

# Posterior Implementation versus Ex-Post Implementation

Philippe Jehiel, Moritz Meyer-ter-Vehn,  
Benny Moldovanu and William R. Zame<sup>1</sup>

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In this short note we discuss how *posterior implementation*, due to Green and Laffont (1987), relates to *ex-post implementation* for which Jehiel et. al (2005) have established a strong impossibility result.

In contrast to most other notions of implementation, posterior implementation is defined with respect to the information released by the mechanism. It requires that agents' strategies are optimal against others' strategies, given the precise information made available by the mechanism. It can be interpreted as a *regret-free* form of implementation: even after agents gain new information by observing some specified features of the mechanism's outcome, they are still willing to stick to their original strategy.

This notion of regret-free implementation (which can be viewed as robustness with respect to the timing of choice) is different from the notion of *robust-implementation* that refers to the agent's prior beliefs and higher-order beliefs. As shown by Bergemann and Morris (2004), robust implementation coincides with ex-post implementation in a large class of environments with quasi-linear preferences, and thus it constitutes the main motivation for ex-post implementation.

Despite the stark difference in interpretation, note that, if the information released by the mechanism includes all players' types, then posterior implementation coincides with ex post implementation. At the other extreme, if no further information is made available to agents, then posterior implementation coincides with Bayes-Nash implementation.

Particularly in settings where signals spaces are of higher dimension than actions spaces, some form of posterior implementation may be possible even

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Jehiel: UCL London and PSE, Paris; jehiel@enpc.fr; Meyer-ter-Vehn : McKinsey Company; moritz\_mtv@web.de; Moldovanu: University of Bonn; mold@uni-bonn.de; Zame: UCLA & Caltech; zame@econ.ucla.edu

if ex-post implementation fails (since in such settings agents are able to make only limited inferences about the signals behind particular actions).

For any setting where Bayes-Nash implementation of some social choice function is possible with respect to some priors, but where ex-post implementation fails, there will be some "maximal information mechanism" that allows for posterior implementation, and one could study the properties of these mechanisms.

To illustrate the idea of posterior implementation, consider the following auction setup:

- There are two bidders  $i \in \{1, 2\}$  competing for a single unit of an indivisible object.
- Bidders have two-dimensional signals  $s^i = (p^i, c^i) \in [0, 1]^2$ ;
- The signals  $(p^i, c^i)$  are uniformly and independently distributed on  $[0, 1]^2$ .
- The valuation of bidder  $i$  is given by  $v^i(s^i, s^{-i}) = p^i + c^i c^{-i}$ .

**Claim:**

- *The second-price auction has a symmetric Bayes-Nash equilibrium with an associated non-trivial allocation function.*
- *This equilibrium is also posterior with respect to the policy of revealing bids.*
- *If the object must be sold, only trivial allocation functions are ex-post implementable<sup>2</sup>.*

**Proof. 1)** We construct a symmetric equilibrium in continuous, strictly monotonic increasing bid functions  $b : [0, 1]^2 \rightarrow \mathbb{R}$ . A necessary and sufficient condition for equilibrium is that each type  $(p, c)$  is indifferent between winning or losing the auction at a tie. This gives the usual condition:

$$b = b(s^i) = \mathbb{E}_{s^{-i}} [v^i(s^i, s^{-i}) | b(s^{-i}) = b(s^i)]$$

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<sup>2</sup>If the object is allowed to stay with the seller (an alternative constantly valued at zero by both buyers), Bikchandani (2004) observes that ex-post implementation is possible.

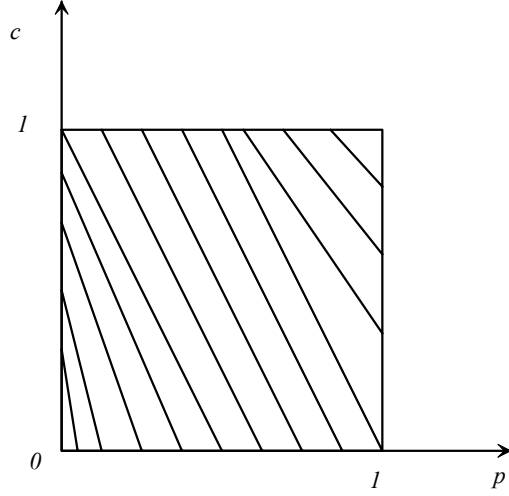


Figure 1:

Given signals' independence, the right hand side is equal to  $p^i + c^i \bar{c}(b)$ . Here,  $\bar{c}(x) = \mathbb{E}_{s^{-i}} [c | b(p, c) = x]$  is the expectation of the opponent's common values signal given that he makes bid  $x$ . This shows that the iso-bid curves  $b^{-1}(x)$  must be straight lines with slope  $-\frac{1}{\bar{c}(x)}$  in the  $(p, c)$ -space  $[0, 1]^2$ . Some tedious calculations show that the iso-bid lines are as follows:

$$b^{-1}(x) := \begin{cases} \left[ (0, \sqrt{2x}), (x, 0) \right] & \text{for } x \in \left[ 0, \frac{1}{2} \right] \\ \left[ \left( x - \frac{1}{2}, 1 \right), (x, 0) \right] & \text{for } x \in \left[ \frac{1}{2}, 1 \right] \\ \left[ \left( x - \frac{1}{4} - \frac{1}{4}\sqrt{8x-7}, 1 \right), \left( 1, \frac{1}{2}\sqrt{8x-7} - \frac{1}{2} \right) \right] & \text{for } x \in [1, 2] \end{cases}$$

These iso-bid lines are drawn in figure 1.

**2)** Assume that bids (but nothing else) are made public after the auction. Posterior implementation with respect to this disclosure policy requires optimality of each player's bid given other players' actual (rather than expected) bids. It is readily verified that the equilibrium exhibited above has this property: conditional on observing the bid  $x$  of bidder  $-i$ , bidder  $i$  makes the inference that the expected value of  $c^{-i}$  is  $E_{s^{-i}}[c^{-i} | b(p^{-i}, c^{-i}) = x]$ , and therefore she is not willing to modify her bid.

**3)** It is easily checked that the above is **not** an ex post equilibrium: assume that  $(-i)$ 's signal is  $s^{-i} = (p^{-i}, c^{-i})$ , while  $i$ 's signal  $s^i = (p^i, c^i)$  satisfies  $p^i + c^i c^{-i} - b(s^{-i}) > 0$  and  $b(s^i) < b(s^{-i})$ . Then, after being informed about  $(-i)$ 's signal,  $i$  prefers bidding above  $b(s^{-i})$  rather than  $b(s^i)$ .

For the impossibility result, we define the two alternatives "i gets the object" and "(-i) gets the object", and we apply Proposition 3.6 of Jehiel et al. (2005). The relative valuations are given by  $\mu^i = p^i + c^i c^{-i}$  and  $\mu^{-i} = -p^{-i} - c^i c^{-i}$ . Assume that  $(\psi, t)$  is a non-trivial ex-post incentive compatible mechanism with continuous relative transfers. Condition (1) of Proposition 3.6 in Jehiel et al. (2005) requires the existence of an indifference signal  $\hat{s} \in (0, 1)^4$ , of a vector  $(y_a, y_b)^T$ , and of a function  $\lambda(c^{-i}) \in \mathbb{R}^+$  such that:

$$\lambda(c^{-i}) \begin{pmatrix} 1 \\ c^{-i} \end{pmatrix} = \begin{pmatrix} 0 - y_a \\ -c^{-i} - y_b \end{pmatrix}$$

for all  $c^{-i}$  in a neighborhood of  $\hat{c}^{-i}$ . By the first equation,  $\lambda(c^{-i})$  is independent of  $c^{-i}$  and equal to  $-y_a$ . But the second equation  $\lambda(c^{-i}) c^{-i} = -c^{-i} - y_b$  can be satisfied for a continuum of  $c^{-i}$  only if  $\lambda(c^{-i}) \equiv -1$ . This is in contradiction to  $\lambda(c^{-i}) \in \mathbb{R}^+$ . Alternatively, a consideration of the cross product  $-c^{-i} - y_b + y_a c^{-i} = 0$  yields  $y_b = 0$  and  $y_a = 1$ . This shows again that  $\nabla_{s^i} \mu^i(s)$  and  $(\nabla_{s^i} \mu^{-i}(s) - (1, 0)^T)$  are co-linear (but point in opposite directions).<sup>3</sup> ■

## References

- [1] Bergemann, D. and S. Morris (2004), "Robust Mechanism Design," Discussion Paper, Yale University.
- [2] Bikhchandani, S., "The Limits of Ex-Post Implementation Revisited," Mimeo, UCLA.
- [3] Green, J. and J. J. Laffont (1987), "Posterior Implementation in a Two-Person Decision Problem," *Econometrica* **55**, 69-94.
- [4] Jehiel, P., M. Meyer-ter-Vehn, and B. Moldovanu and W. Zame (2005), "The Limits of Ex Post Implementation", mimeo.

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<sup>3</sup>To see that condition (2) of that Proposition isn't satisfied either, note that the direction of  $\nabla_{s^i} \mu^i(s) = (1, c^{-i})^T$  cannot be locally independent of  $s^{-i}$ . Thus, non-trivial implementation fails also with discontinuous transfers.