

**Proof that Example 1 (modified) admits no equilibrium with  
type-independent tie-breaking rules.**

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We have modified( Example 1 so that with small probability  $\delta < 1/200$ , there is a second object available. In this event, the auctioneer chooses a number which is uniform on  $[0, 6]$  (which can be taken as the space of allowable bids), and each player receives an object *at price* 0 as long as their bid exceeds the number chosen. One can think of this as a small reward from the auctioneer for bidding. Various other modifications will also work. However, the problem is more delicate than one might initially think, and it is not the case that all “sensible” perturbations rule out equilibria.)

With this perturbation, monotonicity is easy to see. Consider any pair of bids  $b', b''$  with  $b'' > b'$ , and consider player 1. Because types are independent, and because we are considering a tie-breaking rule that does not depend on player types, the set of events (on the types and bids of the other player, the outcome of any randomizations by the seller in the event of ties, and the randomization over whether a second object is available) in which player 1 wins with  $b''$  but loses with  $b'$ , and the probabilities over those events are unaffected by  $t_i$ . But then, if  $b''$  does at least as well as  $b'$  for  $t_1 = t'$ , then it does strictly better for  $t_1 = t''$  for any  $t'' > t'$ . This follows since, given the perturbation,  $b''$  has a strictly higher probability than  $b'$  of winning, and own type only enters expected utility calculations multiplied by probability of winning, while the rest of the expected utility calculation is independent of own type.

Thus, there is no loss in assuming the bidding strategies  $b_1, b_2$  are monotone, in the sense that if  $t'_i > t_i$  then every bid in the support of  $b_i(t'_i)$  is at least as large as every bid in the support of  $b_i(t_i)$ . It follows immediately that there is an at most countable set of signals  $t_i$  for which the support of  $b_i(t_i)$  is not a singleton. For such  $t_i$ , replace  $b_i$  by the infimum of the support of  $b_i(t_i)$ . The modified bid functions  $b_1, b_2$  again constitute an equilibrium. Thus we have an equilibrium in monotone, pure behavioral strategies. Altering bids at signals 0, 1 if necessary, there is no loss in assuming that  $b_1, b_2$  are continuous at 0, 1.

Note first then, that  $b_1(0) = b_2(0) \equiv \underline{b}$ . For instance, if  $b_1(0) < b_2(0)$ , then a small increase in bid never hurts, and helps if there happens to be a second object (the value of which is always positive).

Imagine that  $\underline{b} \geq 4\frac{3}{4}$ . The average value of the object, even if allocated optimally to the player with larger  $t$ , is  $5 + 2/3 - 4(1/3) = 4\frac{2}{3}$ , since  $2/3$  and  $1/3$  are the expected higher and lower values of two draws from the uniform distribution. So, conditional on there being one object, at least  $1/12$  is lost. On the other hand, when there are two objects, each player earns at most 6 (the maximum possible value of the object), for an expectation of  $12\delta$ . Since  $\delta < 1/200$ , someone is thus losing money on average, and would be better off to bid 0 always. Hence,  $\underline{b} < 4\frac{3}{4}$ .

Suppose that one player, say player 2, bids  $\underline{b}$  with probability 0. Let  $b' \in (\underline{b}, 4\frac{3}{4})$  be such that  $b_2 \leq b' \implies t_2 < 1/32$ . When there are two objects  $b'$  is a better bid than any lower bid. Let  $P' > 0$  be the probability that player 2 bids less than  $b'$ . When there is a single object and bidding  $b'$  wins, the object is worth at least  $5 + 0 - 4(1/32) = 4\frac{7}{8}$  to player 1, and so profits conditional on there being a single object are at least  $P'/8 > 0$ . But, for types near 0, 1's equilibrium bid is arbitrarily close to  $\underline{b}$  (and so is less than  $b'$ ) and wins with probability arbitrarily close to 0, since 2 bids in some small neighborhood of  $\underline{b}$  with arbitrarily small probability. So, an interval of 1's types have a profitable deviation, a contradiction.

Thus, both players bid  $\underline{b}$  with positive probability. For each  $i$ , let  $\tau_i = \text{supp}\{t|b_i(t) \leq \underline{b}\} > 0$ .

Assume that ties at  $\underline{b}$  are broken with probability  $p \in (0, 1)$  in favor of player 1. Let  $t'$  and  $t''$ ,  $t' < t''$ , be two values of  $t$  for which  $b_1(t) = \underline{b}$ . It follows that  $5 + t' - 4E(t_2|b_2(t_2) = \underline{b}) \geq \underline{b}$ , else 1 would be better to bid  $\underline{b} - \varepsilon$  with  $t'$  (note that by taking  $\varepsilon$  small enough, the cost of lowering the bid in the event there are two objects can be made arbitrarily small). But then,  $5 + t'' - 4E(t_2|b_2(t_2) = \underline{b}) > \underline{b}$ , and so 1 should deviate to  $\underline{b} + \varepsilon$  with  $t''$ . This is a contradiction.

So, all ties at  $\underline{b}$  are broken in favor of one player, say player 1. Thus, with  $t = \tau_2 - \varepsilon$ , player 2 never wins when there is a single object, while by bidding  $\varepsilon$  more, he can also win when  $t_1 \in [0, \tau_1)$ . For this not to be a profitable deviation, it must be that  $\underline{b} \geq 5 + \tau_2 - 4(Et_1|t_1 \leq \tau_1) = 5 + \tau_2 - 2\tau_1$ . Note that this implies that  $\tau_1 > 1/8$ , since  $\underline{b} < 4\frac{3}{4}$ .

Suppose that  $\tau_2 < 1$ . Pick  $t = \tau_2 + \varepsilon$ , and consider replacing  $b_2(t)$  (which is by definition greater than  $\underline{b}$ ) by any bid  $b'$  in  $(\underline{b}, b_2(t))$ . When this changes a win into a loss,  $t_1 \geq \tau_1$ , and hence  $v_2$  is at best  $5 + \tau_2 + \varepsilon - 4\tau_1 < \underline{b}$ , and

so the change is profitable. At least  $(1 - \delta)/8$  of the time, there was a single object and  $t_1 < \tau_1$ . In this event,  $b'$  still wins, and pays  $b_2(t) - b'$  less. Finally, lowering the bid costs at most 6 (the maximal value of the object) at most  $\delta(b_2(t) - b')/6$  of the time (as this is the probability that there is a second object and the “reserve” is between  $b'$  and  $b_2(t)$ ). As  $\delta < 1/200$ ,  $(1 - \delta)/8 > \delta$ , and so this is a profitable deviation, a contradiction. Thus,  $\tau_2 = 1$ .

Since  $\tau_2 = 1$ ,  $E(t_2|b_2 = \underline{b}) = 1/2$ . Hence,  $\underline{b} \leq 3 = 5 + 0 - 4(1/2)$ , otherwise 1 is better to bid  $\underline{b} - \varepsilon$  with  $t_1$  near 0. But then, player 2 can profitably bid  $\underline{b} + \varepsilon$  when he has  $t$  above 0, a contradiction. Thus, there is no equilibrium to this game.