Subdifferentiability and the Duality Gap

Neil E. Gretsky (neg@math.ucr.edu) *
Department of Mathematics, University of California, Riverside

Joseph M. Ostroy (ostroy@econ.ucla.edu) †
Department of Economics, University of California, Los Angeles

William R. Zame (zame@econ.ucla.edu) ‡
Department of Economics, University of California, Los Angeles

Abstract. We point out a connection between sensitivity analysis and the fundamental theorem of linear programming by characterizing when a linear programming problem has no duality gap. The main result is that the value function is subdifferentiable at the primal constraint if and only if there exists an optimal dual solution and there is no duality gap. To illustrate the subtlety of the condition, we extend Kretschmer's gap example to construct (as the value function of a linear programming problem) a convex function which is subdifferentiable at a point but is not continuous there. We also apply the theorem to the continuum version of the assignment model.

Keywords: duality gap, value function, subdifferentiability, assignment model

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1. Introduction

The purpose of this note is to point out a connection between sensitivity analysis and the fundamental theorem of linear programming. The subject has received considerable attention and the connection we find is remarkably simple. In fact, our observation in the context of *convex* programming follows as an application of conjugate duality [11, Theorem 16]. Nevertheless, it is useful to give a separate proof since the conclusion is more readily established and its import for linear programming is more clearly seen.

The main result (Theorem 1) is that in a linear programming problem there exists an optimal dual solution and there is no duality gap if and only if the value function is subdifferentiable at the primal constraint.

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The result is useful because the value function is convex, so there are simple sufficient conditions that it be subdifferentiable at a given (constraint) point. In particular, if the value function of the linear programming problem is a proper convex function, then the value function is lower semicontinuous and standard results from convex analysis guarantee that the value function is locally bounded, locally Lipschitz, and subdifferentiable at every interior point of its domain. When applied to infinite-dimensional linear programming, Theorem 1 contains the Duffin-Karlovitz no-gap theorem for constraint spaces whose positive cone has interior [3], and the Charnes-Cooper-Kortanek no-gap theorem for semi-infinite programs [1, 2]. However, our results have the greatest "bite"on domains with no interior. We also discuss conditions of Lipschitz continuity for a convex function which guarantee that the function be subdifferentiable as well as structural conditions on the data of the LP problem which guarantee that the value function be (locally) Lipschitz. Finally, we apply this circle of ideas to the assignment model [5], which was the original motivation for the present work.

The subdifferentiability of a convex function and its Lipschitz properties are related, but there is an important distinction. We give below a modification of Kretschmer's well-known no-gap example [8]) to show that the value function can be subdifferentiable at a particular constraint (and hence that the dual problem has a solution and there is no gap for that constraint) even though the value function is nowhere continuous.

Our results highlight two kinds of sensitivity analysis which are distinguished by the nature of the perturbations of the primal constraint: primal sensitivity, i.e. stability of the value function at nearby points, and dual sensitivity, i.e. stability of the subdifferential correspondence at nearby points. The moral of this paper is a graded one: if the value function is not subdifferentiable at the primal constraint, then neither kind of analysis is available; if the value function is subdifferentiable at the primal constraint, then primal sensitivity analysis is available; and if the value function is locally Lipschitz (and, hence, subdifferentiable) at all points close to the primal constraint, then both primal and dual sensitivity analysis are available.

2. Motivation from Mathematical Economics

The assignment model, in general, and the continuum assignment model, in particular, have a distinguished history in both the economics and mathematics literature. See [5] for a brief literature review. The present study was motivated by the problem of showing that there was no gap in the infinite-dimensional linear programming problem that arose in our studies of the continuum assignment problem in [5]. The no-gap argument given there was incomplete; the current paper rectifies that omission. Even though the functional-analytic argument can be successfully completed, it became apparent that a much simpler argument was available and this led to Theorem 1.

The economic interpretation of the continuum assignment model can be expressed as a special kind of two-sided market. There is a set B of buyer types and a set S of seller types. The market is special in that each buyer can be paired with at most one seller and conversely. Thus, the only trades are between buyer and seller pairs which define a match. The value of a match between a buyer of type b and a seller of type s is given by the number $V(b,s)$. The population of buyers and sellers in the market is described by a measure μ on the disjoint union of B and S. A distributional description of matches between buyers and sellers is a measure x on $B \times S$. Constraining the marginals of x by the population measure means that only buyers and sellers in the original population can be matched. Maximizing $\int V\,dx$ means maximizing the total value of the matchings.

The continuum assignment model can be formalized as a linear programming problem (see the following section for the standard format of a linear program). We are given compact metric spaces B and S, a positive Borel measure μ on the disjoint union $B \cup S$, and a measurable function $V : B \times S \to [0,1]$. Write μ_B, μ_S for the restrictions of μ to B, S. For a non-negative Borel measure x on the product $B \times S$ denote by x_1 and x_2 the marginal measures $x_1(E) = x(E \times S)$ and $x_2(F) = x(B \times F)$ for all Borel sets $E \subseteq B$ and $F \subseteq S$. The *assignment problem* is to find a positive measure x on $B \times S$ so as to maximize $\int V\,dx$, subject to the constraint that the marginals of x on B, S do not exceed μ_B, μ_S.

An associated problem (the *dual* problem) is to find a bounded measurable function q on $B \cup S$ to minimize $\int q\,d\mu$, subject to the constraint that $q(b) + q(s) \geq V(b,s)$ for all b, s. The interpretation is that q represents divisions of the social gain among participants; the constraint means that no buyer/seller pair could jointly obtain a gain greater than they are assigned by q.

The formulation of this problem will be completed in Section 7 where the existence of dual solutions and no-gap are established as an application of Theorem 1.

3. Linear Programming

The classical linear programming problem in standard form may be written as follows. Let X and Y be ordered locally convex topological vector spaces with dual spaces X^* and Y^*, respectively. Let $A : X \to Y$ be a continuous linear operator, $b \in Y$, and $c^* \in X^*$. The primal problem is (by convention)
(P) Find $x \in X$ so as to

$$\begin{aligned} \text{minimize} \quad & c^*(x) \\ \text{subject to} \quad & Ax \geq b \\ \text{and} \quad & x \geq 0 \end{aligned}$$

The dual problem is
(D) Find $y^* \in Y^*$ so as to

$$\begin{aligned} \text{maximize} \quad & y^*(b) \\ \text{subject to} \quad & A^*y^* \leq c^* \\ \text{and} \quad & y^* \geq 0 \end{aligned}$$

Note that we could have as easily taken the maximum problem as the primal and the minimum problem as the dual. That is, in fact, the situation for the assignment model.

The *value* of each problem is the number obtained by computing, respectively, the infimum for problem (P) and the supremum for problem (D). A *feasible solution* for either problem is a vector in the appropriate space satisfying the constraints; an *optimal solution* is a feasible solution which achieves the extremum value. It is always true that $c^*(x) \geq y^*(b)$ for any feasible primal solution $x \in X$ and feasible dual solution $y^* \in Y^*$. Thus the value of the primal minimization problem is always greater than or equal than the value of the dual maximization problem.

The values of the primal and dual problems need not be finite. The value of the primal problem is $+\infty$ if and only if the primal problem has no feasible solutions; the value of the dual problem is $-\infty$ if and only if the dual problem has no feasible solutions. The value of the primal problem can be $-\infty$ in which case the value of the dual must also be $-\infty$ (but not conversely); the value of the dual problem can be $+\infty$ in which case the the value of the primal must also be $+\infty$ (but not conversely).

If both X and Y are finite-dimensional spaces, then the Fundamental Theorem of Linear Programming states that (i) the primal and dual values are equal to each other, and (ii) if the values are finite then there exist optimal solutions for both problems. In infinite-dimensional linear programming, however, it is well-known that (i) a "gap" can exist between the values of the two problems, and (ii) there need not exist a solution to either of these problems even when both values are finite.

4. The Subdifferential of the Value Function

Denote by $\bar{\mathbb{R}}$ the extended real line $\mathbb{R}\cup\{-\infty,+\infty\}$. Fixing the operator A and the functional c^* and viewing the constraint b as a variable, define the *value function* of the problem (P) to be $v : Y \to \bar{\mathbb{R}}$ as

$$v(b) = \inf\{c^*(x) : Ax \geq b \text{ and } x \geq 0\}$$

Write $\operatorname{dom} v$ for the *effective domain* of v, the convex cone of those $b \in Y$ for which there is some $x \in A$ with $x \geq 0$ and $Ax \geq b$. Equivalently, $\operatorname{dom} v = \{b : v(b) < \infty\}$. (See [10].)

The value function v is known to be increasing, positive homogeneous, and subadditive — hence, convex. Note that either $v(0) = 0$ or $v(0) = -\infty$; moreover, if $v(0) = -\infty$ then $v(b) = -\infty$ for all $b \in \operatorname{dom} v$. To see that the first assertion holds, note that $0 \in X$ is always feasible for the case $b = 0 \in Y$. Thus $v(0) \leq 0$. But, if $v(0) < 0$, then there is some feasible x such that $c^*(x) < 0$; in which case tx is also feasible for every $t > 0$ and it follows that $v(0) = -\infty$ whenever $v(0) \neq 0$. To see that the second assertion holds, note that if $v(0) = -\infty$ then there exist a sequence $x_n \geq 0$ in X such that $Ax_n \geq 0$ and $c^*(x_n) \to -\infty$. If b is any member of Y with $v(b) < +\infty$ then there is some $x \geq 0$ with $Ax \geq b$. Thus, $x+x_n$ is also feasible for b and $c^*(x_n+x) = c^*(x_n)+c^*(x) \to -\infty$. It follows that $v(b) = -\infty$ for any $b \in \operatorname{dom} v$. To avoid degeneracies, we will henceforth assume that $v(0) = 0$.

Recall that the *subdifferential* of a convex function $v : Y \to \bar{\mathbb{R}}$ at a point $b \in \operatorname{dom} v$ is the set

$$\partial v(b) = \{y^* \in Y^* | y^*(y - b) \leq v(y) - v(b) \text{ for all } y \in Y\}$$

When, as in the present case, v is positively homogeneous, the defining condition is equivalent to $y^*(y) \leq v(y)$ for all $y \in Y$ and $y^*(b) = v(b)$. We say v is *subdifferentiable at* b if $\partial v(b)$ is non-empty.

Our main result characterizes existence of an optimal solution to the dual problem and the absence of a gap between the primal and dual values by the subdifferentiability of the value function.

THEOREM 1. *Suppose that a linear programming problem is given in standard form with data A, b, and c^*. Both the dual problem (D) has a solution and there is no gap if and only v is subdifferentiable at b.*

Proof. We first dispose of two trivial cases. If $v(b) = -\infty$, then the defining equality for the subdifferential can never be satisfied; hence, v is not subdifferentiable at b. The value of the dual problem must also be $-\infty$; thus, there can be no feasible solutions for (D). If $v(b) > -\infty$ but $v(y) = -\infty$ for some y, then the defining inequality for the subdifferential is not satisfied for this y. Hence, v is not subdifferentiable at b. Again, there can be no dual feasible solutions. Thus, the theorem is vacuously true in either of these cases.

It remains to consider the case that $v(y) > -\infty$ for all y. There are two steps in the proof: (i) y^* is a feasible dual solution if and only if $y^*(y) \leq v(y)$ for all y (the subdifferential inequality for y^*); and (ii) given that y^* is a feasible dual solution, then y^* is an optimal dual solution and there is no gap if and only if $y^*(b) = v(b)$ (the subdifferential equality for y^*).

To see (i), assume that $y^*(y) \leq v(y)$ for all $y \in Y$. We have assumed *ab initio* that $v(0) = 0$ and that, in our present case, $v(y) > -\infty$ for all y. Then for any $y \geq 0$, $y^*(-y) \leq v(-y)$ and $y^*(y) \geq -v(-y) \geq -v(0) = 0$. Thus, $y^* \geq 0$. Moreover, for any $x \in X_+$ we have

$$\begin{aligned} (A^*y^*)(x) &= y^*(Ax) \\ &\leq v(Ax) \text{ by hypothesis} \\ &\leq c^*(x) \text{ since } x \text{ is feasible for problem (P)} \\ &\qquad \text{with righthand constraint } Ax \end{aligned}$$

Consequently, $A^*y^* \leq c^*$. Conversely, assume that y^* is a feasible solution for problem (D). Consider the linear programming problem with b replaced by y. The modified problem has the same set of feasible dual solutions as the original. Hence

$$\begin{aligned} v(y) &= \inf\{c^*(x) : Ax \geq y \text{ and } x \geq 0\} \\ &\geq \sup\{z^*(y) : A^*z^* \leq c^* \text{ and } z^* \geq 0\} \\ &\geq y^*(y) \end{aligned}$$

as desired.

To see (ii), note that, since $v(b) \geq y^*(b)$ for any feasible dual solution y^*, $v(b)$ is a upper bound for the value of the dual; in symbols, $y^*(b) \leq v_d \leq v(b)$ where v_d is the value of the dual problem. If $v(b) = y^*(b)$, then there is equality throughout and we conclude there is no gap and that y^* is a dual solution. Conversely, if the dual problem (D) has a solution y^* and there is no gap, then it is immediate that $v(b) = y^*(b)$.

Note that in the absence of subdifferentiability of the value function at the constraint point b, either there is no dual solution or there is a duality gap. Either of these cases precludes sensitivity analysis of any kind.

5. Lipschitz Conditions for Subdifferentiability

Theorem 1 says that existence of the subdifferential of the value function at the point b guarantees that there is no gap for the LP problem with right hand side constraint b. What makes this fact useful in applications is that the value function is convex and there are simple and easily verifiable sufficient conditions for subdifferentiability of a convex function.

We wish to avoid pathological behavior for the linear programming problem. Recall that in the discussion preceding the statement of Theorem 1, we showed that we could assume that $v(0) = 0$ and that in doing so we would rule out only the case that $v(y) = -\infty$ for all y. Similarly, at the start of the proof of Theorem 1, we showed that we could assume that $v(b) > -\infty$ for all y, since otherwise v could not be subdifferentiable anywhere. Finally, we will be interested only in LP problems which have feasible solutions for some constraint b. Thus, we will consider only LP problems for which the value function is *proper*; i.e. we consider only problems for which the value function takes values greater than $-\infty$ (in particular, $v(0) = 0$) and has at least one point for which the value is less than ∞ (i.e. has at least one point in the effective domain $\operatorname{dom} v$ of the function).

Suppose that $f : Y \to \mathbb{R} \cup \{+\infty\}$ is a proper convex function defined on the normed linear space Y and that $b \in \operatorname{dom} f$. Each of the following conditions implies the next and the last is equivalent to the subdifferentiability of f at b.

1. f is lower semicontinuous and b is an interior point of $\operatorname{dom} f$;

2. f is locally Lipschitz at the point b, i.e. there exists $\delta > 0$ such that f is Lipschitz on $\operatorname{dom} f \cap B(b; \delta)$;

3. f has bounded steepness at the point b, i.e. the quotients $\frac{f(b)-f(y)}{\|y-b\|}$ are bounded above.

It is well-known (see Phelps [9]) that an extended real-valued proper lower semicontinuous convex function is locally Lipschitz and locally bounded on the interior of its domain. Conditions 2 and 3 are of interest to us precisely because there are natural domains with no interior (such as the positive cones of most infinite-dimensional Banach lattices).

Condition 3 is a *one-sided* Lipschitz condition at b. The geometric duality theorem of Gale [4] shows that 3 is equivalent to the subdifferentiability of f at b. Moreover, condition 3 has a direct analogue in the more general context of a locally convex space. Define the directional derivative function of f around b

$$d(y) = Df(b)(y)$$

for all $y \in Y$ such that $b + ty \in C$ for some $t > 0$. The condition that d is lower semicontinuous at 0 is equivalent to the subdifferentiability of f at b (see [7]).

The applicability of these observations in our present situation stems from the following observation. Note that this lemma distinguishes the proper value function of a linear programming problem from the general class of proper convex homogeneous functions. The latter need not be lower semicontinuous (cf. [9]).

LEMMA 1. *If the value function v for a linear programming problem in standard form on ordered normed linear spaces is proper, then v is a lower semicontinuous extended real-valued (convex and homogeneous) function.*

Proof. Let b be in the effective domain of the value function v. Let b_n be a sequence in Y which converges to b in Y. Without loss of generality, we may suppose that each b_n is in the effective domain of v. Given $\epsilon > 0$, there is $x_n \geq 0$ such that $Ax_n \geq b_n$ and $v(b_n) \geq c^*(x_n) - \epsilon$. Pass to any converging subsequence of (x_n) and re-label so that (x_n) converges to x. Since the positive cones of X and Y are each closed, it follows that $x \geq 0$ and $Ax \geq 0$; i.e. x is a feasible solution of the LP for the constraint b. Thus,

$$v(b_n) \geq c^*(x_n) - \epsilon \rightarrow c^*(x) - \epsilon \geq v(b) - \epsilon$$

Since this is true for any $\epsilon > 0$ and any converging subsequence of the x_n, it follows that $\liminf_{n\to\infty} v(b_n) \geq v(b)$; i.e. v is lower semicontinuous.

As promised in the Introduction, we can quickly derive the Duffin-Karlovitz [3] and the Charnes-Cooper-Kortanek [1, 2] no-gap theorems from Theorem 1. Let X and Y be ordered normed linear spaces and consider a linear programming problem in standard form with data A, b, c^*. The Duffin-Karlovitz theorem asserts that if the positive cone Y_+ has non-empty interior, if there is a feasible solution $\hat{x}$ for the primal problem such that $\hat{x} \geq 0$ and $A\hat{x} - b$ is in the interior of Y_+, and if the value of the primal is finite, then the dual problem has a solution and there is no gap. Since $\hat{x}$ is feasible for b, b is in the domain of the value function v and, hence, $v(b) < +\infty$. By hypothesis there is an open ball U around $A\hat{x} - b$ within F_+. Hence, $\hat{x}$ is feasible for $b + u$ for all $u \in U$; viz. b is an interior point of the domain of v. The hypothesis that the value of the primal is finite is just the statement that $v(b) > -\infty$. Consequently, v is subdifferentiable at b and we conclude by Theorem 1 that there exists a dual solution and that there is no gap, as asserted.

A semi-infinite linear programming problem can be given in the following form. Let F be a partially ordered normed linear space whose positive cone F_+ is closed and let n and r be positive integers with $0 \le r \le n$. Equip $\mathbb{R}^n$ with its usual ordering. Fix $c_1, ..., c_n$ in $\mathbb{R}$, $f_1, ..., f_n$ in F, and b in F. The primal and dual semi-infinite linear programming problems in standard form are

(P) Find $x \in \mathbb{R}^n$ so as to

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} f_j x_j \ge b \\ \text{and} \quad & x \ge 0 \end{aligned}$$

(D) Find $y^* \in Y^*$ so as to

$$\begin{aligned} \text{maximize} \quad & y^*(b) \\ \text{subject to} \quad & y^*(f_j) \le c_j \text{ for } j = 1, ..., r \\ \text{and} \quad & y^*(f_j) = c_j \text{ for } j = r+1, ..., n \\ \text{and} \quad & y^* \ge 0 \end{aligned}$$

The Charnes-Cooper-Kortanek theorem asserts that if the positive cone of F has non-empty interior, if there exists $\hat{x} \ge 0$ in $\mathbb{R}^n$ such that $\sum_{j=1}^{n} f_j \hat{x}_j - b$ is in the interior of the positive cone of F, and if the value of the primal is finite, then the dual problem has a solution and there is no-gap. This theorem follows from Theorem 1 directly.

The upshot of the discussion at the start of this section is that local Lipschitz continuity of the value function at the primal constraint implies that the value function is subdifferentiable and, hence, such a local Lipschitz condition is sufficient for the conclusion that there is no gap and that there exists a dual solution. Theorem 1 states that the subdifferentiability of the value function is a necessary and sufficient condition for the conclusion that there is no gap and that there exists a dual solution. To see the relationship between the sufficient condition that the value function is locally Lipschitz and the necessary and sufficient condition that value function be subdifferentiable, we change our point of view from considering a single point in the constraint space to considering the domain of the value function.

A. Verona and M. E. Verona have given in [12] a necessary and sufficient connection for a convex function to be locally Lipschitz in terms of the function being subdifferentiable. Suppose that $W \subseteq C \subseteq Z$ where Z is a Banach space, that C is a convex subset of Z with the property that the closure of the affine span of C is all of Z, and that W is a relatively open subset of C. For a convex function $f : C \to \mathbb{R}$, $f|_W$ is locally Lipschitz if and only if $\partial f(z) \neq \emptyset$ for each $z \in W$ and ∂f has a locally bounded selection on W.

If the set W is actually the interior of C in the above discussion, then the condition of a non-empty subdifferential is equivalent to the condition of f being locally Lipschitz. However, if C has no interior, the condition of a non-empty subdifferential at a point can be weaker than that of f being locally Lipschitz on all of W. We give an example of a convex function which is subdifferentiable but not locally Lipschitz by exhibiting a linear programming problem for which the value function has this property. The example takes place in the Banach lattice $L^2[0,1]$ (a space for which the positive cone has empty interior) and for which $\operatorname{dom} v \supseteq L^2[0,1]_+$.

Example. We start by recalling Kretschmer's no-gap example [8]. Let $X = L^2[0,1] \times \mathbb{R}$ equipped with the norm $\|x\| = \|(f,r)\| = \|f\|_2 + |r|$ and pointwise a.e. order. Let $Y = L^2[0,1]$ equipped with the norm $\|f\|_2$ and pointwise a.e. order. Define the continuous linear map $A : X \to Y$ by

$$Ax = (A(f,r))(t) = \int_t^1 f(s)ds + r \text{ for } f \in L^2[0,1] \text{ and } r \in \mathbb{R}$$

Define $b \in Y$ by $b = \mathbf{1}$ and $c^* \in X^*$ by $c^*(x) = c^*(f,r) = \int_0^1 tf(t)dt + 2r$. Finally, we use in place of the generic linear functional $y^* \in L^2[0,1]^*$ the representing function $h \in L^2[0,1]$.

The primal problem is to find $x = (f,r) \in L^2[0,1] \times \mathbb{R}$ so as to

$$\begin{aligned} \text{minimize} \quad & \int_0^1 tf(t)dt + 2r \\ \text{subject to} \quad & \int_t^1 f(s)ds + r \geq 1 \text{ a.e.} \\ \text{and} \quad & f \geq 0, r \geq 0 \end{aligned}$$

The dual problem is to find $h \in L^2[0,1]$ so as to

$$\begin{aligned} \text{maximize} \quad & \int_0^1 h(t)dt \\ \text{subject to} \quad & \int_0^t h(s)ds \leq t \text{ a.e.} \\ \text{and} \quad & h \geq 0 \end{aligned}$$

The optimal primal solution is $x = (f, r) = (0, 1)$ with value 2; the optimal dual solution is $h = \mathbf{1}$ with value 1. Thus, even though both problems have solutions and finite values there is a gap.

By Theorem 1 the subdifferential of the value function at $\mathbf{1}$ must be empty. Consequently, the value function v cannot be locally Lipschitz at $\mathbf{1}$. We examine a modification of this example by changing the constraint in such a way that the value function does have a subdifferential but is not locally Lipschitz. Consider $b_0 = \chi_{[0,\frac{1}{2}]}$, then we see that the optimal solution for the primal problem "wants to be" the point mass at $\frac{1}{2}$; more precisely, there is no optimal solution for the primal but the value can be computed as the infimum of the objective function at feasible solutions $f_\epsilon = \frac{1}{\epsilon}\chi_{[\frac{1}{2},\frac{1}{2}+\epsilon]}$. The primal value is $\frac{1}{2}$. The optimal dual solution is $h = \mathbf{1}$ and the dual value is $\frac{1}{2}$. Thus, we have directly verified both that there is no gap and that a dual solution exists for this b_0. By Theorem 1 the subdifferential $\partial v(b_0)$ is non-empty.

On the other hand, we will establish that v is not locally Lipschitz at b_0; in fact, v is not even continuous there (or anywhere). To see this, fix $0 < \epsilon < \frac{1}{2}$. Define $b_\epsilon = b_0 + \chi_{[1-\epsilon^2,1]}$. Then $\|b_\epsilon - b_0\|_2 \leq \epsilon$. Yet the constraint for the primal requires that $\int_t^1 f(s)ds + r \geq 1$ for almost all $t \in [1-\epsilon^2, 1]$. This forces r to be (at least) 1 and, hence, $v(b_\epsilon) \geq 2$. Thus, v is not continuous at b_0. Similar perturbations show that v is not continuous anywhere on $L^2[0,1]_+$. (Of course, v is lower semicontinuous.) $\square$

Recall that subdifferentiability of the value function is equivalent to the bounded steepness condition (a one-sided Lipschitz property); this condition is enough to make primal sensitivity analysis meaningful even when the value function is not continuous. However, as the theorem of Verona and Verona cited above makes clear, it is the (two-sided) local Lipschitz property of the value function in some relative neighborhood of b_0 that is needed for dual sensitivity analysis. The example illustrates this distinction.

6. Structural Conditions for Subdifferentiability.

As a useful adjunct to this discussion it is interesting to note that, in the context of X and Y being Banach lattices, there are structural conditions on the data A, b, c^* of the LP problem that are sufficient for the value function to be (locally) Lipschitz.

Suppose the operator A is positive and onto. Consider two cases: either c^* is positive or not. If c^* is not positive, then there exists some $x_0 \geq 0$ in X such that $c^*(x_0) < 0$. Hence, $\lim_{n\to\infty} c^*(nx_0) = -\infty$. Since A is onto and positive, there exists a feasible u, i.e. $Au \geq b$ and $u \geq 0$. (There is x such that $Ax = b$. Thus $Ax^+ - Ax^- = b$ and, consequently, x^+ is feasible.) It is clear that $u + nx_0$ is also feasible for each n. We compute

$$\lim_{n\to\infty} c^*(u + nx_0) = \lim\{c^*(u) + nc^*(x_0)\} = -\infty$$

Consequently, if c^* is not positive, $v(b) = -\infty$ for all b.

On the other hand, suppose that c^* is positive. We already know that $v(b) < +\infty$ since there exists a feasible solution. By the positivity of c, we have that $v(b) \geq 0$ since any feasible solution is positive. Thus, in this case, the convex function v is real-valued for all b, so that every point is an interior point of the domain of v. This ensures that v is locally Lipschitz (and locally bounded) for every b. (This, of course, is enough to guarantee that v is subdifferentiable for every b and that, by Theorem 1, there is no gap and there is a dual solution.)

PROPOSITION 1. *Let X and Y be Banach lattices and let A, b, and c be the data for a LP problem in standard form. Assume that A is a positive and onto operator. Then*

- *if c^* is not a positive functional, then the value function $v(b) = -\infty$ for all b;*
- *if c^* is a positive functional, then the value function $v(b)$ is finite for all b and is Lipschitz on Y.*

Proof. The discussion before the statement of the proposition establishes that (i) if c^* is not a positive functional then the value function is identically $-\infty$, and (ii) if c^* is a positive functional then the value function is locally Lipschitz (and locally bounded) on Y. It remains to show that v is actually Lipschitz.

We have seen above that there is a feasible solution for every $b \in Y$. Let $b_1, b_2 \in Y$. Suppose that x_1 is feasible for the constraint b_1; i.e. assume that $Ax_1 \geq b_1$ and that $x_1 \geq 0$. We will construct x_2 which is feasible for b_2 ($Ax_2 \geq b_2$, $x_2 \geq 0$) such that $\|x_1 - x_2\| \leq M\|b_1 - b_2\|$. The construction of x_2 proceeds as follows. Since A is a bounded linear onto map between Banach spaces, we have by the Open Mapping Theorem a constant $M > 0$ which depends only on A such that for each $y \in Y$ there is $x \in X$ with $Ax = y$ and $\|x\| \leq M\|Ax\|$. Thus, there exists $z \in X$ such that $Az = b_2 - b_1$ and $\|z\| \leq M\|b_2 - b_1\|$. Define $x_2 = x_1 + z^+$. We note that $x_2 \geq 0$ and

$$\begin{aligned} Ax_2 &= Ax_1 + Az^+ \\ &\geq b_1 + Az \text{ since } z^+ \geq z \text{ and } A \geq 0 \\ &= b_1 + b_2 - b_1 \\ &= b_2 \end{aligned}$$

so that x_2 is feasible for b_2. Also, since $\|z^+\| \leq \|z\|$, we have that $\|x_1 - x_2\| \leq \|z\| \leq M\|b_1 - b_2\|$.

Since we have made no assumption about the existence of primal solutions, we use an elementary approximation to complete the proof. Fix $\epsilon > 0$. Then there exists x_1^ϵ such that x_1^ϵ is feasible for b_1 and $g(b_1) > c^*(x_1^\epsilon) - \epsilon$. Construct x_2^ϵ as in the preceding paragraph. Then

$$\begin{aligned} v(b_2) - v(b_1) &< g(b_2) - c^*x_1^\epsilon + \epsilon \\ &\leq c^*x_2^\epsilon - c^*x_1^\epsilon + \epsilon \\ &\leq \|c^*\|\|x_2^\epsilon - x_1^\epsilon\| + \epsilon \\ &\leq \|c^*\|M\|b_1 - b_2\| + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$v(b_2) - v(b_1) \leq \|c^*\|M\|b_1 - b_2\|$$

Reversing the roles of b_1 and b_2 yields the Lipschitz inequality

$$|v(b_2) - v(b_1)| \leq \|c^*\|M\|b_1 - b_2\|$$

as desired.

Another structural condition is useful for application to the assignment model. We will use the condition in the context of a maximization problem and will state it as such.

PROPOSITION 2. *Let X and Y be Banach lattices and let A, b, and c be the data for an LP maximization problem. Assume that*

- *A is a positive operator which maps the positive cone X_+ onto Y_+;*
- *the order interval $[0, x_0]$ is mapped onto the order interval $[0, Ax_0]$ for every $x_0 \geq 0$;*

- *A is bounded below on the positive cone X_+, i.e. there exists a constant $M > 0$ such that $\|Ax\| \geq M\|x\|$ for all $x \geq 0$.*

Then the value function is Lipschitz on the positive cone X_+.

Proof. Start with $b_1 \geq 0$ and $b_2 \geq 0$. First, consider the case that $b_2 \leq b_1$. Given $\epsilon > 0$, there is an almost optimal x_1 for b_1, viz. there is $x_1 \geq 0$ such that $Ax_1 = b_1$, and $c^*(x_1) + \epsilon > v(b_1)$. Since $0 \leq b_2 \leq b_1 = Ax_1$ and since the positive operator A maps $[0, x_1]$ onto $[0, Ax_1]$, there is x_2 such that $0 \leq x_2 \leq x_1$ with $Ax_2 = b_2$. Clearly, x_2 is feasible for b_2; hence, $v(b_2) \geq c^*(x_2)$.

We compute

$$\begin{aligned} v(b_1) - v(b_2) &\leq c^*(x_1) + \epsilon - c^*(x_2) \\ &\leq \|c^*\|\|x_1 - x_2\| + \epsilon \\ &\leq \|c^*\|\frac{1}{M}\|Ax_1 - Ax_2\| + \epsilon \\ &= \|c^*\|\frac{1}{M}\|b_1 - b_2\| + \epsilon \end{aligned}$$

Since this true for arbitrary $\epsilon > 0$, we have that

$$v(b_1) - v(b_2) \leq \frac{1}{M}\|c^*\|\|b_1 - b_2\|$$

Switching the roles of b_1 and b_2 gives us

$$|v(b_1) - v(b_2)| \leq \frac{1}{M}\|c^*\|\|b_1 - b_2\|$$

as desired.

For the general case in which we do not assume any order dominance between x_1 and x_2, define $x_3 = x_1 \wedge x_2$. Then $b_3 = b_1 - (b_1 - b_2)^+$; i.e., $b_1 - b_3 = (b_1 - b_2)^+$. Consequently,

$$\|b_1 - b_3\| \leq \|(b_1 - b_2)^+\| \leq \|b_1 - b_2\|$$

Since $0 \leq b_3 \leq b_2$ and v is an increasing function, we have that

$$v(b_1) - v(b_2) \leq v(b_1) - v(b_3) \leq c \leq \|c^*\|\frac{1}{M}\|b_1 - b_2\|$$

The same x_3 works for $v(b_2) - v(b_1)$ and we have shown that v is Lipschitz on Y.

7. Application to the Assignment Model

We formulated the assignment model in Section 2. As given there, it is not quite a linear programming problem in standard form, but it is easily turned into one. Write $M(B \cup S)$ for the Banach lattice of Borel measures on $B \cup S$ and $M_\mu(B \cup S)$ for the sublattice of measures x whose marginals x_1, x_2 on B, S are absolutely continuous with respect to μ_B, μ_S. Using the Radon-Nikodým theorem, identify absolutely continuous measures with elements of $L^1(B \cup S)$. Write $X = M_\mu(B \times S)$, $Y = L^1(\mu_B) \oplus L^1(\mu_S)$ and define the operator $A : X \to Y$ by $Ax = (x_1, x_2)$. Note that every bounded Borel function f on $B \times S$ defines, via integration, a linear functional on $M_\mu(B \times S)$, and that f, f' define the same functional exactly if the set where they differ is of measure 0 relative to every measure in $M_\mu(B \times S)$ (we write $f = f'$ $M_\mu(B \times S)$-a.e.).

Thus the primal and dual problems for the assignment model may be written as

(P) Find $x \in M_\mu(B \times S)$ so as to

$$\begin{array}{rl} \text{maximize} & \int_{B\times S} V dx \\ \text{subject to} & (x_1, x_2) \leq (\mathbf{1}, \mathbf{1}) \ \mu\text{-a.e.} \\ \text{and} & x \geq 0 \end{array}$$

(D) Find $q \in L^\infty(\mu)$ so as to

$$\begin{array}{rll} \text{minimize} & \int_{B\cup S} q d\mu & \\ \text{subject to} & q(b) + q(s) \geq V(b, s) & M_\mu(B \times S)\text{-a.e.} \\ \text{and} & q \geq 0 & \end{array}$$

Since the primal for the assignment model is a maximization problem, the value function is concave rather than convex. To apply Theorem 1 when the primal is a maximization problem, one recognizes that the value function is concave, positively homogeneous, and increasing. By convention, the definition for the subdifferential of a concave function has the inequality reversed from the definition for a convex function. By Proposition 2 the value function v is Lipschitz continuous on the positive cone $[L^1(\mu_B) \oplus L^1(\mu_S)]_+$. (It was shown in [5, 6] specifically for the assignment model that the value function v is Lipschitz continuous on the positive cone $[L^1(\mu_B) \oplus L^1(\mu_S)]_+$. Here we have applied a structural result from the preceding section whose proof is an abstract version of the assignment model proof.) Thus, we may conclude that the value function is subdifferentiable at any point of the positive cone (and that the subdifferential map is locally efficient). In particular, this is true at the point $(\mathbf{1}, \mathbf{1})$ and we have established by Theorem 1 that the linear programming formulation of the assignment problem has a dual solution and that there is no gap.

Remark. The Lipschitz property of the value function v in the assignment model ensures that the set of dual solutions coincides with the subdifferential of the value function. This, of course, need not be true in more general economic models.

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Address for Offprints:
Neil Gretsky
Department of Mathematics
University of California
Riverside, CA 90254