential members of a club and let  $\Gamma$  be a finite set of club *activities*. An element  $\omega \in \Omega$  is a complete description of the external characteristics of an individual that are relevant for the other members of a club. A *profile* (of a club) is a function  $\pi : \Omega \rightarrow \mathbf{Z}_+ = \{0, 1, \dots\}$ . For  $\pi$  a profile and  $\omega \in \Omega$ ,  $\pi(\omega)$  is the number of members of the club having external characteristic  $\omega$ , and  $|\pi| = \sum_{\omega \in \Omega} \pi(\omega)$  is the total number of members in the club.

A *club type* is a pair  $(\pi, \gamma)$  consisting of a profile and an activity  $\gamma \in \Gamma$ . We take as given a finite set of possible club types  $\mathbf{Clubs} = \{(\pi, \gamma)\}$ .<sup>3</sup>

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<sup>3</sup>Because the set of club types is finite, there is a bound on the number of members of

Formation of the club  $(\pi, \gamma)$  requires a total input of private goods equal to  $\mathbf{inp}(\pi, \gamma) \in \mathbb{R}_+^N$ .

A *club membership* is an opening in a particular club type for an agent with a particular external characteristic; i.e., a triple  $m = (\omega, \pi, \gamma)$  such that  $(\pi, \gamma) \in \mathbf{Clubs}$  and  $\pi(\omega) \geq 1$ . Write  $\mathcal{M}$  for the set of club memberships.

Each agent may belong to many clubs or to none. A *list* is a function  $\ell : \mathcal{M} \rightarrow \{0, 1, \dots\}$ ;  $\ell(\omega, \pi, \gamma)$  specifies the number of memberships of type  $(\omega, \pi, \gamma)$ . Write  $\mathbf{Lists}$  for the set of lists.  $\mathbf{Lists}$  is a set of functions from  $\mathcal{M}$  to  $\{0, 1, \dots\}$ , but we frequently view it as a subset of  $\mathbb{R}^{\mathcal{M}}$ , which is the set of functions from  $\mathcal{M}$  to  $\mathbb{R}$ . For an arbitrary list  $\ell$  we define:

$$\tau(\ell) = \sum_{(\omega, \pi, \gamma) \in \mathcal{M}} \ell(\omega, \pi, \gamma) \frac{1}{|\pi|} \mathbf{inp}(\pi, \gamma)$$

$\tau(\ell) \in \mathbb{R}_+^N$  is the total bundle of inputs that an agent choosing list  $\ell$  would have to contribute if inputs to every club were imputed equally to each member of that club.

## 2.3 Agents

A complete description of an agent  $a$  consists of his/her external characteristics, choice set, endowment of private goods, and utility function.

The *external characteristic* of an agent  $a$  is an element  $\omega_a \in \Omega$ .

The *choice set* of an agent  $a$ , which specifies the feasible bundles of private goods and club memberships, is a subset  $X_a \subset \mathbb{R}^N \times \mathbf{Lists}$ . For simplicity, we assume that the only restriction on private good consumption is that it be non-negative, so that  $X_a = \mathbb{R}_+^N \times \mathbf{Lists}(a)$  for some subset  $\mathbf{Lists}(a) \subset \mathbf{Lists}$ . We assume that  $0 \in \mathbf{Lists}(a)$  for every  $a$  (so agents may choose to belong to no clubs) and that  $\ell(\omega, \pi, \gamma) = 0$  if  $(\omega, \pi, \gamma) \in \mathcal{M}$  and  $\omega \neq \omega_a$  (so that no individual may choose membership in any club type containing no members  


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any club).

of his/her external characteristic). We assume that there is a bound  $M$  on the number of memberships any agent may choose. Write

$$\mathbf{Lists}_M = \{\ell \in \mathbf{Lists} : |\ell| \leq M\}$$

so  $\mathbf{Lists}(a) \subset \mathbf{Lists}_M$  for each  $a$ .

The *endowment* of agent  $a$  is a bundle  $e_a \in \mathbb{R}_+^N$  of private goods; agents are not endowed with club memberships.

The *utility function* of agent  $a$  is defined over consumption of private goods and club memberships;  $u_a : X_a = \mathbb{R}_+^N \times \mathbf{Lists}(a) \rightarrow \mathbb{R}$ . We assume throughout that utility functions are continuous and strictly monotone in private goods; that is, for each list  $\ell \in \mathbf{Lists}(a)$ ,  $u_a(\cdot, \ell) : \mathbb{R}_+^N \rightarrow \mathbb{R}$  is continuous and strictly monotone. We make no assumption about the utility of choices  $(x, \ell), (x', \ell')$  when  $\ell \neq \ell'$ . In particular, club memberships may be desirable or undesirable, and club memberships and private goods may be complements or substitutes.

## 2.4 Club Economies

A *club economy*  $\mathcal{E}$  is specified by a finite set of private goods, a finite set of club types and corresponding inputs, a finite set of agents  $A$ , and a mapping  $a \mapsto (\omega_a, X_a, e_a, u_a)$  that assigns to each agent  $a \in A$  his/her external characteristic, choice set, endowment, and utility function.

## 2.5 Feasible States

A *state* of the club economy  $\mathcal{E}$  is a mapping  $(x, \mu) : A \rightarrow \mathbb{R}_+^N \times \mathbf{Lists}$ ;  $x : A \rightarrow \mathbb{R}_+^N$  specifies consumption of private goods and  $\mu : A \rightarrow \mathbf{Lists}$  specifies choices of club memberships. Feasibility of a state  $(x, \mu)$  entails feasibility of individual choices, market clearing for private goods, and feasibility of the list assignment  $\mu$ , in the sense of the following definition.

**Definition** The list assignment  $\mu : B \rightarrow \mathbf{Lists}$  is *feasible for B* if for every club type  $(\pi, \gamma)$  there is a (possibly empty) family of subsets (clubs)  $C^1, \dots, C^k$  of  $B$  (not necessarily disjoint) such that

- (1) for each  $\omega \in \Omega$  and each  $j$ :

$$\pi(\omega) = |\{a \in C^j : \omega_a = \omega\}|$$

(so the profile of the club  $C^j$  coincides with the profile  $\pi$  of the club type)

- (2) for each  $a \in B$ :

$$\mu_a(\omega, \pi, \gamma) = \begin{cases} |\{j : a \in C^j\}| & \text{if } \omega = \omega_a \\ 0 & \text{if } \omega \neq \omega_a \end{cases}$$

(so  $\mu_a(\omega, \pi, \gamma)$  is the number of clubs to which consumer  $a$  belongs).

Given the definition of feasibility for list assignments, the definition of feasibility for states is straightforward.

**Definition** A state  $(x, \mu)$  is *feasible for B* if :

- (1) individual choices belong to consumption sets:

$$(x_a, \mu_a) \in X_a \text{ for each } a \in B$$

(individual choices are feasible)

- (2) private consumption and inputs to club formation sum to the total endowment of agents in  $B$ :

$$\sum_{a \in B} x_a + \sum_{a \in B} \tau(\mu_a) = \sum_{a \in B} e_a$$

(markets for private goods clear)

- (3) the list assignment  $\mu$  is feasible for  $B$

We say the state  $(x, \mu)$  is *feasible* if it is feasible for the set  $A$  itself.

## 2.6 Consistency

An alternative formulation of feasibility for list assignments will prove to be very convenient. Say that a vector  $\bar{\mu} \in \mathbb{R}^{\mathcal{M}}$  (which will represent the aggregation of individual membership choices) is *consistent* if for each club type  $(\pi, \gamma)$  and external characteristic  $\omega$  with  $\pi(\omega) > 0$  the ratio

$$n(\pi, \gamma) = \frac{\bar{\mu}(\omega, \pi, \gamma)}{\pi(\omega)}$$

is independent of  $\omega \in \Omega$ . Say that  $\bar{\mu}$  is *integer consistent* if it is consistent and, for each club type  $(\pi, \gamma)$ , the ratio  $n(\pi, \gamma)$  is a non-negative integer. Write  $\mathbf{Cons} \subset \mathbb{R}^{\mathcal{M}}$  for the linear subspace of consistent membership vectors and  $\mathbf{Cons}^*$  for the set of integer consistent membership vectors. Note that  $\mathbf{Cons}$  is the linear subspace spanned by  $\mathbf{Cons}^*$ .

If  $B \subset A$ , we say that the list assignment  $\mu : B \rightarrow \mathbf{Lists}$  is *consistent* if  $\mu_a(\omega, \pi, \gamma) = 0$  whenever  $\omega \neq \omega_a$  (so individuals choose memberships only in clubs to which they might belong) and the corresponding aggregate membership vector

$$\mu_B = \sum_{a \in B} \mu_a \in \mathbb{R}^{\mathcal{M}}$$

is consistent. Similarly,  $\mu$  is *integer consistent* if  $\mu$  is consistent and  $\mu_B$  is integer consistent.

It is easy to see that a feasible list assignment is integer consistent, but an integer consistent list assignment need not be feasible. The difficulty is that integer consistency allows for the possibility that a particular individual is assigned several memberships in the same club. (For instance, suppose there is a single external characteristic and a single club type, consisting of 2 individuals. A list assignment in which one agent chooses 2 memberships and the other chooses none is integer consistent, but could only arise from formation of clubs if we were to allow a single agent to occupy both memberships in the same club.) The following proposition shows how to rule out precisely this problem.

**Proposition 2.1** *The list assignment  $\mu : B \rightarrow \mathbf{Lists}$  is feasible if and only if it is integer consistent and*

$$\mu_b(\pi, \gamma, \omega_b) \leq n(\pi, \gamma) = \frac{\mu_B(\omega, \pi, \gamma)}{\pi(\omega)}$$

for every agent  $b \in B$  and every club type  $(\pi, \gamma)$ . (That is, no agent is assigned more clubs of a given type than are formed by the entire population.)

**Proof** Let  $\mu$  be an integer consistent list assignment for which the given inequalities are satisfied. Fix a club type  $(\pi, \gamma)$ . If  $n(\pi, \gamma) = 0$ , the desired family  $\{C^j\}$  is the empty family, so we henceforth assume  $n(\pi, \gamma) > 0$ .

We construct the family  $\{C^j\}$  by an inductive procedure. Toward this end, define

$$n^0 = n(\pi, \gamma) \tag{1}$$

$$g^0(b, \omega) = \mu_b(\omega, \pi, \gamma) \text{ for each } b \in B, (\omega, \pi, \gamma) \in \mathcal{M} \tag{2}$$

Our definitions and the assumed inequality guarantee that

$$\sum_{b \in B} g^0(b, \omega) = n^0 \pi(\omega) \text{ for each } \omega \in \Omega \tag{3}$$

$$g^0(b, \omega) \leq n^0 \text{ for each } b \in B, \omega \in \Omega \tag{4}$$

For each  $\omega \in \Omega$  for which  $\pi(\omega) > 0$ , let

$$B_\omega^0 = \{b \in B : \omega_b = \omega, g^0(b, \omega) > 0\}$$

The properties (3), (4) guarantee that  $|B_\omega^0| \geq \pi(\omega)$ . Choose distinct individuals  $b_1^\omega, \dots, b_{\pi(\omega)}^\omega \in B_\omega^0$  to maximize the sum

$$\sum_{i=1}^{\pi(\omega)} g^0(b_i^\omega, \omega)$$

Define  $C^1 = \{b_i^\omega : \pi(\omega) > 0\}$ . Our construction guarantees that, for each  $\omega$ ,  $C^1$  contains exactly  $\pi(\omega)$  distinct individuals whose external characteristic

is  $\omega$ , and that  $\mu_b(\pi, \gamma, \omega) > 0$  for each such individual. Hence,  $C^1$  has the desired profile.

If  $n^0 = 1$  we are done. If  $n^0 > 1$ , we continue by defining

$$n^1 = n^0 - 1 \tag{5}$$

$$g^1(b, \omega) = \begin{cases} g^0(b, \omega) - 1 & \text{if } b \in C^1 \\ g^0(b, \omega) & \text{if } b \notin C^1 \end{cases} \tag{6}$$

Our construction guarantees that

$$\begin{aligned} \sum_{b \in B} g^1(b, \omega) &= \sum_{b \in B} g^0(b, \omega) - \sum_{b \in C^1} 1 \\ &= \sum_{b \in B} g^0(b, \omega) - \pi(\omega) \\ &= n^0 \pi(\omega) - \pi(\omega) \\ &= n^1 \pi(\omega) \end{aligned}$$

If  $c \in C^1$  then

$$g^1(c, \omega) = g^0(c, \omega) - 1 \leq n^0 - 1 = n^1$$

If  $d \in B \setminus C^1$  and  $g^1(d, \omega) > n^1$  then

$$g^0(d, \omega) = g^1(d, \omega) \geq n^0$$

Because  $\{b_1^\omega, \dots, b_{\pi(\omega)}^\omega\}$  were chosen to maximize  $\sum g^0(b_i, \omega)$ , it follows that  $g^0(c, \omega) \geq g^0(d, \omega)$  for every  $c \in C^1$ . Hence  $g^0(c, \omega) \geq n^0$  for every  $c \in C^1$ .

But then

$$\sum_{b \in B} g^0(b, \omega) \geq \sum_{b \in C^1 \cup \{d\}} g^0(b, \omega) \geq n^0 [\pi(\omega) + 1]$$

which contradicts (4). We conclude that

$$\sum_{b \in B} g^1(b, \omega) = n^1 \pi(\omega) \quad \text{for each } \omega \in \Omega \tag{7}$$

$$g^1(b, \omega) \leq n^1 \quad \text{for each } b \in B, \omega \in \Omega \tag{8}$$

Hence we may repeat the construction of  $C^1$  to obtain  $C^2, n^2, g^2$ . After  $n^0$  iterations,  $n^{\pi(\omega)} = 0$ ,  $g^{\pi(\omega)} \equiv 0$  and the process stops. We have constructed the desired family of clubs  $C^1, \dots, C^{n(\pi, \gamma)}$ .

The converse is straightforward and left to the reader. ■

## 2.7 Prices

In the club context, we price club memberships as well as private goods, so prices are pairs  $(p, q) \in \mathbb{R}^N \times \mathbb{R}^M$ .<sup>4</sup> Because we assume that preferences are strictly monotone in private goods, prices of private goods must be strictly positive. However, we do not assume that preferences are monotone in club memberships, so prices of club memberships may be positive, negative or zero. (Prices for club memberships incorporate transfers between agents in a given club.) As noted earlier, we find it convenient to normalize private good prices so that they lie in the open simplex  $\Delta$ . We do not normalize club membership prices.

## 2.8 Equilibrium

**Definition** A *club equilibrium* consists of a feasible state  $(x, \mu)$  and prices  $(p, q) \in \Delta \times \mathbb{R}^M$ ,  $p \neq 0$ , satisfying

- (1) **Budget Feasibility for Individuals** For each  $a \in A$ :

$$(p, q) \cdot (x_a, \mu_a) = p \cdot x_a + q \cdot \mu_a \leq p \cdot e_a$$

- (2) **Optimization** For each  $a \in A$ :

$$(x'_a, \mu'_a) \in X_a \text{ and } u_a(x'_a, \mu'_a) > u_a(x_a, \mu_a) \Rightarrow p \cdot x'_a + q \cdot \mu'_a > p \cdot e_a$$

- (3) **Budget Balance for Club Types** For each  $(\pi, \gamma) \in \mathbf{Clubs}$ :

$$\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) = p \cdot \mathbf{inp}(\pi, \gamma)$$

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<sup>4</sup>Note that club membership prices depend on the club type and on the external characteristics, but not on endowments or preferences.



Thus, at an equilibrium, individuals optimize subject to their budget constraints and the sum of membership prices in a given club type is just enough to pay for the inputs to clubs of that type. A *club quasi-equilibrium* consists of a feasible state  $(x, \mu)$  and prices  $(p, q) \in \Delta \times \mathbb{R}^M$ ,  $p \neq 0$ , satisfying (1), (3) and:

(2') **Quasi-Optimization** For all  $a \in A$ :

$$(x'_a, \mu'_a) \in X_a \text{ and } u_a(x'_a, \mu'_a) > u_a(x_a, \mu_a) \Rightarrow (p, q) \cdot (x'_a, \mu'_a) \geq p \cdot e_a$$

That is, nothing feasible and strictly preferred costs strictly less than agent  $a$ 's wealth.

Note that an equilibrium is necessarily a quasi-equilibrium, but not vice versa. EGSZ presents conditions that guarantee that every quasi-equilibrium is an equilibrium.

### 3 Feasible and Near-Feasible Assignments

As noted in the Introduction, our arguments depend on a construction that extracts a list assignment that is feasible from a list assignment that is only “nearly” feasible. To make this statement precise, fix a club structure **Clubs** and a bound  $M$  on the number of clubs to which an individual may belong.

**Lemma 3.1** *There are constants<sup>5</sup>  $K_1(\mathbf{Clubs}, M), K_2(\mathbf{Clubs}, M)$  such that:*

*If  $\mu : B \rightarrow \mathbf{Lists}_M$  is any list assignment, then there is a subset  $B' \subset B$  such that  $\mu|_{B'}$  is feasible and*

$$|B \setminus B'| \leq K_1(\mathbf{Clubs}, M) \text{dist}(\mu_B, \mathbf{Cons}) + K_2(\mathbf{Clubs}, M)$$

A thought exercise will help to understand what this result does *not* say. Suppose  $\mu : B \rightarrow \mathbf{Lists}_M$  is a feasible list assignment; for each club type  $(\pi, \gamma)$  let  $\{C_i^{(\pi, \gamma)}\}$  be a family of clubs whose existence is guaranteed by the definition of feasibility. Choose an arbitrary  $b \in B$  and set  $B^* = B \setminus \{b\}$ . It is evident that the distance from  $\mu_{B^*}$  to **Cons** is at most  $M$ , so Lemma 3.1 guarantees that there is a subset  $B' \subset B^*$  such that  $|B^* \setminus B'| \leq K_1 M + K_2$  and for which the restriction  $\mu|_{B'}$  is feasible. Again, there are families of clubs  $\{D_j^{(\pi, \gamma)}\}$  as guaranteed by the definition of feasibility. However, Lemma 3.1 does *not* guarantee that  $\{D_j^{(\pi, \gamma)}\}$  is a subfamily of  $\{C_i^{(\pi, \gamma)}\}$ , or that there is any particular relationship between the families. As illustrated in Example 5.3, to construct the family  $\{D_j^{(\pi, \gamma)}\}$  of clubs meeting the requirements of feasibility of  $\mu|_{B'}$ , it may be necessary to dissolve *all* of the clubs  $C_i^{(\pi, \gamma)}$  and form a completely different family.

The heart of the construction in Lemma 3.1 is an integer programming problem of some independent interest. To formulate this problem, we need some definitions. The non-empty subset  $H \subset \mathbf{Z}_+^m$  is *closed under addition* if

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<sup>5</sup>We derive the constants later. These constants depend on **Clubs** and  $M$  in a complicated and apparently non-constructive way.

$x, x' \in H \Rightarrow x + x' \in H$ ;  $H$  is relatively closed under subtraction if  $x, x' \in H, x - x' \in \mathbf{Z}_+^m \Rightarrow x - x' \in H$ . (Note that  $\mathbf{Cons}^* \subset \mathbb{R}^M$  enjoys both these properties.)

**Lemma 3.2** *If  $H \subset \mathbf{Z}_+^m$  is closed under addition and relatively closed under subtraction, and  $\mathbf{L} \subset \mathbf{Z}_+^m$  is a finite set, then there is a constant  $c(m, H, \mathbf{L})$  such that:*

*Given coefficients  $\{\beta_\ell : \ell \in \mathbf{L}\} \subset \mathbf{Z}_+$  there are coefficients  $\{\beta'_\ell : \ell \in \mathbf{L}\} \subset \mathbf{Z}_+$  such that*

- (1)  $\beta'_\ell \leq \beta_\ell$  for all  $\ell \in \mathbf{L}$
- (2)  $\sum \beta'_\ell \ell \in H$
- (3)  $\sum |\beta_\ell - \beta'_\ell| \leq c(m, H, \mathbf{L}) \text{dist}(\sum \beta_\ell \ell, H)$

The conclusion of Lemma 3.2 depends crucially on the assumption that  $H$  is relatively closed under subtraction, and need not obtain without this assumption. For instance, if  $m = 2$ ,  $\bar{H} = \{(0, 0)\} \cup \{(i, j) \in \mathbf{Z}_+^2 : i > j\}$  (which is closed under addition but not relatively closed under subtraction),  $\ell = (1, 1)$  and  $\mathbf{L} = \{\ell\}$ , then  $\text{dist}(\beta \ell, \bar{H}) = 1$  for every  $\beta \in \mathbf{Z}_+$ , but if  $\beta' \in \mathbf{Z}_+$  and  $\beta' \ell \in \bar{H}$  then  $\beta' = 0$ .

## 4 Approximate Core and Decentralization

Many notions of approximate core have been used in the literature; ours is the analog of one used in the exchange setting by Anderson (1985).

**Definition** Let  $(x, \mu)$  be a feasible state for the club economy  $\mathcal{E}$  and let  $\varepsilon \geq 0$ . We say  $(x, \mu)$  is in the *fat  $\varepsilon$ -core* of  $\mathcal{E}$  if there does not exist a non-empty coalition  $B \subset A$  and a state  $(y, \nu)$  such that

$$(1) \quad u_a(y_a, \nu_a) > u_a(x_a, \mu_a) \text{ for all } a \in B$$

$$(2) \quad \nu \text{ is feasible for } B$$

$$(3) \quad y_B + \tau(\nu_B) \leq e_B - \varepsilon|A|\mathbf{1}$$

We say  $(x, \mu)$  is in the *weak  $\varepsilon$ -core* of  $\mathcal{E}$  if there does not exist a non-empty coalition  $B \subset A$  and a state  $(y, \nu)$  satisfying (1) and (2) above and

$$(3') \quad y_B + \tau(\nu_B) \leq e_B - \varepsilon|B|\mathbf{1}$$

We say  $(x, \mu)$  is in the *strong  $\varepsilon$ -core* of  $\mathcal{E}$  if there does not exist a non-empty coalition  $B \subset A$  and a state  $(y, \nu)$  satisfying (1) and (2) above and

$$(3'') \quad y_B + \tau(\nu_B) \leq e_B - \varepsilon\mathbf{1}$$

Note that the fat  $\varepsilon$ -core contains the weak  $\varepsilon$ -core, the weak  $\varepsilon$ -core contains the strong  $\varepsilon$ -core, and the strong  $\varepsilon$ -core contains the core. Each of these  $\varepsilon$ -cores shrink as  $\varepsilon \downarrow 0$ , and all coincide with the core when  $\varepsilon = 0$ . We show below that the fat  $\varepsilon$ -core of a club economy is not empty provided that the economy contains sufficiently many individuals. As Example 5.2 demonstrates, both the weak and strong  $\varepsilon$ -cores may be empty, no matter how many individuals the economy may contain.

## 4.1 Approximate Decentralization

Our notion of approximate decentralization parallels that of Anderson (1978, 1985). Following Anderson, we construct two measures of how well a given price system  $(p, q) \in \Delta \times \mathbb{R}^M$  approximately decentralizes a feasible state  $(x, \mu)$  for an economy  $\mathcal{E}$ . For  $a \in A$  define

$$\begin{aligned}\rho_a^1(x, \mu; p, q) &= [(p, q) \cdot (x_a, \mu_a) - p \cdot e_a]^+ \\ \rho_a^2(x, \mu; p, q) &= \sup \left\{ [p \cdot e_a - (p, q) \cdot (x', \mu')]^+ : u_a(x', \mu') > u_a(x_a, \mu_a) \right\}\end{aligned}$$

(where  $r^+ = \max\{r, 0\}$  is the positive part of the real number  $r$ ). The number  $\rho_a^1(x, \mu; p, q)$  measures how far  $(x_a, \mu_a)$  lies outside agent  $a$ 's budget set; the number  $\rho_a^2(x, \mu; p, q)$  measures how much less than the value of his initial endowment the agent  $a$  can spend and still obtain something preferred to  $(x_a, \mu_a)$ . Note that  $\rho_a^1(x, \mu; p, q) = 0$  if and only if  $(x_a, \mu_a)$  lies in agent  $a$ 's budget set and that  $\rho_a^2(x, \mu; p, q) = 0$  if and only if  $(x_a, \mu_a)$  is a quasi-optimizing bundle for agent  $a$ . Our measures of approximate decentralization are the averages:

$$\begin{aligned}\rho^1(x, \mu; p, q) &= \frac{1}{|A|} \sum_{a \in A} \rho_a^1(x, \mu; p, q) \\ \rho^2(x, \mu; p, q) &= \frac{1}{|A|} \sum_{a \in A} \rho_a^2(x, \mu; p, q)\end{aligned}$$

**Definition** For  $\delta \geq 0$ , the feasible state  $(x, \mu)$  can be  $\delta$ -decentralized by prices if there exist  $(p, q) \in \Delta \times \mathbb{R}^M$  such that

- (1)  $\rho^1(x, \mu; p, q) \leq \delta$
- (2)  $\rho^2(x, \mu; p, q) \leq \delta$
- (3) for each  $(\pi, \gamma) \in \mathbf{Clubs}$ :

$$\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) = p \cdot \mathbf{inp}(\pi, \gamma)$$

That is, the average deviation from individual budget sets is small, the average deviation from individual optimization is small, and budgets of clubs balance *exactly*.

Note that if  $(x, \mu)$  is 0-decentralized by prices  $(p, q)$  then  $(p, q), (x, \mu)$  constitute a quasi-equilibrium for  $\mathcal{E}$ .

In what follows,  $K_1(\mathbf{Clubs}, M), K_2(\mathbf{Clubs}, M)$  are the constants determined in Lemma 3.1, which depend only on the club structure and the number of clubs to which an individual may belong. (In particular, these constants do not depend on the number of agents in the economy.) It is convenient to set

$$K_3(\mathbf{Clubs}, M, N) = [1 + M K_1(\mathbf{Clubs}, M)] [N + |\mathcal{M}|] + K_2(\mathbf{Clubs}, M)$$

If  $\mathbf{Clubs}, M, N$  are understood, we simplify notation by writing  $K_3$  rather than  $K_3(\mathbf{Clubs}, M, N)$ . If  $\mathcal{E}$  is a club economy, write

$$W(\mathcal{E}) = \max \{e_{a_n} : a \in A, 1 \leq n \leq N\}$$

**Theorem 4.1** *If  $\mathcal{E}$  is a club economy and  $\varepsilon \geq 0$ , then every state in the fat  $\varepsilon$ -core can be  $K_3 W(\mathcal{E}) |A|^{-1} + \varepsilon$  - decentralized by prices.*

Because the fat  $\varepsilon$ -core coincides with the core when  $\varepsilon = 0$ , the following is an immediate corollary.

**Corollary 4.2** *If  $\mathcal{E}$  is a club economy, then every state in the core of  $\mathcal{E}$  can be  $K_3 W(\mathcal{E}) |A|^{-1}$  - decentralized by prices.*

Note that these results give a “rate of convergence” for decentralization of  $|A|^{-1}$ , exactly as in the exchange case; cf. Anderson (1978, 1985).

## 4.2 Existence

The decentralization Theorem 4.1 is of little interest when the fat  $\varepsilon$ -core is empty. Fortunately, the fat  $\varepsilon$ -core is not empty whenever the number of agents in the economy is sufficiently large. Let  $K_1, K_2$  denote the constants  $K_1(\mathbf{Clubs}, M), K_2(\mathbf{Clubs}, M)$  identified in Lemma 3.1. Let

$$K_4(\mathbf{Clubs}, M, N) = (K_1 M + 1)[(K_1 M + 1)(N + |\mathcal{M}|) + K_2] + K_2$$

Again, if  $\mathbf{Clubs}, M, N$  are understood, we simplify notation by writing  $K_4$  rather than  $K_4(\mathbf{Clubs}, M, N)$ .

**Theorem 4.3** *If  $\mathcal{E}$  is a club economy and  $\varepsilon > K_4 W(\mathcal{E}) |A|^{-1}$ , then the fat  $\varepsilon$ -core of  $\mathcal{E}$  is not empty.*

Again, the “rate of convergence” is  $|A|^{-1}$ , as in the exchange case; cf. Grodal, Trockel and Weber (1984).

## 5 Examples

We begin with an example that illustrates the meaning of approximate decentralization.

### Example 5.1 Approximation decentralization

Consider an economy  $\mathcal{E}$  with one private good, one external characteristic  $\omega$ , and one club type  $c = (\pi, \gamma)$ , with  $\pi(\omega) = 2$  and  $\mathbf{inp}(c) = 2$ . Each agent is endowed with 3 units of the private good, can choose at most two club memberships, and has utility function

$$u(x, \ell) = \begin{cases} x & \text{if no club memberships are chosen;} \\ 2x & \text{if one club membership is chosen;} \\ 5x & \text{if two club memberships are chosen.} \end{cases}$$

Normalize the private good price  $p = 1$ . If equilibrium exists, budget balance requires  $q(\omega, c) = 1$  for any club which forms. At these prices, optimization requires that each agent choose 1 unit of the private good and 2 club memberships. These choices define a feasible state.<sup>6</sup> If  $|A| \geq 3$ , this is the only state in the core — so this economy displays core equivalence.

However, in club economies with a finite number of agents, both core and equilibrium are very fragile. Suppose choosing 2 memberships yields utility  $u(x, \ell) = 3x$ , all else remaining the same. Once again, if an equilibrium exists we must have  $p = q(\omega, c) = 1$ . Facing these prices, each agent chooses 2 units of consumption and 1 membership. If  $|A|$  is even, these choices define a feasible state. (Agents pair into  $|A|/2$  clubs.) And if  $|A| \geq 4$ , this is the only state in the core, so this economy again displays core equivalence. But if  $|A|$  is *odd* and  $|A| \geq 5$ , the core is empty, and no equilibrium exists.<sup>7</sup>

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<sup>6</sup>With respect to list assignments, one possibility is: agent 1 enters into a club with agent 2 and a club with agent  $|A|$ , agent  $|A|$  enters into a club with agent  $|A| - 1$  and a club with agent 1, and every agent  $a$  with  $1 < a < |A|$  enters into one club with agent  $a - 1$  and another with agent  $a + 1$ .

<sup>7</sup>Indeed the core is empty if  $|A| = 3$ , but the argument is somewhat different.



If agents were permitted to choose at most one club membership (as in most of the club literature), this would be easy to see. Suppose  $(x, \mu)$  is a core state and suppose that two agents, say  $a = 1$  and  $a = 2$ , choose no memberships. Individual rationality implies  $x_1 \geq 3$  and  $x_2 \geq 3$ . But now  $(x, \mu)$  will be Pareto dominated if agents 1 and 2 form a club (in addition to the existing clubs) and consume  $x'_1 = x_1 - 1$  and  $x'_2 = x_2 - 1$ . Thus,  $\mu_a = 1$  except for exactly one agent, say agent 1. Individual rationality implies  $x_1 \geq 3$ ; the inability of  $A \setminus \{1\}$  to block implies  $x_1 \leq 3$ . Hence  $x_1 = 3$ . For each  $a \in A \setminus \{1\}$ , the inability of  $\{1, a\}$  to block implies that  $x_1/2 + x_a \geq 4$ , so  $x_a \geq 5/2$ . Because this inequality holds for every  $a \in A \setminus \{1\}$ , it follows that the total consumption of agents in  $A \setminus \{1\}$  is at least  $(|A| - 1)\frac{5}{2}$ . Because these agents all belong to a club, their total cost of club formation is  $\frac{1}{2}(|A| - 1)2 = |A| - 1$ . But then the total of private good consumption plus the input to club formation for all agents is at least  $3 + (|A| - 1)\frac{7}{2}$  while the total endowment of all agents is  $3|A|$ . But  $3 + (|A| - 1)\frac{7}{2} > 3|A|$  (this requires only that  $|A| > 1$ ), so we have arrived at a contradiction. We conclude that the core is empty.

Since we allow agents to choose two memberships, the argument is more subtle. Suppose  $(x, \mu)$  is a core state. As above there is at most one agent who does not choose a club membership. Suppose there are two agents, say 1 and 2, who each choose two memberships. If  $x_1 + x_2 \geq 2$ , then, because  $|A| \geq 5$ , the agents in  $A \setminus \{1, 2\}$  could, using only their own resources, form clubs among themselves, each consume the same number of memberships as in  $\mu$ , and obtain private good consumption as in  $x$  (see Proposition 2.1). That is,  $A \setminus \{1, 2\}$  could block  $(x, \mu)$ . Since  $(x, \mu)$  is in the core, we conclude that  $x_1 + x_2 \leq 2$ . But then  $\{1, 2\}$  can block by forming one club and distributing consumption appropriately; this is a contradiction. Hence there is at most one agent who belongs to two clubs.

As we have shown above, there are no core states in which all agents have at most one membership. Consistency of membership choices implies that it is impossible that one agent belongs to no clubs, one agent belongs to two clubs, and all other agents belong to one club. Hence in a putative core state

$(x, \mu)$  it must be the case that one agent, say 1, belongs to two clubs and all other agents belong to one club. Because  $A \setminus \{1\}$  can block, we can argue as above to show that  $x_1 \leq 1$ . For each  $a \in A \setminus \{1\}$ , the inability of  $\{1, a\}$  to block implies that  $3x_1 + 2x_a \geq 8$  and hence that  $x_2 \geq 5/2$ . As before, this means that the total of private good consumption plus club membership cost for all agents exceeds the social endowment, which is a contradiction. We conclude that the core is empty, as asserted.

However, we can find states in the fat  $\varepsilon$ -core and these states are almost decentralizable by prices, provided the number of agents is large enough. Consider the state  $(y, \nu)$  in which all but one agent chooses one club membership and consumes 2 units of the private good, while the remaining agent, say agent 1, chooses no club memberships and consumes 3 units of the private good. The best blocking opportunities are by two-person coalitions. Suppose agent 1 and agent  $a \neq 1$  form a coalition and are required to pay a coalition formation cost of  $\varepsilon|A|$  in addition to providing the input 2. They can obtain utilities summing to  $8 - 2\varepsilon|A|$ . Since their utilities in the state  $(y, \nu)$  sum to 7, these agents cannot both improve if  $\varepsilon|A| \geq 1/2$ . Put differently,  $(y, \nu)$  is in the fat  $\varepsilon$ -core if  $\varepsilon|A| \geq 1/2$  or  $|A| \geq 1/2\varepsilon$ . (Note that the number of agents necessary is inversely proportional to  $\varepsilon$ .) At prices  $p = q(\omega, c) = 1$ , all agents except agent 1 are optimizing, so our measures of decentralization are

$$\rho^1(y, \nu; p, q) = 0 \quad \text{and} \quad \rho^2(y, \nu; p, q) = \frac{1}{2|A|}$$

That is,  $(y, \nu)$  is  $1/2|A|$ -decentralized by the given prices.

Although this state is “almost” an equilibrium (only one agent is not optimizing), there is no reason to attach special significance to this particular member of the fat- $\varepsilon$  core; other states in the fat- $\varepsilon$  core might be more plausible. Consider for example the state  $(y', \nu')$  in which agent 1 is again the only one left out of a club, but agents in clubs transfer enough of the private good to agent 1 so that everyone achieves the same level of utility. That is, private good consumptions are

$$y'_1 = 3 + \frac{|A| - 1}{|A| + 1}$$

$$y'_a = 2 - \frac{1}{|A| + 1} \quad \text{if } a \neq 1$$

We leave to the reader to check that this state is in the fat  $\varepsilon$ -core if  $\varepsilon \geq (|A| - 1)/(|A|(|A| + 1))$ , in the weak  $\varepsilon$ -core if  $\varepsilon \geq 1/(|A| + 1)$ , in the strong  $\varepsilon$ -core if  $\varepsilon \geq (|A| - 1)/(|A| + 1)$  and that, at prices  $p = 1$  and  $q(\omega, c) = 1$ , the measures of approximate decentralization are

$$\rho^1(x, \mu; p, q) = \rho^2(x, \mu; p, q) = \frac{|A| - 1}{|A|^2 + |A|} \leq \frac{1}{|A|}$$

In the example above, the weak  $\varepsilon$ -core is not empty, provided the number of agents is sufficiently large. In the following example, the weak  $\varepsilon$ -core is empty, no matter how large the number of agents is, but the fat  $\varepsilon$ -core is not empty.

### Example 5.2 Empty approximate cores

Consider an economy  $\mathcal{E}$  with one private good, one external characteristic  $\omega$ , and one club type  $c = (\pi, \gamma)$ , with  $\pi(\omega) = 2$  and  $\mathbf{inp}(c) = 0$ . Each agent is endowed with 1 unit of the private good, can choose at most one club membership, and has utility function

$$u(x, \mu) = \begin{cases} 1 - e^{-x} & \text{if no membership is chosen} \\ 4x & \text{if one membership is chosen} \end{cases}$$

We assert that if the number of agents  $|A|$  is odd then the weak  $\varepsilon$ -core is empty for all  $\varepsilon < 1/4$ , regardless of the size of the economy. To see this, consider a feasible state  $(x, \mu)$ . Because there are an odd number of agents, at least one, say agent 1, is not in a club. Since he cannot weakly  $\varepsilon$ -block,  $x_1 \geq 1 - \varepsilon$ . Because the state is feasible, there exists an agent, say agent 2, such that  $x_2 \leq 1 + \varepsilon$ . Since clubs require no inputs and  $1/4 + (1 + \varepsilon) < 2 - 2\varepsilon$ , agents 1 and 2 can form a club and consume  $x'_1 \geq 1/4$ ,  $x'_2 > 1 + \varepsilon$ . Because both agents get higher utility than in the state  $(x, \mu)$ , the coalition  $\{1, 2\}$  can

weakly  $\varepsilon$ -block the state. We conclude that the weak  $\varepsilon$ -core (and *a fortiori* the strong  $\varepsilon$ -core) is empty, regardless of the number of agents.

However, for any  $\varepsilon > 0$  the fat  $\varepsilon$ -core is nonempty provided the number of agents is sufficiently large. Indeed, direct calculation shows that the state in which each agent consumes his endowment and as many agents as possible choose clubs (one agent being left-out if the number of agents is odd) is in the fat  $\varepsilon$ -core provided  $|A| \geq 1/\varepsilon$ .

Our final example illustrates the meaning of Lemma 3.1 (see especially the discussion of the thought experiment).

**Example 5.3 Feasible list assignments.**

Consider an economy populated by agents with two distinct external characteristics, male and female:  $\Omega = \{M, F\}$ . There is a single private good. Three clubs are possible, distinguished by the club profile:

- $c_1 = (\pi_1, \gamma_1)$ :  $\pi_1(M) = 2$  and  $\pi_1(F) = 0$
- $c_2 = (\pi_2, \gamma_2)$ :  $\pi_2(M) = 0$  and  $\pi_2(F) = 2$
- $c_3 = (\pi_3, \gamma_3)$ :  $\pi_3(M) = 1$  and  $\pi_3(F) = 1$

All clubs require 2 units of input, agents can belong to at most two clubs, and everyone is endowed with 3 units of the private good.

Assume that agents choosing no club membership have utility function

$$u_M(x, 0) = u_F(x, 0) = x$$

(The subscript  $M$  or  $F$  indicates that the agent is male or female.) Agents choosing one club membership have utility function

$$u_M(x, c_1) = u_M(x, c_3) = u_F(x, c_3) = u_F(x, c_2) = 2x;$$

agents choosing two club memberships with members of the same sex have utility function

$$u_M(x, 2c_1) = u_F(x, 2c_2) = 5x;$$

and agents choosing two memberships, at least one with a member of the opposite sex, have utility function

$$u_M(x, c_1 + c_3) = u_M(x, 2c_3) = u_F(x, 2c_3) = u_F(x, c_2 + c_3) = 6x$$

Figure 1 goes here

Suppose  $|A| = 6$  with three males and three females. There are many equilibrium states, but all assign every agent one of their top-ranked lists: i.e., two memberships, at least one with a member of the opposite sex. The left-hand diagram in Figure 1 illustrates one of these equilibrium assignments corresponding to 1 club of type  $c_1$ , 1 club of type  $c_2$  and 4 clubs of type  $c_3$  with

$$\mu_A(\omega, c) = \begin{cases} 2 & \text{if } (\omega, c) = (M, c_1) \\ 2 & \text{if } (\omega, c) = (F, c_2) \\ 4 & \text{if } (\omega, c) = (M, c_3) \\ 4 & \text{if } (\omega, c) = (F, c_3) \end{cases}$$

Suppose we form a coalition  $B^*$  by dropping the male on the left; the assignments are as in the middle diagram in Figure 1. The restriction  $\mu|_{B^*}$  is not feasible since

$$\mu_{B^*}(\omega, c) = \begin{cases} 2 & \text{if } (\omega, c) = (M, c_1) \\ 2 & \text{if } (\omega, c) = (F, c_2) \\ 2 & \text{if } (\omega, c) = (M, c_3) \\ 4 & \text{if } (\omega, c) = (F, c_3) \end{cases}$$

so there is a mismatch in the number of male and female memberships in clubs of type  $c_3$ . Lemma 3.1 seeks a  $B' \subset B^*$  for which the restriction  $\mu_{B'}$  is feasible. We can construct  $B'$  by deleting a single agent, the female on the right, obtaining

$$\mu_{B'}(\omega, c) = \begin{cases} 2 & \text{if } (\omega, c) = (M, c_1) \\ 2 & \text{if } (\omega, c) = (F, c_2) \\ 2 & \text{if } (\omega, c) = (M, c_3) \\ 2 & \text{if } (\omega, c) = (F, c_3) \end{cases}$$

However, actual assignments to clubs must now be quite different: see the right-hand diagram in Figure 1.

## 6 Proofs

### 6.1 Feasible and Near-Feasible Assignments

To establish Lemma 3.2, which will be used repeatedly in the proof of Lemma 3.1), we need a preliminary notion and result.

Let  $H \subset \mathbf{Z}_+^m$  be a non-empty subset that is closed under addition and relatively closed under subtraction, and let  $G \subset H$  be a finite subset. We say that  $H$  is *generated by*  $G$  if every element of  $H$  is the sum of elements of  $G$ ; that is, for every  $x \in H$  there are non-negative integers  $\{\gamma_g(x) : g \in G\}$  such that  $x = \sum_{g \in G} \gamma_g(x) g$ .

**Lemma 6.1** *Every subset  $H \subset \mathbf{Z}_+^m$  that is closed under addition and relatively closed under subtraction is generated by a finite set.*

**Proof** For each integer  $k > 0$ , write  $H_k = \{x \in H : |x| \leq k\}$ . We assert that there is an integer  $k$  with the property that every element of  $H$  dominates some element of  $H_k$ ; that is, for each  $x \in H$  there is a  $y \in H_k, y \neq 0$  such that  $x \geq y$ . Suppose it were not so. Then for each integer  $k$  there would be an  $x^k \in H$  which does not dominate any element of  $H_k$ . In particular,  $x^k \notin H_k$ , so  $|x^k| > k$ . For each coordinate  $1 \leq i \leq m$ , the sequence  $(x_i^k)$  is either bounded or not. If it is bounded we may use the fact that elements of  $H$  have non-negative integer coordinates to extract a subsequence that is constant; if it is unbounded we may extract a subsequence that is strictly increasing to infinity. Applying the same reasoning to each coordinate in turn, we may extract a subsequence  $(x^{k_j})$  that is non-decreasing:  $x^{k_j} \leq x^{k_{j+1}}$  for each  $j$ . Set  $k^* = |x^{k_1}|$ . Because  $k_j \rightarrow \infty$ , there is an index  $j^*$  such that  $k_{j^*} > k^*$ . Since  $x^{k_j} \geq x^{k_1}$  for every  $j$ , it follows that  $x^{k_{j^*}} \geq x^{k_1}$ . Because  $x^{k_1}$  is an element of  $H_{k^*}$  and  $H_{k^*}$  is a subset of  $H_{k_{j^*}}$ , it follows that  $x^{k_{j^*}}$  dominates an element of  $H_{k_{j^*}}$ . This is a contradiction, so we conclude the existence of such an integer  $k$ , as asserted.

We claim that  $H_k$  (which is a finite set) generates  $H$ ; that is, every element

of  $H$  is a sum of elements of  $H_k$ . To see this, we proceed by induction on  $|x|$ . The assertion is evidently true when  $|x| \leq k$  (because then  $x \in H_k$ ). Assume then that the assertion is true for all  $x \in H$  with  $|x| < r$  (with  $r \geq k$ ) and consider an element  $y \in H$  with  $|y| = r$ . In view of the assertion established in the previous paragraph, there is a nonzero element  $z \in H_k$  with  $y \geq z$ . Because  $H$  is closed under conditional subtraction, it follows that  $y - z \in H$ , and it is evident that  $|y - z| < |y| = r$ . Our induction hypothesis guarantees that  $y - z$  is the sum of elements of  $H_k$ . Because  $y = (y - z) + z$  and  $z \in H_k$ ,  $y$  is also the sum of elements of  $H_k$ . The conclusion follows by induction. ■

We note that, as with Lemma 3.2, the conclusion of Lemma 6.1 depends crucially on the assumption that  $H$  is relatively closed under subtraction. Our counterexample in Section 3 will work here too: it is easily seen that  $\bar{H} = \{(0, 0)\} \cup \{(i, j) \in \mathbf{Z}_+^2 : i > j\}$  is not finitely generated.

With this result in hand, we turn to our integer programming problem.

**Proof of Lemma 3.2** Set

$$\mathbf{L}^* = \{y \in H : \exists \ell \in \mathbf{L}, y \leq \ell\}$$

Because  $H \subset \mathbf{Z}_+^m$  and  $\mathbf{L}$  is finite,  $\mathbf{L}^*$  is also finite. View elements  $\ell \in \mathbf{L}^*$  as unit vectors  $\delta_\ell \in \mathbb{R}^{\mathbf{L}^*}$ . We work in  $\mathbb{R}^{\mathbf{L}}$  and in  $\mathbb{R}^{\mathbf{L}^*}$ ; it is convenient to use Latin letters for elements of  $\mathbb{R}^{\mathbf{L}}$  and Greek letters for elements of  $\mathbb{R}^{\mathbf{L}^*}$ . Define the linear map  $T : \mathbb{R}^{\mathbf{L}^*} \rightarrow \mathbb{R}^m$  by  $T(\delta_\ell) = \ell$  and set

$$\Upsilon = \{v \in \mathbf{Z}_+^{\mathbf{L}^*} : T(v) \in H\}$$

Define  $s \in \mathbf{Z}_+^m, \sigma \in \mathbf{Z}_+^{\mathbf{L}^*}$  by

$$s = \sum_{\ell \in \mathbf{L}} \beta_\ell \ell, \quad \sigma = \sum_{\ell \in \mathbf{L}} \beta_\ell \delta_\ell$$

Our ultimate goal is to construct  $\zeta \in \Upsilon$  with  $\zeta \leq \sigma$  for which  $|\sigma - \zeta|$  is small. Because  $\sigma$  is a sum of unit basis vectors, so is  $\zeta$ ; we will therefore be able to find coefficients  $\beta'_\ell \leq \beta_\ell$  such that  $\zeta = \sum \beta'_\ell \delta_\ell$  and  $\sum |\beta_\ell - \beta'_\ell| = |\sigma - \zeta|$ . The



definitions of  $T$  and of  $\Upsilon$  entail that  $T(\zeta) \in H$ , and because our estimate of  $|\sigma - \zeta|$  entails an estimate for  $|s - T(\zeta)|$ , this will yield the required estimate.

Because the construction of  $\zeta$  requires several intermediate constructions and estimations, the proof is in several steps: 1) choose sets of generators for  $H, \Upsilon$  and, from these sets of generators, define  $c(m, H, L)$ ; 2) construct  $x \in H$  with  $x \leq s$ , and estimate  $|s - x|$ ; 3) from  $x$ , construct  $\chi \in \Upsilon$  with  $T(\chi) = x$ , and estimate  $|\sigma - \chi|$ ; 4) from  $\chi$ , construct  $\zeta$ , and derive the required estimate.

**Step 1** We have assumed  $H$  is closed under addition and relatively closed under subtraction; it is evident that  $\Upsilon \subset \mathbf{Z}_+^{L^*}$  also enjoys these properties. Use Lemma 6.1 to choose finite sets of generators  $G, \Gamma$  for  $H, \Upsilon$ .<sup>8</sup> Define

$$c(m, H, \mathbf{L}) = 2 \left( \max_{g \in G} |g| + 1 \right) \left( \max_{\gamma \in \Gamma} |\gamma| + 1 \right)$$

**Step 2** We wish to construct  $x \in H$  with  $s \geq x$  for which  $|s - x|$  is small. To this end, fix an element  $w \in H$  such that  $|s - w| = \text{dist}(s, H)$ . If  $w \leq s$ , take  $x = w$ . If  $w \not\leq s$ , use Lemma 6.1 to write

$$w = \sum_{g \in G} \gamma_g(w) g$$

Because  $w \not\leq s$ , there is an index  $i$  with  $w_i > s_i$ . Pick  $g^1 \in G$  such that  $\gamma_{g^1}(w) > 0$  and  $g_i^1 > 0$ , and set

$$x^1 = [\gamma_{g^1}(w) - 1] g^1 + \sum_{g \neq g^1} \gamma_g(w) g$$

By construction,  $x^1 \in H$ ,  $x^1 \leq w$  and  $x_i^1 < w_i$ . If  $x^1 \leq s$ , set  $x = x^1$ . Otherwise continue in this way to construct a decreasing sequence  $x^1, x^2, \dots$  of elements of  $H$ . After at most  $|s - w|$  iterations, we obtain a vector  $x = x^j \in H$

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<sup>8</sup>In our application,  $H = \mathbf{Cons}^*$ . As we shall see later (see paragraph before Lemma 6.4), it is easy to write down a simple set of generators for  $\mathbf{Cons}^*$ , but it does not seem easy to write down a set of generators for the corresponding set  $\Upsilon$ .

with  $x \leq s$ . Since we subtract an element of  $G$  at each iteration, it follows that

$$|w - x| \leq |s - w| \left( \max_{g \in G} |g| \right)$$

whence

$$|s - x| \leq |s - w| + |w - x| \leq |s - w| \left( \max_{g \in G} |g| + 1 \right) \quad (9)$$

**Step 3** Given  $x$  we wish to construct  $\chi \in \Upsilon$  with  $T(\chi) = x$ , and estimate  $|\sigma - \chi|$ . To construct  $\chi$ , list the elements of  $\mathbf{L}$ , with each element  $\ell$  occurring  $\beta_\ell$  times:  $\ell_1, \ell_2, \dots, \ell_n$ , where  $n = \sum \beta_\ell$ . Now proceed inductively to define  $y_1, y_2, \dots, y_n \in \mathbf{L}^*$ :

$$\begin{aligned} y_1 &= \min\{\ell_1, x\} \\ y_2 &= \min\{\ell_2, x - y_1\} \\ &\vdots \\ y_{n-1} &= \min\left\{\ell_{n-1}, x - \sum_{i=1}^{n-2} y_i\right\} \\ y_n &= x - \sum_{i=1}^{n-1} y_i \end{aligned}$$

This construction guarantees that

$$x - \sum_{i=1}^k y_i \leq \sum_{j=k+1}^n \ell_j$$

for each  $k$ ; hence  $0 \leq y_i \leq \ell_i$  for each  $i$  and  $\sum_{i=1}^n y_i = x$ . Set

$$\chi = \sum_{i=1}^n \delta_{y_i}$$

The definition of  $T$  entails that  $T(\chi) = x$  so  $\chi \in \Upsilon$ .

Let  $I = \{i : 1 \leq i \leq n, y_i = \ell_i\}$  and let  $J = \{i : 1 \leq i \leq n, y_i \neq \ell_i\}$  be the complementary set of indices. Because  $0 \leq y_i \leq \ell_i$  for each  $i$  and

$\ell_i - y_i \in \mathbf{Z}_+^m$ , it follows that

$$\begin{aligned} |\ell_i - y_i| &= 0 & \text{if } i \in I \\ |\ell_i - y_i| &\geq 1 & \text{if } i \in J \end{aligned}$$

The distance between distinct unit basis vectors in  $\mathbb{R}^{\mathbf{L}^*}$  is 2, so  $|\delta_{\ell_i} - \delta_{y_i}| = 2$  whenever  $i \in J$ . Because  $\sigma = \sum \beta_\ell \ell = \sum \delta_{\ell_i}$ , it follows that

$$|\sigma - \chi| = 2|J| \tag{10}$$

and

$$|J| \leq \sum_{i \in J} |\ell_i - y_i| = \sum_{i=1}^n |\ell_i - y_i| = |s - x| \tag{11}$$

**Step 4** We proceed exactly as in Step 2 to construct  $\zeta \in \Upsilon$  such that  $\zeta \leq \sigma$  and

$$|\zeta - \sigma| \leq |\sigma - \chi| \left( \max_{\gamma \in \Gamma} |\gamma| + 1 \right) \tag{12}$$

For each  $\ell$ , set  $\beta'_\ell = \zeta_{\delta_\ell}$ ; note that  $\zeta = \sum \beta'_\ell \delta_\ell$ . Because  $\zeta \leq \sigma$ , it follows that

$$\beta'_\ell = \zeta_{\delta_\ell} \leq \sigma_{\delta_\ell} = \beta_\ell$$

for each  $\ell$ . By construction,  $\zeta \in \Upsilon$  so  $T(\zeta) \in H$ . Linearity of  $T$  entails that

$$T(\zeta) = T \left( \sum_{\ell \in \mathbf{L}} \beta'_\ell \delta_\ell \right) = \sum_{\ell \in \mathbf{L}} \beta'_\ell \ell$$

Moreover

$$\sum_{\ell \in \mathbf{L}} |\beta_\ell - \beta'_\ell| = \sum_{\ell \in \mathbf{L}} |\sigma_{\delta_\ell} - \zeta_{\delta_\ell}| = |\sigma - \zeta| \tag{13}$$

Combining (9) – (13), expanding, substituting the definition of  $c(m, H, \mathbf{L})$ , and recalling that  $|s - w| = \text{dist}(s, H)$  yields:

$$\begin{aligned} \sum_{\ell \in \mathbf{L}} |\beta_\ell - \beta'_\ell| &= |\sigma - \zeta| \\ &\leq |\sigma - \chi| \left( \max_{\gamma \in \Gamma} |\gamma| + 1 \right) \end{aligned}$$

$$\begin{aligned}
&= 2|J| \left( \max_{\gamma \in \Gamma} |\gamma| + 1 \right) \\
&\leq 2|s - x| \left( \max_{\gamma \in \Gamma} |\gamma| + 1 \right) \\
&\leq 2|s - w| \left( \max_{g \in G} |g| + 1 \right) \left( \max_{\gamma \in \Gamma} |\gamma| + 1 \right) \\
&= c(m, H, \mathbf{L}) \text{dist}(s, H)
\end{aligned}$$

which is the required estimate. ■

Using Lemma 3.2, we derive Lemma 3.1 in three steps: 1) given an arbitrary list assignment, extract an integer consistent list assignment and estimate the number of agents excluded; 2) given an integer consistent list assignment, extract a feasible list assignment and estimate the number of agents excluded; 3) combine the estimates. The following direct application of Lemma 3.2 accomplishes the first of these steps.

**Lemma 6.2** *If  $\mu : B \rightarrow \mathbf{Lists}_M$  is an arbitrary list assignment then there is a subset  $B' \subset B$  such that  $\mu_{B'}$  is integer consistent and*

$$|B \setminus B'| \leq c(|\mathcal{M}|, \mathbf{Cons}^*, \mathbf{Lists}_M) \text{dist}(\mu_B, \mathbf{Cons}^*)$$

**Proof** Note first that  $\mathbf{Cons}^*$  is not empty and is closed under addition and relatively closed under subtraction. For each  $\ell \in \mathbf{Lists}_M$ , set  $B_\ell = \{b \in B : \mu_b = \ell\}$  and  $\beta_\ell = |B_\ell|$ . Applying Lemma 3.2 with  $m = |\mathcal{M}|$ ,  $H = \mathbf{Cons}^*$ ,  $\mathbf{L} = \mathbf{Lists}_M$  provides non-negative coefficients  $\beta'_\ell \leq \beta_\ell$  such that  $\sum \beta'_\ell \ell \in \mathbf{Cons}^*$  and

$$\sum |\beta_\ell - \beta'_\ell| \leq c(|\mathcal{M}|, \mathbf{Cons}^*, \mathbf{Lists}_M) \text{dist}(\sum \beta_\ell \ell, \mathbf{Cons}^*) \quad (14)$$

(both summations are over  $\ell \in \mathbf{Lists}_M$ ). For each  $\ell$ , choose a subset  $B'_\ell \subset B_\ell$  with  $|B'_\ell| = \beta'_\ell$ , and set  $B' = \cup B'_\ell$ . Observing that  $\mu_B = \sum \beta_\ell \ell$  and that  $\mu_{B'} = \sum \beta'_\ell \ell$  and substituting into the inequality (14) yields the desired conclusion. ■

The next lemma accomplishes the second step.

**Lemma 6.3** *If  $\mu : B \rightarrow \mathbf{Lists}_M$  is an integer consistent list assignment then there is a subset  $B' \subset B$  such that  $\mu_{B'}$  is feasible and*

$$|B \setminus B'| \leq [c(|\mathcal{M}|, \mathbf{Cons}^*, \mathbf{Lists}_M) + 1] M^2 \sum_{(\pi, \gamma) \in \mathbf{Clubs}} |\pi|$$

**Proof** For each club type  $(\pi, \gamma)$ , let  $n(\pi, \gamma) = \mu_B(\omega, \pi, \gamma) / \pi(\omega)$ . (Because  $\mu$  is integer consistent, the right hand side is independent of  $\omega$ .) This is the total number of clubs of type  $(\pi, \gamma)$  that are assigned by  $\mu$ . If  $\mu_a(\omega_a, \pi, \gamma) \leq n(\pi, \gamma)$  for all  $a$ , all  $(\pi, \gamma)$  then  $\mu$  is feasible. Suppose  $(\pi, \gamma)$  is a club type for which  $\mu_a(\omega_a, \pi, \gamma) > n(\pi, \gamma)$  for some  $a \in B$ . Let

$$B^1 = \{b \in B : \mu_b(\omega_b, \pi, \gamma) = 0\}$$

Individuals in  $B \setminus B^1$  are assigned memberships in the club type  $(\pi, \gamma)$ . The number of such memberships is at most  $n(\pi, \gamma)|\pi|$ . Because no individual is assigned more than  $M$  memberships and there exists an agent for which  $\mu_a(\omega_a, \pi, \gamma) > n(\pi, \gamma)$ , it follows that  $n(\pi, \gamma) < M$ . Hence

$$|B \setminus B^1| \leq M|\pi|$$

Because  $\mu$  is integer consistent and no individual is assigned more than  $M$  memberships,

$$\text{dist}(\mu_{B^1}, \mathbf{Cons}^*) \leq M^2|\pi|$$

Lemma 6.2 guarantees that there is a subset  $B^2 \subset B^1$  such that the restriction  $\mu|_{B^2}$  is integer consistent and

$$|B^1 \setminus B^2| \leq c(|\mathcal{M}|, \mathbf{Cons}^*, \mathbf{Lists}_M) M^2 |\pi|$$

and hence

$$|B \setminus B^2| \leq [c(|\mathcal{M}|, \mathbf{Cons}^*, \mathbf{Lists}_M) + 1] M^2 |\pi| \quad (15)$$

By construction,  $\mu|_{B^2}$  assigns no memberships in the club type  $(\pi, \gamma)$ . If  $\mu|_{B^2}$  is not feasible, we need only repeat this argument once (at most) for

each club type in turn, eventually arriving at a subset  $B' \subset B$  such that  $\mu_{|B'}$  is feasible. Applying the estimate (15) at each stage and summing over all possible club types yields

$$|B \setminus B'| \leq [c(|\mathcal{M}|, \mathbf{Cons}^*, \mathbf{Lists}_M) + 1] M^2 \sum_{(\pi, \gamma) \in \mathbf{Clubs}} |\pi|$$

which is the desired result. ■

To take the third step in our proof of Lemma 3.1, we need a few more notions. For each club type  $(\pi, \gamma)$ , define the vector  $g^{(\pi, \gamma)} \in \mathbb{R}^M$  by

$$g^{(\pi, \gamma)} = \sum_{\omega \in \Omega} \pi(\omega) \delta_{(\omega, \pi, \gamma)}$$

Note that if  $(\pi, \gamma) \neq (\pi', \gamma')$  then the vectors  $g^{(\pi, \gamma)}$  and  $g^{(\pi', \gamma')}$  are orthogonal. In particular,  $G \equiv \{g^{(\pi, \gamma)} : (\pi, \gamma) \in \mathbf{Clubs}\}$  is a linearly independent collection. As we have noted,  $x \in \mathbb{R}^M$  belongs to  $\mathbf{Cons}$  if and only if the quotients  $x(\omega, \pi, \gamma)/\pi(\omega)$  are independent of  $\omega$ . It follows immediately that  $x \in \mathbf{Cons}$  if and only if

$$x = \sum_{(\pi, \gamma)} \frac{x(\omega, \pi, \gamma)}{\pi(\omega)} g^{(\pi, \gamma)}$$

In particular,  $G = \{g^{(\pi, \gamma)}\}$  is an orthogonal basis for  $\mathbf{Cons}$ . Note that  $x \geq 0$  if and only if  $x(\omega, \pi, \gamma)/\pi(\omega) \geq 0$  for each  $\omega, \pi, \gamma$ . Moreover, orthogonality of the basis  $G$  entails that  $x \in \mathbf{Cons}^*$  if and only if  $x(\omega, \pi, \gamma)/\pi(\omega)$  is a positive integer for each  $\omega, \pi, \gamma$ . In particular,  $G$  is a set of generators for  $\mathbf{Cons}^*$ .

The third step in our program follows easily from these considerations.

**Lemma 6.4** *If  $x \in \mathbf{Cons} \cap \mathbb{R}_+^M$  then*

$$\text{dist}(x, \mathbf{Cons}^*) \leq \sum_{(\pi, \gamma)} |\pi|$$

**Proof** Because  $x \in \mathbf{Cons}$  we can write

$$x = \sum_{(\pi, \gamma)} \frac{x(\omega, \pi, \gamma)}{\pi(\omega)} g^{(\pi, \gamma)}$$

Because  $x \geq 0$ , the coefficients  $x(\omega, \pi, \gamma)/\pi(\omega)$  are non-negative. For each  $(\omega, \pi, \gamma)$ , let  $\beta^{(\pi, \gamma)}$  be the greatest integer less than or equal to  $x(\omega, \pi, \gamma)/\pi(\omega)$ . Set

$$\bar{x} = \sum_{(\pi, \gamma)} \beta^{(\pi, \gamma)} g^{(\pi, \gamma)}$$

It is evident that  $\bar{x} \in \mathbf{Cons}^*$  and

$$\text{dist}(x, \mathbf{Cons}^*) \leq |x - \bar{x}| \leq \sum_{(\pi, \gamma)} |g^{(\pi, \gamma)}|$$

Since  $|g^{(\pi, \gamma)}| = |\pi|$ , the number of members of the club  $(\pi, \gamma)$ , we obtain the desired result. ■

All that remains before turning to the proof of Lemma 3.1 is to define the constants:

$$\begin{aligned} K_1(\mathbf{Clubs}, M) &= c(|\mathcal{M}|, \mathbf{Cons}^*, \mathbf{Lists}_M) \\ K_2(\mathbf{Clubs}, M) &= [(1 + K_1(\mathbf{Clubs}, M)) M^2 + K_1(\mathbf{Clubs}, M)] \sum_{(\pi, \gamma) \in \mathbf{Clubs}} |\pi| \end{aligned}$$

**Proof of Lemma 3.1** Applying Lemma 6.2 guarantees that there is a subset  $B^1 \subset B$  such that  $\mu_{|B^1}$  is integer consistent and

$$|B \setminus B^1| \leq K_1(\mathbf{Clubs}, M) \text{dist}(\mu_B, \mathbf{Cons}^*) \quad (16)$$

Lemma 6.4 guarantees that

$$\text{dist}(\mu_B, \mathbf{Cons}^*) \leq \text{dist}(\mu_B, \mathbf{Cons}) + \sum_{(\pi, \gamma) \in \mathbf{Clubs}} |\pi| \quad (17)$$

Lemma 6.3 guarantees that there is a subset  $B' \subset B^1$  such that  $\mu_{|B'}$  is feasible and

$$|B^1 \setminus B'| \leq [1 + K_1(\mathbf{Clubs}, M)] M^2 \sum_{(\pi, \gamma) \in \mathbf{Clubs}} |\pi| \quad (18)$$

Combining the inequalities (17), (16) and (18), and substituting the definitions of the various constants yields the desired result. ■

## 6.2 Approximate Decentralization

We now use the Shapley-Folkman Theorem and Lemma 3.1 as the main tools to show that approximate core states can be approximately decentralized. As we have discussed earlier, the proof follows Anderson (1978) rather closely, but with some subtle differences.

An additional notion will be convenient. Say that  $q \in \mathbb{R}^M$  is a *pure transfer* if  $q \cdot \mu = 0$  for each  $\mu \in \mathbf{Cons}$ . Note that  $q$  is a pure transfer if and only if

$$\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) = 0$$

Write  $\mathbf{Trans}$  for the space of pure transfers.

**Proof of Theorem 4.1** Let  $(x^*, \mu^*)$  be a state in the fat  $\varepsilon$ -core of  $\mathcal{E}$ . We construct a pure transfer by separating the net preferred set from an appropriate cone, and use this pure transfer to construct the approximately decentralizing price. The argument is in 4 steps.

**Step 1** For each agent  $a \in A$ , write

$$\varphi(a) = \{(x, \ell) \in X_a : u_a(x, \ell) > u_a(x_a^*, \mu_a^*)\}$$

for agent  $a$ 's preferred set and

$$\psi(a) = \{(z, \ell) \in \mathbb{R}^N \times \mathbb{R}^M : (z + e_a - \tau(\ell), \ell) \in \varphi(a)\}$$

for agent  $a$ 's preferred set net of endowment and share of club input requirements. Set  $\Psi(a) = \psi(a) \cup \{0\}$  and let

$$Z = \sum_{a \in A} \Psi(a)$$

Note that  $0 \in Z$ .

**Step 2** We henceforth suppress the dependence of the constants  $K_1, K_2, K_3$  on  $\mathbf{Clubs}, M, N$ . Let  $W = W(\mathcal{E})$  and define

$$C^* = \{(\bar{x}, \bar{\mu}) \in \mathbb{R}^N \times \mathbb{R}^M : x + [WK_3 + \varepsilon|A|] \mathbf{1} \ll -WK_1 \text{dist}(\mu, \mathbf{Cons}) \mathbf{1}\}$$



Note that  $C^*$  is an open convex cone; the vertex of this cone is the point  $(-[WK_3 + \varepsilon|A|]\mathbf{1}, 0)$ . We want to separate  $C^*$  from  $Z$ ; to accomplish this, we show first that  $C^* \cap \text{conv } Z = \emptyset$ .

Suppose that  $(\bar{z}, \bar{\nu}) \in C^* \cap \text{conv } Z$ ; we reach a contradiction by constructing a blocking coalition. The Shapley-Folkman theorem guarantees that we can choose elements  $(z_a, \nu_a) \in \text{conv } \Psi(a)$  for each  $a \in A$  such that:

$$\begin{aligned} (\bar{z}, \bar{\nu}) &= \sum_{a \in A} (z_a, \nu_a) \\ |\{a \in A : (z_a, \nu_a) \notin \Psi(a)\}| &\leq N + |\mathcal{M}| \end{aligned}$$

Set  $B_1 = \{a \in A : (z_a, \nu_a) \in \Psi(a)\}$ . Agents in  $B_1$  are assigned bundles in  $\Psi(a)$ ; agents in  $A \setminus B_1$  are assigned bundles in  $\text{conv } \Psi(a)$ . By assumption, agents are constrained to choose no more than  $M$  memberships. Because  $|A \setminus B_1| \leq N + |\mathcal{M}|$  it follows that

$$\text{dist}(\nu_{B_1}, \mathbf{Cons}) \leq \text{dist}(\nu_A, \mathbf{Cons}) + (N + |\mathcal{M}|)M$$

We can therefore use Lemma 3.1 to choose a subset  $B_2 \subset B_1$  such that the restriction  $\nu|_{B_2}$  is feasible and

$$\begin{aligned} |B_1 \setminus B_2| &\leq K_1 \text{dist}(\nu_{B_1}, \mathbf{Cons}) + K_2 \\ &\leq K_1 [\text{dist}(\nu_A, \mathbf{Cons}) + (N + |\mathcal{M}|)M] + K_2 \end{aligned}$$

Hence

$$|A \setminus B_2| \leq K_1 [\text{dist}(\nu_A, \mathbf{Cons}) + (N + |\mathcal{M}|)M] + K_2 + N + |\mathcal{M}| \quad (19)$$

Define the coalition

$$B = \{a \in B_2 : (z_a, \nu_a) \neq (0, 0)\}$$

We assert that  $B$  is a blocking coalition. To see this, we need to verify three conditions:

(i) Our construction guarantees that  $(z_b, \nu_b) \in \psi(b)$  for each  $b \in B$ , so

$$u_b(z_b + e_b - \tau(\nu_b), \nu_b) > u_b(x_b^*, \mu_b^*)$$

for each  $b \in B$ .

(ii) Because  $\nu_{|B_2}$  is feasible and agents in  $B \setminus B_2$  are assigned no club memberships,  $\nu_{|B}$  is also feasible.

(iii) By construction,  $(\bar{z}, \bar{\nu}) = \sum(z_a, \nu_a) \in C^*$ , so

$$\sum_{a \in A} z_a \ll [-(WK_3 + \varepsilon|A|) - WK_1 \text{dist}(\mu_A, \mathbf{Cons})] \mathbf{1} \quad (20)$$

For each  $a \in A$ ,  $z_a \geq -e_a \geq -W\mathbf{1}$ ; together with (19) this yields

$$\sum_{a \in A \setminus B_2} z_a \geq -\{K_1[(N + |\mathcal{M}|)M + \text{dist}(\mu_A, \mathbf{Cons})] + K_2 + N + |\mathcal{M}|\} W\mathbf{1}$$

Using (20) and the definition of  $K_3$ , and keeping in mind that agents in  $B \setminus B_2$  are assigned 0 net trades, we obtain

$$z_B = \sum_{b \in B} z_b = \sum_{a \in B_2} z_a = \sum_{a \in A} z_a - \sum_{a \in A \setminus B_2} z_a \ll -\varepsilon|A|\mathbf{1},$$

The last inequality certainly implies that  $B$  is not empty, so we conclude that  $B$  is a blocking coalition; this contradicts the assumption that the state  $(x^*, \mu^*)$  is in the fat  $\varepsilon$ -core. We conclude that  $C^* \cap \text{conv } Z = \emptyset$ , as asserted.

**Step 3** We now use the separation theorem to find prices  $(p, q) \in \mathbb{R}^N \times \mathbb{R}^{\mathcal{M}}$ ,  $(p, q^*) \neq (0, 0)$  and a real number  $s$  such that

$$\begin{aligned} (p, q^*) \cdot (\bar{z}, \bar{\nu}) &\leq s && \text{for each } (\bar{z}, \bar{\nu}) \in C^* \\ (p, q^*) \cdot (\bar{z}, \bar{\nu}) &\geq s && \text{for each } (\bar{z}, \bar{\nu}) \in Z \end{aligned}$$

Because  $0 \in Z$ , it follows that  $s \leq 0$ . Because  $C^*$  contains a translate of  $-\mathbb{R}_{++}^N \times \{0\}$ , it follows that  $p \geq 0$ . Because  $C^*$  contains a translate of  $\{0\} \times \mathbf{Cons}$ , it follows that  $q^*$  vanishes on  $\mathbf{Cons}$ . In particular, for each club type  $(\pi, \gamma)$ ,

$$\sum_{\omega \in \Omega} \pi(\omega) q_{(\omega, \pi, \gamma)}^* = 0$$

Thus,  $q^*$  represents pure transfers within each club.

We assert that  $p \neq 0$ . To see this, suppose to the contrary that  $p = 0$ . By construction,  $(p, q^*) \neq (0, 0)$  so  $q^* \neq 0$ . Hence there is a  $\bar{\eta} \in \mathbb{R}^M$  such that  $q^* \cdot \bar{\eta} > 0$ . For  $r$  sufficiently large,  $(-r\mathbf{1}, \bar{\eta})$  belongs to  $C^*$ , so

$$(p, q^*) \cdot (-r\mathbf{1}, \bar{\eta}) \leq s \leq 0$$

Because  $p = 0$ , it follows that

$$(p, q^*) \cdot (-r\mathbf{1}, \bar{\eta}) = q^* \cdot \bar{\eta} > 0$$

This is a contradiction, so we conclude that  $p \neq 0$ , as asserted.

Normalize  $p$  such that  $p \in \Delta$ . The price  $q^*$  represents pure transfers within each club; to define membership prices  $q$ , set

$$q_{(\omega, \pi, \gamma)} = q_{(\omega, \pi, \gamma)}^* + \frac{1}{|\pi|} p \cdot \mathbf{inp}(\pi, \gamma)$$

Note that if  $(y, \ell) \in \mathbb{R}^N \times \mathbf{Lists}_M$ ,

$$(p, q) \cdot (y, \ell) = (p, q^*) \cdot (y + \tau(\ell), \ell)$$

**Step 4:** It remains to show that  $(x^*, \mu^*)$  is  $K_3 W |A|^{-1} + \varepsilon$ -decentralized by the prices  $(p, q)$ .

The construction of  $q$  from the pure transfer  $q^*$  guarantees that budgets of clubs balance exactly. Moreover, because  $(-[WK_3 + \varepsilon|A|]\mathbf{1}, 0)$  belongs to the closure of  $C^*$ , it follows that

$$s \geq (p, q^*) \cdot (-[WK_3 + \varepsilon|A|]\mathbf{1}, 0) = -[WK_3 + \varepsilon|A|] \quad (21)$$

To estimate  $\rho^1(x^*, \mu^*, p, q)$ , write

$$E_1 = \{a \in A : (p, q) \cdot (x_a^*, \mu_a^*) > p \cdot e_a\}$$

for the set of individuals for whom expenditure exceeds income, and  $E_2 = A \setminus E_1$  for the complementary set. Because  $(x^*, \mu^*)$  is feasible for  $A$  and  $q^*$  is a pure transfer,

$$(p, q) \cdot (x_A^*, \mu_A^*) = p \cdot (x_A^* + \tau(\mu_A^*)) = p \cdot e_A$$

Since  $A = E_1 \cup E_2$  and  $E_1 \cap E_2 = \emptyset$ , it follows that

$$(p, q) \cdot (x_{E_1}^*, \mu_{E_1}^*) - p \cdot e_{E_1} = - \left[ (p, q) \cdot (x_{E_2}^*, \mu_{E_2}^*) - p \cdot e_{E_2} \right] \quad (22)$$

For each  $a$ , the net trade  $(x_a^* + \tau(\mu_a^*) - e_a, \mu_a^*)$  is in the closure of  $\Psi(a)$ . The separation property of prices and equation (21) therefore entail:

$$(p, q) \cdot (x_{E_2}^*, \mu_{E_2}^*) - p \cdot e_{E_2} \geq -WK_3 - \varepsilon|A| \quad (23)$$

Using equations (22) and (23) and keeping in mind that expenditure minus income is positive for agents in  $E_1$  and no others yields:

$$\begin{aligned} \rho^1(x^*, \mu^*, p, q) &= \frac{1}{|A|} \sum_{a \in A} \rho_a^1(x^*, \mu^*, p, q) \\ &= \frac{1}{|A|} \sum_{a \in E_1} [(p, q) \cdot (x_a^*, \mu_a^*) - p \cdot e_a] \\ &= -\frac{1}{|A|} [(p, q) \cdot (x_{E_2}^*, \mu_{E_2}^*) - p \cdot e_{E_2}] \\ &\leq \frac{WK_3}{|A|} + \frac{\varepsilon|A|}{|A|} \\ &= \frac{WK_3}{|A|} + \varepsilon \end{aligned}$$

This is the required estimate for  $\rho^1$ .

To estimate  $\rho^2$ , let  $E_3$  be the set of agents for whom there is a choice vector  $(y_a, v_a)$  in the budget set such that  $u_a(y_a, v_a) > u_a(x_a^*, \mu_a^*)$ . As before, separation implies

$$(p, q) \cdot [(y_{E_3}, v_{E_3}) - (e_{E_3}, 0)] \geq -WK_3 - \varepsilon|A|$$

Rearranging yields

$$\begin{aligned} \frac{1}{|A|} (p, q) \cdot [(e_{E_3}, 0) - (y_{E_3}, v_{E_3})] &\leq \frac{WK_3}{|A|} + \frac{\varepsilon|A|}{|A|} \\ &= \frac{WK_3}{|A|} + \varepsilon \end{aligned}$$

This is the required estimate for  $\rho^2$ , so the proof is complete. ■

### 6.3 Existence

As discussed in the Introduction, we construct a state in the fat  $\varepsilon$ -core from a quasi-equilibrium for a continuum economy; Lemma 3.1 enters crucially in the construction of the desired state and in the verification that it belongs to the fat  $\varepsilon$ -core.

**Proof of Theorem 4.3** Consider the continuous representation  $\mathcal{E}^c$  of the finite economy  $\mathcal{E}$ . The space of agents in  $\mathcal{E}^c$  is  $A^c = A \times [0, 1]$ ; we equip  $A^c$  with the product of counting measure on  $A$  and Lebesgue measure on  $[0, 1]$ . Agent  $(a, t) \in A^c$  has the external characteristic, endowment, and utility function of agent  $a \in A$ . EGSZ guarantees that  $\mathcal{E}^c$  has a quasi-equilibrium  $(p, q); (x, \mu)$ .

For each  $a \in A$ , define

$$(x_a^1, \mu_a^1) = \int_{[0,1]} (x_{(a,t)}, \mu_{(a,t)}) d\lambda(t)$$

Because agent  $(a, t)$  chooses in his quasi-demand set and has the endowment and utility function of agent  $a$ , the average choice  $(x_a^1, \mu_a^1)$  belongs to the convex hull  $\text{conv } D_a$  of agent  $a$ 's quasi-demand set  $D_a$ .

Because  $\sum_{a \in A} \text{conv } D_a = \text{conv } \sum_{a \in A} D_a$ , we can use the Shapley-Folkman theorem to find choices  $(x_a^2, \mu_a^2) \in \text{conv } D_a$  and an exceptional set  $E \subset A$  such that  $(x_a^2, \mu_a^2) \in D_a$  for all  $a \in A \setminus E$  and

$$\begin{aligned} \sum_{a \in A} (x_a^2, \mu_a^2) &= \sum_{a \in A} (x_a^1, \mu_a^1) \\ |E| &\leq N + |\mathcal{M}| \end{aligned}$$

Our construction guarantees that

$$\mu_A^2 = \mu_A^1 = \sum_{a \in A} \int_{[0,1]} \mu_{(a,t)} d\lambda(t)$$

Because  $(x, \mu)$  is a quasi-equilibrium,  $\mu$  is a consistent list assignment, so  $\mu_A^2 \in \mathbf{Cons}$ . Because individuals are constrained to choose at most  $M$  memberships,  $|\mu_a^2| \leq M$  for each  $a \in E$ . Hence,

$$\text{dist}(\mu_{A \setminus E}^2, \mathbf{Cons}) \leq M|E|$$

Lemma 3.1 guarantees that there is a subset  $F \subset A \setminus E$  such that  $\mu_F^2$  is a feasible assignment,

$$|A \setminus (E \cup F)| = |(A \setminus E) \setminus F| \leq K_1 M |E| + K_2$$

and hence

$$|A \setminus F| \leq K_1 M |E| + K_2 + |E| \leq (K_1 M + 1)(N + |\mathcal{M}|) + K_2 \quad (24)$$

Set

$$\bar{y} = \frac{1}{|A \setminus F|} \left( \sum_{a \in A} e_a - \sum_{a \in A \setminus F} [x_a^2 + \tau(\mu_a^2)] \right)$$

Our construction guarantees that

$$\begin{aligned} \sum_{a \in F} [x_a^2 + \tau(\mu_a^2)] &\leq \sum_{a \in A} [x_a^2 + \tau(\mu_a^2)] \\ &= \sum_{a \in A} [x_a^1 + \tau(\mu_a^1)] \\ &= \sum_{a \in A} \int_{[0,1]} [x_{(a,t)} + \tau(\mu_{(a,t)})] d\lambda(t) \\ &= \sum_{a \in A} \int_{[0,1]} e_{(a,t)} d\lambda(t) \\ &= \sum_{a \in A} e_a \end{aligned}$$

In particular,  $\bar{y} \geq 0$ . Hence we may define a feasible state  $(y, \nu)$  by

$$(y_a, \nu_a) = \begin{cases} (x_a^2, \mu_a^2) & \text{if } a \in F \\ (\bar{y}, 0) & \text{if } a \in A \setminus F \end{cases}$$

Note that  $(p, q), (y, \nu)$  constitutes an approximate equilibrium, in the sense that most individuals — those in  $F$  — are quasi-optimizing.

We assert that, for  $\varepsilon > K_4 W |A|^{-1}$ ,  $(y, \nu)$  is in the fat  $\varepsilon$ -core. To see this, suppose not, and let  $B$  be a blocking coalition. By definition, there is a state  $(y', \nu')$  which is feasible for  $B$ , unanimously preferred to  $(y, \nu)$  by all members of  $B$ , and which satisfies the resource inequality:

$$y'_B + \tau(\nu'_B) \leq e_B - \varepsilon |A| \mathbf{1} \quad (25)$$

Write  $B_0 = B \cap F$ . By construction, for each  $b \in B_0$ , the choice  $(y_b, \nu_b)$  is quasi-optimal at prices  $(p, q)$  and

$$|B \setminus B_0| \leq |A \setminus F| \quad (26)$$

By assumption, agents are constrained to choose at most  $M$  club memberships, so  $|\nu'_B - \nu'_{B_0}| \leq M|B \setminus B_0|$ . By assumption,  $\nu'_B$  is feasible, so Lemma 3.1 provides a subset  $B_1 \subset B_0$  such that  $\nu'_{B_1}$  is feasible and

$$|B_0 \setminus B_1| \leq K_1 M |B \setminus B_0| + K_2 \quad (27)$$

Recalling the definition of  $K_4(\mathbf{Clubs}, M, N)$  and combining the estimates (24), (26) and (27) yields

$$\begin{aligned} |B \setminus B_1| &\leq (K_1 M + 1)|B \setminus B_0| + K_2 \\ &\leq (K_1 M + 1)|A \setminus F| + K_2 \\ &\leq (K_1 M + 1)[(K_1 M + 1)(N + |\mathcal{M}|) + K_2] + K_2 \\ &= K_4 \end{aligned} \quad (28)$$

Because  $(p, q)$  is a quasi-equilibrium price for  $\mathcal{E}^c$ , budgets of *all* club types balance. Because  $\nu'_B$  and  $\nu'_{B_1}$  are feasible, it follows that  $q \cdot \nu'_B = p \cdot \tau(\nu'_B)$  and  $q \cdot \nu'_{B_1} = p \cdot \tau(\nu'_{B_1})$ , so

$$\sum_{b \in B} (p, q) \cdot (y'_b, \nu'_b) = p \cdot [y'_B + \tau(\nu'_B)] \quad (29)$$

$$\sum_{b \in B_1} (p, q) \cdot (y'_b, \nu'_b) = p \cdot [y'_{B_1} + \tau(\nu'_{B_1})] \quad (30)$$

and hence that

$$\sum_{b \in B \setminus B_1} (p, q) \cdot (y'_b, \nu'_b) = p \cdot [y'_{B \setminus B_1} + \tau(\nu'_{B \setminus B_1})] \geq 0 \quad (31)$$

Because endowments are bounded by  $W\mathbf{1}$ ,

$$p \cdot e_{B \setminus B_1} \leq W|B \setminus B_1| \quad (32)$$

Recalling that  $\varepsilon > K_4 W(\mathcal{E}) |A|^{-1}$ , and combining the inequalities (28)-(31), we obtain

$$\begin{aligned}
\sum_{b \in B_1} (p, q) \cdot (y'_b, \nu'_b) &\leq \sum_{b \in B} (p, q) \cdot (y'_b, \nu'_b) \\
&= p \cdot [y'_B + \tau(\nu'_B)] \\
&\leq p \cdot e_B - \varepsilon |A| \\
&= p \cdot e_{B_1} + p \cdot e_{B \setminus B_1} - \varepsilon |A| \\
&\leq p \cdot e_{B_1} + W(\mathcal{E}) |B \setminus B_1| - \varepsilon |A| \\
&\leq p \cdot e_{B_1} + W(\mathcal{E}) K_4 - \varepsilon |A| \\
&< p \cdot e_{B_1}
\end{aligned}$$

Hence there is at least one agent  $b \in B_1$  for whom

$$(p, q) \cdot (y'_b, \nu'_b) < p \cdot e_b$$

This agent prefers  $(y'_b, \nu'_b)$  to  $(y_b, \nu_b)$ , so is not quasi-optimizing at the prices  $(p, q)$ , contradicting the construction of  $B_1$ . We conclude that there is no blocking coalition, and hence that  $(y, \nu)$  is in the fat  $\varepsilon$ -core, as asserted. ■



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