Quadratic Concavity and Determinacy of Equilibrium*

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Abstract

One of the central features of classical models of competitive markets is the generic determinacy of competitive equilibria. For smooth economies with a finite number of commodities and a finite number of consumers, almost all initial endowments admit only a finite number of competitive equilibria, and these equilibria vary (locally) smoothly with endowments; thus equilibrium comparative statics are locally determinate. This paper establishes parallel results for economies with finitely many consumers and infinitely many commodities. The most important new condition we introduce, quadratic concavity, rules out preferences in which goods are perfect substitutes globally, locally, or asymptotically. Our framework is sufficiently general to encompass many of the models that have proved important in the study of continuous-time trading in financial markets, trading over an infinite time horizon, and trading of finely differentiated commodities.

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1 Introduction

One of the central features of classical models of competitive markets is the generic determinacy of competitive equilibria. For smooth economies with a finite number of consumers and a finite number of commodities, almost all initial endowments yield an economy that admits only a finite number of competitive equilibria, and these equilibria vary (locally) smoothly with endowments. These results, based on Debreu's (1970, 1972) seminal work, guarantee that equilibrium and local comparative statics in the Walrasian model are meaningful.

In this paper we establish parallel results for economies with finitely many consumers and infinitely many commodities. Our framework is sufficiently general to encompass most of the models that have proved important in the study of continuous-time trading in financial markets, trading over an infinite time horizon, and trading of finely differentiated commodities.

Debreu's approach to determinacy in the finite-dimensional setting relies on the familiar characterization of equilibrium as a zero of aggregate excess demand. His assumptions — that preferences are differentiably strictly convex and satisfy a boundary condition — guarantee that aggregate excess demand is a smooth mapping on the open domain of strictly positive prices, so the machinery of smooth analysis — the implicit function theorem and the transversality theorem in particular — may be applied. By now it is well-understood that this approach cannot work in general in the infinite-dimensional setting for many reasons, including the fact that demand may be undefined for many prices, the domain of strictly positive prices is not open, and demand may not be smooth even where it is defined.[1,2]

To address some of these problems, Kehoe and Levine (1985) pioneered an approach to determinacy in the infinite-dimensional setting that relies on the Negishi characterization of equilibrium as a zero of the excess spending map. Because they consider consumption over a discrete infinite horizon

[1]See Shannon (1998a) for a more detailed discussion.

[2]Kehoe, Levine, Mas-Colell, and Zame (1989) treat a model in which smooth analysis can be applied. They take the commodity space to be a Hilbert space, specify a consumer by a smooth demand function (rather than by a preference relation), and require that price and consumption sets be open. However, the last requirement means that they allow for negative consumptions and negative prices, which are difficult to interpret economically.

and assume that utility functions are additively separable across time, they can decompose the infinite-dimensional planner's problem into an independent sequence of finite-dimensional planning problems. Because they assume period utility functions are differentiably strictly concave and satisfy Inada conditions, the solutions to these finite-dimensional planning problems are smooth. From this it follows that the excess spending map is smooth, so the machinery of smooth analysis again applies to yield generic determinacy in a fairly straightforward fashion.

Much of the relatively small body of existing work on determinacy in infinite-dimensional models adopts both Kehoe and Levine's approach of using the excess spending map and their assumption of additively separable preferences (see Kehoe, Levine, and Romer (1990), Balasko (1997), and Chichilnisky and Zhou (1998)). Additive separability is clearly economically restrictive, ruling out habit formation, any disentangling of risk aversion and intertemporal substitution, or interpretation of nearby characteristics as close substitutes for example, yet it is crucial for their results.

Shannon (1998a) gives the first determinacy results in infinite dimensions applicable to a broad class of non-separable preferences. As Shannon (1998a) points out, such results must overcome two main problems. The Inada or boundary conditions on utility functions used by Debreu (1970, 1972) and Kehoe and Levine (1985) are inconsistent with the assumptions ("properness") generally required to guarantee the existence of equilibrium in infinite-dimensional models. Without these assumptions, however, the solution to the planner's problem, and hence the excess spending map, need not be smooth. To address this problem, Shannon (1998a) introduces techniques from nonsmooth analysis and demonstrates that Lipschitz continuity of the excess spending map is sufficient to yield generic determinacy. A more fundamental problem involves finding conditions on general preferences in infinite dimensions sufficient to ensure that distinct goods are not perfect substitutes either locally or asymptotically, and that the planner's problem – and hence excess spending map – is Lipschitz continuous. Shannon (1998a) gives one such set of conditions. The method Shannon (1998a) uses to analyze the planner's problem involves approximation by increasing finite truncations of the commodity space, however, and the results apply only if the number of commodities is countable. Shannon (1998a) therefore does not apply to a variety of important commodity spaces, including those that arise in continuous-time models of finance, which have proved to be among the most useful and successful applications of general equilibrium theory, or in models of commodity differentiation.

In this paper, we introduce simple and natural restrictions on utility functions that generalize Debreu's differentiable strict concavity to the infinite-dimensional setting. The most important of these restrictions is a condition we call *quadratic concavity*, which requires that near any feasible bundle, utility differs from the linear approximation by an amount that is at least quadratic in the distance to the given bundle. Quadratic concavity provides a quantitative measure of the extent to which distinct commodities are not perfect substitutes — globally, locally, or asymptotically. We use quadratic concavity to give a direct analysis of Pareto optima and supporting prices. A simple geometric argument shows that the solution to the planner's problem is Lipschitz; a parallel analysis of supporting prices establishes that the excess spending mapping is Lipschitz. Generic determinacy then follows by arguments similar to those in Shannon (1998a). The direct, geometric nature of our arguments means we require neither a countable number of commodities nor separability of preferences. Our methods also allow us to appeal to the infinite-dimensional genericity notion developed by Anderson and Zame (1997), Christensen (1974) and Hunt, Sauer, and Yorke (1992), and apply the transversality results of Shannon (1998b) to establish that determinacy is generic with respect to the (infinite-dimensional) set of all possible endowment distributions.

Because our approach does not depend on the number of commodities, our results apply equally well to all commodity spaces, regardless of whether they have a finite, countably infinite, or uncountably infinite number of commodities. Our results encompass the results of Debreu (1970, 1972) and Shannon (1994) for finite-dimensional commodity spaces, the results of Kehoe and Levine (1985) for ℓ_∞, and the results of Shannon (1998a) for ℓ_2. Our results are not strictly comparable to the results of Shannon (1998a) for ℓ_∞, although in spirit both our assumptions and our conclusions are weaker.[3]

The remainder of the paper is organized as follows. In Section 2 we detail

[3] In particular, Shannon (1998a) uses a stronger notion of differentiability and a different notion of genericity, but obtains determinacy with respect to the ℓ_∞ norm, while we obtain determinacy with respect to the Mackey topology. We discuss this in more detail in Section 3.

the basic assumptions maintained throughout. In Section 3 we introduce the notion of quadratic concavity. In Section 4 we characterize equilibrium in terms of welfare weights as the zeroes of the excess spending mapping. In Section 5 we study the social planner's problem characterizing Pareto optimal allocations, and in Section 6 we study supporting prices and show that the excess spending map is Lipschitz. We use these results in Section 7 to show that equilibria are generically determinate. In Section 8 we discuss several illustrative examples, including models of continuous-time trading, trading in differentiated commodities, and trading over an infinite horizon. Some of the more technical proofs are relegated to the Appendix.

2 The Economy

We consider an exchange economy $\mathcal{E}$ with m consumers. Throughout we maintain the following quite standard assumptions on the commodity and price spaces and on consumer characteristics:

A1 the commodity space X is a vector lattice endowed with a Hausdorff, locally convex topology τ [4]

A2 the price space X^* is the topological dual of X and is a sublattice of the order dual of X [5]

A3 order intervals in X are weakly compact

A4 each consumer's consumption set is the positive cone X_+

A5 each individual endowment e_i is positive and the social endowment $\bar{e} = \sum e_i$ is strictly positive [6]

[4]Following Mas-Colell and Richard (1991), we do *not* assume X is a topological vector lattice, so the lattice operations may not be continuous.

[5]In particular, prices are τ-continuous and the supremum and infimum of prices in X^* are again in X^*.

[6]Recall that $\bar{e} \in X_+$ is *strictly positive* if the order ideal $\{x \in X : \exists k > 0, |x| \leq k\bar{e}\}$ is weakly dense in X. If X is a topological vector lattice, this is equivalent to the more familiar requirement that $p \cdot \bar{e} > 0$ for every $p \in X^*_+ \setminus \{0\}$.

A6 each consumer's utility function $U_i : X_+ \to \mathbf{R}$ is τ-continuous, strictly monotone, and strictly concave

We view the social endowment as fixed and treat the distributions of endowments as parameters. Let $P(\bar{e}) \subset X^m$ denote the set of feasible Pareto optimal allocations of the social endowment $\bar{e}$ and $P^0(\bar{e}) \subset P(\bar{e})$ the subset of allocations $(x_1, \ldots, x_m)$ for which each $x_i \neq 0$. Let $P_i(\bar{e})$ and $P_i^0(\bar{e})$ denote the projections of $P(\bar{e})$ and $P^0(\bar{e})$ onto the i-th coordinate. In addition to the above we assume:

A7 for each i, U_i is Gateaux differentiable at each $x \in P_i^0(\bar{e})$ and the Gateaux derivative $DU_i(x) \in X^*_{++}$ [7]

We call an economy satisfying assumptions A1-A7 a *basic economy*.

These assumptions represent standard conditions needed to ensure existence of equilibrium in economies with infinitely many commodities. The assumption that consumers' utilities are Gateaux differentiable plays the role of uniform properness here in ensuring the existence of prices supporting each Pareto optimal allocation. While it might seem strange to require differentiability only on the Pareto set, rather than on the entire consumption set, our weaker requirement allows us to include preferences satisfying Inada conditions, which might otherwise be excluded.[8] Of course differentiability on the entire consumption set or on the order interval $[0, \bar{e}]$ would suffice.

3 Quadratic Concavity

To motivate the central new notions we use, consider the simplest examples of robust indeterminacy of equilibrium: a two person, two commodity Edgeworth square in which both consumers find the two commodities to be perfect

[7]Recall that U_i is *Gateaux differentiable* at $x \in X_+$ if there is a continuous linear functional $DU_i(x)$ such that

$$\left[\lim_{h \to 0^+} \frac{U_i(x + hy) - U_i(x)}{h} - DU_i(x) \cdot y\right] = 0$$

for each $y \in X$ having the property that $x + hy \in X_+$ for h sufficiently small.

[8]For more on this point see Duffie and Zame (1989) and Araujo and Monteiro (1991).

complements or both consumers find the two commodities to be perfect substitutes. Smoothness of utility functions rules out perfect complements, while differential strict concavity rules out perfect substitutes. Moreover, in finite dimensions these assumptions are sufficient to rule out not only these simple examples of robust indeterminacy but all robust indeterminacies. Our assumptions are intended to have the same effect, but the precise formulation requires some care in our infinite-dimensional setting. A little background will help to understand our definitions.

Let $U : \mathbf{R}^n_+ \to \mathbf{R}$ be twice continuously differentiable and differentiably strictly concave, let $Y \subset \mathbf{R}^n_+$ be a compact set and let $Z \subset \mathbf{R}^n_+$ be a bounded set. For our purposes, these assumptions have three important implications. Continuity of the first derivative implies that it is bounded on compact sets, thus:

(i) there is a constant B such that, for each $x \in Y$ and $z \in \mathbf{R}^n$

$$|DU(x) \cdot z| \leq B \, \|z\|$$

Continuity of the second derivative implies that the gradient map $x \mapsto DU(x)$ is Lipschitz on Y. That is, there is a constant c such that

$$\|DU(x) - DU(y)\| \leq c \, \|x - y\|$$

for all $x, y \in Y$.[9] In particular, for $z \in Z$,

$$|DU(x) \cdot z - DU(y) \cdot z| \leq \|DU(x) - DU(y)\| \, \|z\| \leq c \, \|z\| \, \|x - y\|$$

Because Z is a bounded set, we conclude:

(ii) there is a constant C such that, for each $x, y \in Y$ and $z \in Z$:

$$|DU(x) \cdot z - DU(y) \cdot z| \leq C \, \|x - y\|$$

[9]To see this, apply Taylor's theorem to the first derivative. Given $x, y \in Y$, there is some $\tilde{x}$ on the line segment from x to y such that $DU(x) - DU(y) = D^2U(\tilde{x})(x - y)$. Hence $\|DU(x) - DU(y)\| \leq \|D^2U(\tilde{x})\| \, \|x - y\|$. Because $\tilde{x} \mapsto D^2U(\tilde{x})$ is continuous and Y is compact, there is a constant c such that $\|DU(x) - DU(y)\| \leq c \, \|x - y\|$.

In other words, the evaluation map $x \mapsto DU(x) \cdot z$ is Lipschitz on Y, uniformly for $z \in Z$.

Finally, Taylor's theorem implies that for $x, y \in Y$,

$$U(y) = U(x) + DU(x) \cdot (y - x) + \frac{1}{2}[D^2U(\hat{x})(y - x)] \cdot [y - x]$$

for some $\hat{x}$ on the line segment between x and y. Strict differential concavity together with continuity of the second derivative means that the second derivative matrix is strictly negative definite, uniformly on compact sets, so:

(iii) there is a constant $K > 0$ such that, for each $x, y \in Y$:

$$U(y) \leq U(x) + DU(x) \cdot (y - x) - K\|y - x\|^2$$

The properties (i)-(iii) all refer to a particular norm on $\mathbf{R}^n$, but they are in fact unambiguous because all norms on $\mathbf{R}^n$ are equivalent. In an infinite-dimensional setting, the given commodity space X may not admit any norm or may admit many inequivalent norms. Our key assumptions abstract the properties (i)-(iii) of differentiably strictly concave functions in $\mathbf{R}^n$, but a crucial feature of our approach is that we do not require that X be normed, or that the conditions be satisfied with respect to the given norm of X even if X is normed. Rather, we require only that there be *some* norm with respect to which these conditions are satisfied, that also induces the given topology τ on the set $[0, \bar{e}]$ of feasible consumptions.

The following two definitions abstract the properties we need.

Definition Let $U : X_+ \to \mathbf{R}$ be Gateaux differentiable on $Y \subset X_+$. We say the norm $\|\cdot\|$ is *adapted to U on Y* if the topology induced by $\|\cdot\|$ coincides with τ on the order interval $[0, \bar{e}]$,[10] and

(i) there is a constant B such that, for each $x \in Y$ and $z \in X$:

$$|DU(x) \cdot z| \leq B\|z\|$$

[10]For most of our purposes, it would suffice to assume that the topology induced by $\|\cdot\|$ is stronger than τ on the order interval $[0, \bar{e}]$.

(ii) there is a constant C such that for each $x, y \in Y$ and $z \in [0, \bar{e}]$

$$|DU(x) \cdot z - DU(y) \cdot z| \leq C\|x - y\|$$

That is, the evaluation map $x \mapsto DU(x) \cdot z : Y \to \mathbf{R}$ is Lipschitz on Y, uniformly for $z \in [0, \bar{e}]$.[11]

Next we adapt condition (iii) above. This condition implies that a significant change in consumption, measured by $\|x - y\|$, must have a significant effect on marginal utility,[12] which in turn provides a measure of the extent to which distinct goods are not perfect substitutes either locally or globally. The following abstraction of this condition will be central for our determinacy results.

Definition Let $U : X_+ \to \mathbf{R}$ be a concave function and let $\|\cdot\|$ be a norm on X. We say U is *quadratically concave* on $Y \subset X_+$ with respect to $\|\cdot\|$ if U is Gateaux differentiable on Y and there is a constant $K > 0$ such that for each $x, y \in Y$:

$$U(y) \leq U(x) + DU(x) \cdot (y - x) - K \;\|y - x\|^2$$

To understand this condition, recall that a differentiable concave function is bounded above by the linear approximation given by the gradient, that is, $U(y) \leq U(x) + DU(x) \cdot (y - x)$ for all $x, y \in X_+$. Quadratic concavity requires in addition that the error in this linear approximation is at least quadratic, uniformly for $x, y \in Y$.

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[11] As in the finite-dimensional setting, the second condition is implied by either of the simpler and more familiar conditions:

(iia) the gradient map $x \mapsto DU(x) : X \to X^*$ is Lipschitz on Y

(iib) U is twice continuously Gateaux differentiable on Y and $D^2U(x)$ is uniformly bounded with respect to $\|\cdot\|$ on Y

We use the less restrictive, although more complicated, condition given rather than either of these simpler conditions because the difference is important in a number of applications. In particular, **?** presents an example of an environment in which the natural utility functions satisfy our conditions but are not twice differentiable and for which the gradient mappings are not Lipschitz.

[12] See also the discussion in Example 3.1.

As a simple illustration, note that quadratic concavity is implied by differential strict concavity. Indeed, if $U : X_+ \to \mathbf{R}$ is twice continuously Gateaux differentiable and differentiably strictly concave on a convex set $Y \subset X_+$, then there is a constant $K > 0$ such that $z \cdot D^2U(y)z \leq -K\|z\|^2$ for all $z \in X$ and $y \in Y$. It follows immediately from Taylor's theorem that U is quadratically concave on Y, as we argued above. On the other hand, many natural utility functions in some of the most basic infinite-dimensional models are quadratically concave but not differentiably strictly concave. See **?** for examples of quadratically concave utility functions in environments that admit no differentiably strictly concave utility functions.

Shannon (1998a) introduces a different condition, called uniform concavity, that is also meant to generalize differential strict concavity to infinite dimensions. For the commodity space ℓ_2 or L_2, uniform concavity is essentially equivalent to differential strict concavity, as it amounts to requiring the utility function to be C^2 with uniformly strictly negative definite second derivative. Thus in ℓ_2 or L_2 quadratic concavity is implied by uniform concavity, and our results encompass those of Shannon (1998a). For the commodity space ℓ_∞, uniform concavity requires instead that the utility function have a second derivative that, after an appropriate rescaling, is uniformly negative definite. This rescaling essentially amounts to a local renormalization similar in spirit to the global renormalization we permit with quadratic concavity. The conditions are not directly comparable, however. In spirit quadratic concavity is a weaker condition, and yields weaker conclusions, as we obtain determinacy with respect to the Mackey topology while Shannon (1998a) obtains determinacy with respect to the ℓ_∞ norm. The additional generality we get by stating our condition only in terms of the directional derivatives and the first-order approximation error will be very useful, as our examples illustrate.

We will assume that each consumer's utility function is quadratically concave on weakly compact subsets of $P_i^0(\bar{e})$ with respect to some norm $\|\cdot\|_i$ that is adapted to U_i on weakly compact subsets of $P_i^0(\bar{e})$. The flexibility both to choose a norm different from the underlying norm when X is a normed space, and to choose a different norm for each consumer, will be important in a number of applications, as the following example illustrates; see also Examples 8.2 and 8.3 and **?**, Example 8.4.

Example 3.1 Let $X = \ell_\infty$, the space of bounded real sequences, with the Mackey topology. Let $u : \mathbf{R}_+ \to \mathbf{R}$ be twice continuously differentiable and differentiably strictly concave. Fix β with $0 < \beta < 1$, and define

$$U(x) = \sum_{t=0}^{\infty} \beta^t u(x_t)$$

We claim that U is *not* quadratically concave with respect to the ℓ_∞ norm $\|\cdot\|_\infty$ on $[0, \bar{e}]$ for any positive social endowment $\bar{e}$.

The intuition is simple. Discounting means that big changes in consumption in the distant future have small effects on utility and on marginal utility. Discounting thus generates the same insensitivity of marginal utility to changes in consumption ordinarily associated with goods with a high degree of substitutability, and therefore suggests the potential for robust indeterminacies.

To make the point more formally, note that Gateaux differentiability and quadratic concavity of a function U on some set Y require that there exist some $K > 0$ such that for each $x, y \in Y$

$$\begin{aligned} U(x) &\leq U(y) + DU(y) \cdot (x - y) - K\|x - y\|^2 \text{ and} \\ U(y) &\leq U(x) + DU(x) \cdot (y - x) - K\|y - x\|^2 \end{aligned}$$

Combining and simplifying yields the following inequality for all $x, y \in Y$:

$$[DU(y) - DU(x)] \cdot (x - y) \geq 2K\|x - y\|^2 \tag{1}$$

On the other hand, discounting entails that the directional derivatives $DU(y) \cdot (x - y)$ and $DU(x) \cdot (x - y)$ will be close, even if the consumption bundles x and y are far apart, provided the change in consumption $x - y$ occurs far in the future. That is, the left hand side in (1) may be small while the right hand side is large, provided the change occurs in the distant future. To see this formally, set $x = (1, 1, \ldots)$. For each T, let $\chi^T \in \ell_\infty$ be the sequence which has 1 in the T-th coordinate and 0 elsewhere. Write $y^T = x + \chi^T$, and note that $\|y^T - x\| = 1$ for each T. Applying Taylor's theorem and noting that $DU(x) \cdot (y^T - x) = \beta^T u'(1)$, we conclude that there exists $\zeta \in (1, 2)$ such that

$$U(y^T) = U(x) + \beta^T (u(2) - u(1))$$

$$
\begin{aligned}
&= U(x) + \beta^T u'(1) + \frac{1}{2}\beta^T u''(\zeta) \\
&= U(x) + DU(x)\cdot(y^T - x) - \frac{1}{2}\beta^T |u''(\zeta)| \|y^T - x\|_\infty^2
\end{aligned}
$$

Because u'' is bounded on the interval $[1,2]$, $\beta^T u''(\zeta)\|y^T - x\|_\infty^2 \to 0$ as $T \to \infty$. In particular, U is certainly not quadratically concave with respect to the ℓ_∞ norm.

Nonetheless there is an adapted norm with respect to which U is quadratically concave.[13] The appropriate norm is the β weighted norm defined, for $z \in \ell_\infty$, by

$$
\|z\|_\beta = \sum_{t=0}^{\infty} \beta^t |z_t|
$$

This norm reflects the same impatience as the utility function, measuring as close bundles that differ only in the distant future.

Fix the social endowment $\bar{e} \in \ell_{\infty+}$. The computations required to verify that $\|\cdot\|_\beta$ is adapted to U on $[0,\bar{e}]$ are straightforward, and left to the reader. To see that U is quadratically concave on $[0,\bar{e}]$ with respect to $\|\cdot\|_\beta$, fix $x, y \in [0,\bar{e}]$. Applying Taylor's theorem to utility in period t yields

$$
u(y_t) = u(x_t) + u'(x_t)(y_t - x_t) + \frac{1}{2}u''(z_t)(y_t - x_t)^2
$$

for some z_t between x_t and y_t. Because u is differentiably strictly concave, there is a constant $c > 0$ such that $u''(\zeta) < -c$ for $\zeta \le \sup_t e_t$. Hence

$$
\begin{aligned}
U(y) - U(x) &= \sum \beta^t (u(y_t) - u(x_t)) \\
&= \sum \beta^t u'(x_t)(y_t - x_t) + \sum \beta^t \frac{1}{2} u''(z_t)(y_t - x_t)^2 \\
&= DU(x)\cdot(y - x) + \frac{1}{2}\sum \beta^t u''(z_t)(y_t - x_t)^2 \\
&\le DU(x)\cdot(y - x) - \frac{c}{2}\sum \beta^t (y_t - x_t)^2 \\
&\le DU(x)\cdot(y - x) - cb\left(\sum \beta^t |y_t - x_t|\right)^2 \\
&= DU(x)\cdot(y - x) - cb\|y - x\|_\beta^2
\end{aligned}
$$

[13] See also Example 8.3 in Section 8.

for some $b > 0$, where the second inequality follows from the fact that in a finite measure space, there exists $B > 0$ such that $\|f\|_2 \geq B\|f\|_1$ for all f, where $\|\cdot\|_p$ denotes the L_p norm for $1 \leq p \leq \infty$. Thus U is quadratically concave with respect to $\|\cdot\|_\beta$ on $[0, \bar{e}]$.

4 Equilibrium and the Excess Spending Map

Given a distribution $e = (e_1, \ldots, e_m)$ of the social endowment $\bar{e}$, an equilibrium can be characterized, using the welfare theorems, as a Pareto optimal allocation x and a supporting price p for which the budget equations

$$\begin{aligned} p \cdot (x_1 - e_1) &= 0 \\ &\vdots \\ p \cdot (x_m - e_m) &= 0 \end{aligned}$$

are satisfied. Because x is a feasible allocation, we henceforth suppress the last (redundant) equation. Central to our approach is an amplification of this characterization, following Negishi, in which *pairs* consisting of Pareto optimal allocations and supporting prices are parametrized uniquely by "welfare weights." The parametrization of Pareto optima is of course very familiar, while, surprisingly, the parametrization of allocation/price pairs seems to have escaped attention, even in the finite-dimensional setting.

The first step in this program involves the familiar parametrization of Pareto optima as the solutions to a social planner's problem. Given a social endowment bundle $\bar{e}$ and a vector of "welfare weights" $\lambda \in \mathbf{R}^m_+$ with $\sum \lambda_i = 1$, the social planner's problem is to choose a feasible allocation $x(\lambda) = (x_1(\lambda), \ldots, x_m(\lambda))$ to maximize the weighted sum of utilities $\sum \lambda_i U_i(x_i)$. The following result records several basic properties of the solution to the planner's problem under our assumptions; we omit the familiar proof. We write

$$\begin{aligned} \Lambda &\equiv \{\lambda \in \mathbf{R}^m_+ : \textstyle\sum \lambda_i = 1\} \\ \Lambda^0 &\equiv \{\lambda \in \Lambda : \lambda_i > 0 \text{ for all } i\} \end{aligned}$$

for the sets of welfare weights and strictly positive welfare weights.

Lemma 4.1 *If $\mathcal{E}$ is a basic economy then* *(i) for each $\lambda \in \Lambda$ the planner's problem has a unique solution $x(\lambda) \in P(\bar{e})$*

(ii) the mapping $x : \Lambda \to P(\bar{e})$ is continuous when $P(\bar{e})$ is equipped with the weak topology of X^m

(iii) $x(\Lambda) = P(\bar{e})$ and $x(\Lambda^0) = P^0(\bar{e})$

The second step is the characterization of supporting prices in terms of welfare weights. Our assumptions imply that every Pareto optimum $x(\lambda)$ can be supported by some price. If $x = x(\lambda)$ is interior, the unique supporting price, up to normalization, is $\lambda_i DU_i(x_i)$ for any i. If $x(\lambda)$ is not interior, however, it may admit many supporting prices, a difficulty that is particularly acute in the infinite-dimensional setting, and any one of these might be an equilibrium price. Mas-Colell and Richard (1991) show that $\bigvee_i \lambda_i DU_i(x_i)$ is always one supporting price, but there may be many other supporting prices.[14] Because we seek to show that a given economy admits only finitely many equilibria, we must be sure to identify *all* of the equilibria. The following lemma, which shows that pairs consisting of Pareto optima and supporting prices can be parametrized uniquely by the welfare weights, is just what we need. We defer the proof to the Appendix.

Lemma 4.2 *If $\mathcal{E}$ is a basic economy, x is a feasible allocation for which $x_i \neq 0$ for each i, and $q \in X^*_+$ is a non-zero price, then the following statements are equivalent:*

(i) x is a Pareto optimal allocation and q supports x

(ii) there is a vector of welfare weights $\lambda \in \Lambda^0$ and a constant $\beta > 0$ such that x solves the planner's problem for the weights λ and

$$q = \beta \bigvee_i \lambda_i DU_i(x_i)$$

[14] Indeed, if the map from welfare weights to Pareto optima is not one-to-one, so that $x(\lambda) = x(\lambda')$ for some $\lambda \neq \lambda'$, then $\bigvee_i \lambda_i DU_i(x_i)$ and $\bigvee_i \lambda'_i DU_i(x_i)$ are certainly distinct prices supporting the Pareto optimal allocation $x = x(\lambda) = x(\lambda')$. Lemma 4.2 shows that this is the only possible multiplicity.

In other words, the map $\lambda \mapsto \big(x(\lambda), \bigvee_i \lambda_i DU_i(x_i)\big)$ is a parametrization of pairs consisting of Pareto optima and supporting prices by non-zero welfare weights, and this parametrization is one-to-one and onto (up to scalar multiplication of prices).

In view of Lemma 4.2 and the redundancy of the budget equations, we obtain immediately the following characterization of equilibrium in terms of welfare weights.

Lemma 4.3 *Let $\mathcal{E}$ be a basic economy. The allocation x and the price p constitute an equilibrium if and only if there exists a vector of welfare weights $\lambda \in \Lambda^0$ such that*

(a) x solves the planner's problem with weights λ

(b) $p = \beta \bigvee_i \lambda_i DU_i(x_i(\lambda))$ for some $\beta > 0$

(c) the budget equations

$$\begin{aligned} p \cdot (x_1(\lambda) - e_1) &= 0 \\ &\vdots \\ p \cdot (x_{m-1}(\lambda) - e_{m-1}) &= 0 \end{aligned}$$

are satisfied

Given these results, we can characterize equilibrium in terms of the zeroes of the excess spending mapping. Given the social endowment, write

$$D^0(\bar{e}) \equiv \Big\{ e \in X_+^m : e_i \neq 0 \text{ for each } i \text{ and } \sum e_i = \bar{e} \Big\}$$

for the set of distributions of the social endowment that give no consumer zero endowment. Write $p(\lambda) = \bigvee_i \lambda_i DU_i(x_i(\lambda))$, and define the *excess spending mapping*

$$S : \Lambda^0 \times D^0(\bar{e}) \to \mathbf{R}^{m-1}$$

by $S_i(\lambda, e) = p(\lambda) \cdot (x_i(\lambda) - e_i)$ for each i. If e is a distribution of the social endowment $\bar{e}$, write $\mathcal{E}(e)$ for the economy with endowment profile e. In view of the discussion above, we may identify an equilibrium of the economy $\mathcal{E}(e)$ with a zero of $S(\cdot, e)$. In the following sections, we first show that the planner's problem is Lipschitz and then that the excess spending mapping is Lipschitz; we then use versions of Sard's theorem and the transversality theorem for Lipschitz functions to obtain our generic determinacy results.

5 The Social Planner's Problem

In this section, we carry out the first step in our program, analyzing the solution to the social planner's problem. As we show below, under the additional assumption of quadratic concavity with respect to an adapted norm, the solution to the planner's problem is locally Lipschitz continuous. This result will become the key to all of our determinacy results.

Lemma 5.1 *If $\mathcal{E}$ is a basic economy and for each i there is a norm $\|\cdot\|_i$ such that*

(a) $\|\cdot\|_i$ is adapted to U_i on weakly compact subsets of $P_i^0(\bar{e})$

(b) U_i is quadratically concave with respect to $\|\cdot\|_i$ on weakly compact subsets of $P_i^0(\bar{e})$

then the solution $x(\cdot)$ to the planner's problem is locally Lipschitz on Λ^0 with respect to these norms and continuous with respect to the topology τ.

Proof: We first show that each U_i is Lipschitz on weakly compact subsets of $P_i^0(\bar{e})$. For $\delta > 0$ set $\Lambda^\delta \equiv \{\lambda \in \Lambda : \lambda_i \geq \delta \text{ for all } i\}$, write $P^\delta(\bar{e}) = x(\Lambda^\delta)$, and let $P_i^\delta(\bar{e})$ be the projection of $P^\delta(\bar{e})$ onto the i-th coordinate. Lemma 4.1 guarantees that $P_i^\delta(\bar{e})$ is weakly compact, so by adaptedness, for each i there is a constant B_i such that

$$|DU_i(x) \cdot z| \leq B_i \|z\|_i$$

for each $x \in P_i^\delta(\bar{e})$ and $z \in X$. If $x, y \in P_i^\delta(\bar{e})$ then concavity of U_i guarantees that

$$U_i(y) - U_i(x) \leq DU_i(x) \cdot (y - x) \leq B_i \|y - x\|_i$$

Reversing the roles of x and y yields

$$|U_i(y) - U_i(x)| \leq \max\left(|DU_i(x) \cdot (y - x)|, |DU_i(y) \cdot (x - y)|\right) \leq B_i \|y - x\|_i$$

which is the desired Lipschitz estimate.

Now fix $\delta > 0$ and let $\lambda, \lambda' \in \Lambda^\delta$. Write $x = x(\lambda)$ and $x' = x(\lambda')$. For each i, quadratic concavity of U_i with respect to $\|\cdot\|_i$ on $P_i^\delta(\bar{e})$ means there is a constant $C_i > 0$ such that

$$U_i(x_i') \leq U_i(x_i) + DU_i(x_i) \cdot (x_i' - x_i) - C_i \|x_i' - x_i\|_i^2$$

Multiplying by λ_i and summing over i yields

$$\sum \lambda_i U_i(x_i') \leq \sum \lambda_i U_i(x_i) + \sum \lambda_i DU_i(x_i) \cdot (x_i' - x_i) - \sum C_i \lambda_i \|x_i' - x_i\|_i^2 \quad (2)$$

By assumption, x solves the social planner's problem for weights λ, so the first order conditions imply that $\sum \lambda_i DU_i(x_i) \cdot (x_i' - x_i) \leq 0$. Substituting into (2) gives

$$\sum \lambda_i U_i(x_i') \leq \sum \lambda_i U_i(x_i) - \sum C_i \lambda_i \|x_i' - x_i\|_i^2 \quad (3)$$

Because x' solves the social planner's problem for weights λ', weighted utility is no greater at x, so:

$$0 \leq \sum \lambda_i' U_i(x_i') - \sum \lambda_i' U_i(x_i) \quad (4)$$

Adding (3) and (4) yields:

$$\sum \lambda_i U_i(x_i') \leq \sum \lambda_i U_i(x_i) - \sum C_i \lambda_i \|x_i' - x_i\|_i^2 + \sum \lambda_i' U_i(x_i') - \sum \lambda_i' U_i(x_i)$$

Rearranging terms gives

$$\begin{aligned} \sum C_i \lambda_i \|x_i' - x_i\|_i^2 &\leq \sum (\lambda_i - \lambda_i')[U_i(x_i) - U_i(x_i')] \\ &\leq \sum |\lambda_i' - \lambda_i|\, |U_i(x_i') - U_i(x_i)| \quad (5) \end{aligned}$$

Because utility functions are Lipschitz on $P_i^\delta(\bar{e}) = x(\Lambda^\delta)$, for each i there is a constant $K_i > 0$ such that $|U_i(x_i') - U_i(x_i)| \leq K_i \|x_i' - x_i\|_i$. Substituting into (5) yields

$$\sum C_i \lambda_i \|x_i' - x_i\|_i^2 \leq \sum K_i |\lambda_i' - \lambda_i|\, \|x_i' - x_i\|_i \quad (6)$$

Let $C = \min C_i$ and $K = \max K_i$. The left hand side of (6) is the summation of m positive terms, so is at least as large as any one of them. Because $\lambda, \lambda' \in \Lambda^\delta$, it follows that

$$C\delta \left(\max_i \|x_i' - x_i\|_i \right)^2 \leq K \max_i \|x_i' - x_i\|_i \left(\sum |\lambda_i' - \lambda_i| \right)$$

Rearranging terms yields

$$\max_i \|x_i' - x_i\|_i \leq \frac{K}{C\delta} \sum |\lambda_i' - \lambda_i|$$

which gives the desired Lipschitz estimate.

Finally, because the topology induced by each of the norms $\|\cdot\|_i$ coincides with the topology τ on the set $[0, \bar{e}]$ of feasible consumptions, the solution $x(\cdot)$ to the planner's problem is continuous in the topology τ as well. □

6 Spending, Wealth and Excess Spending

In this section we turn to the second step in our program, demonstrating the Lipschitz continuity properties of the spending map, the wealth map and the excess spending map. The proof is deferred to the Appendix.

Lemma 6.1 *If $\mathcal{E}$ is a basic economy and for each i there is a norm $\|\cdot\|_i$ such that*

(a) $\|\cdot\|_i$ *is adapted to U_i on weakly compact subsets of $P_i^0(\bar{e})$*

(b) U_i *is quadratically concave with respect to $\|\cdot\|_i$ on weakly compact subsets of $P_i^0(\bar{e})$*

then

(i) for each i the spending map $\lambda \mapsto p(\lambda) \cdot x_i(\lambda)$ is locally Lipschitz on Λ^0

(ii) the wealth map $\lambda \mapsto p(\lambda) \cdot w$ is locally Lipschitz on Λ^0, uniformly for $w \in [0, \bar{e}]$

(iii) $S(\cdot, e)$ is locally Lipschitz on Λ^0, uniformly for $e \in D^0(\bar{e})$

(iv) $S(\cdot, \cdot)$ is jointly continuous on $\Lambda^0 \times D^0(\bar{e})$

7 Generic Determinacy

In this section we use the results of Sections 5 and 6 to establish generic determinacy of equilibria. We treat the basic features of the economy $\mathcal{E}$ — commodity space, price space, utility functions, social endowment — as fixed, and consider variations in the initial endowment profile e over the set of all distributions of the social endowment $\bar{e}$. As before, we write $D^0(\bar{e})$ for the set of non-zero endowment distributions. For $e \in D^0(\bar{e})$, let $\mathcal{E}(e)$ denote the economy $\mathcal{E}$ with initial endowment profile e, and let $E(e)$ denote the set of equilibrium allocations of $\mathcal{E}(e)$. Our basic notion of determinacy involves finiteness of the number of equilibria and continuity of the equilibrium allocation correspondence. Formally:

Definition The economy $\mathcal{E}(e)$ is *determinate* if the number of equilibria is finite and the equilibrium allocation correspondence $E : D^0(\bar{e}) \to X_+^m$ is continuous at e.

In view of our discussion at the end of Section 4, we may identify an equilibrium of $\mathcal{E}(e)$ with a zero of $S(\cdot, e)$. It is convenient to define the equilibrium weight correspondence $E_\Lambda : D^0(\bar{e}) \to \Lambda$ by

$$E_\Lambda(e) \equiv \{\lambda \in \Lambda : S(\lambda, e) = 0\}$$

Since Lemma 5.1 guarantees that the solution to the planner's problem x is continuous with respect to the topology τ, $\mathcal{E}(e)$ is determinate if and only if $E_\Lambda(e)$ is finite and E_Λ is continuous at e.[15]

Our goal is to show that, given our assumptions, almost all endowment distributions lead to determinate economies. To make this statement precise, we need to explain what we mean by "almost all" endowment distributions. In a finite-dimensional setting, it is natural to interpret "almost all" to mean having full Lebesgue measure in the set of all endowment distributions. In an infinite-dimensional setting, however, there is no natural measure on the set of endowment distributions. We provide two alternatives; the first makes use of a finite-dimensional parameterization of endowment distributions, while the second uses an infinite-dimensional analogue of Lebesgue measure 0.

For our first determinacy result, fix a profile $e^* = (e_1^*, \ldots, e_m^*) \in X^m$ for which $\sum e_i^* = \bar{e}$ and a vector $v \in X_+ \setminus \{0\}$.[16] Set

$$A(e^*, v) = \left\{\alpha \in \mathbf{R}^m : e_i^* + \alpha_i v > 0 \text{ all } i \text{ and } \sum \alpha_i = 0\right\}$$

To each vector $\alpha \in A(e^*, v)$ corresponds an initial endowment distribution e^α of the social endowment $\bar{e}$ defined by $e_i^\alpha = e_i + \alpha_i v$ for each i. We view e^α as a perturbation of the initial profile e^*. Considering the family of such perturbations gives us a simple finite-dimensional parameterization of initial endowments indexed by $A(e^*, v)$. Our first determinacy result shows that

[15] We do not insist that the equilibrium price correspondence be continuous; this may be a delicate issue. However, it follows from Lemma 6.1 and Theorem 7.2 that if $\mathcal{E}(e)$ is determinate then the "evaluation" correspondence $P_z : D^0(\bar{e}) \to \mathbf{R}$ defined by $P_z(e) = \{p(\lambda) \cdot z : \lambda \in E_\Lambda(e)\}$ is continuous at e for each $z \in [0, \bar{e}]$.

[16] For applicability in Theorem 7.2, we allow for the possibility that e^* is not positive and even that $A(e^*, v)$ may be empty for some e^* and v.

those perturbations for which the economy $\mathcal{E}(e^\alpha)$ is determinate form a set of full $(m-1)$-dimensional Lebesgue measure.

Theorem 7.1 *If $\mathcal{E}$ is a basic economy and for each i there is a norm $\|\cdot\|_i$ such that*

(a) $\|\cdot\|_i$ is adapted on weakly compact subsets of $P_i^0(\bar{e})$

(b) U_i is quadratically concave with respect to $\|\cdot\|_i$ on weakly compact subsets of $P_i^0(\bar{e})$

then for each $e^* \in X^m$ with $\sum e_i^* = \bar{e}$ and each $v \in X_+ \setminus \{0\}$, almost all parameters $\alpha \in A(e^*, v)$ lead to a determinate economy; i.e.,

$$A_d(e^*, v) \equiv \{\alpha \in A(e^*, v) : \mathcal{E}(e^\alpha) \text{ is determinate } \}$$

is a set of full $(m-1)$-dimensional Lebesgue measure in $A(e^*, v)$.

Proof: Normalize prices by defining

$$\widehat{p}(\lambda) = \left(\frac{1}{p(\lambda)\cdot v}\right) p(\lambda)$$

for each $\lambda \in \Lambda$. Let $\widehat{S} : \Lambda^0 \times D^0(\bar{e}) \to \mathbf{R}^{m-1}$ be the corresponding normalized excess spending mapping whose i-th coordinate is:

$$\widehat{S}_i(\lambda, e) = \widehat{p}(\lambda) \cdot [x_i(\lambda) - e_i]$$

Note that S and $\widehat{S}$ have the same zeroes. Define $\sigma : \Lambda^0 \to \mathbf{R}^{m-1}$ by $\sigma(\lambda) = \widehat{S}(\lambda, e^*)$. Note that

$$\widehat{S}_i(\lambda, e^\alpha) = \widehat{S}_i(\lambda, e^*) - \alpha_i = \sigma_i(\lambda) - \alpha_i$$

for each i. Now for each $\alpha \in A(e^*, v)$, write $\alpha_{-m} = (\alpha_1, \ldots, \alpha_{m-1}) \in \mathbf{R}^{m-1}$. Note that $\widehat{S}(\lambda, e^\alpha) = \sigma(\lambda) - \alpha_{-m}$. Lemma 6.1 guarantees that the excess spending map $S(\cdot, e)$ is locally Lipschitz on Λ^0 for each e, and that $\lambda \mapsto p(\lambda)\cdot v$ is locally Lipschitz on Λ^0. It follows that $\widehat{S}(\cdot, e)$ is locally Lipschitz on Λ^0 for each e and thus that σ is locally Lipschitz on Λ^0.

If U is an open subset of $\mathbf{R}^{m-1}$ and $f : U \to \mathbf{R}^{m-1}$ is a mapping, recall that $\gamma \in \mathbf{R}^{m-1}$ is said to be a regular value of f if $Df(\zeta)$ exists and is nonsingular whenever $\zeta \in U$ and $f(\zeta) = \gamma$. Sard's theorem for locally Lipschitz functions (see Rader (1973) Lemma 2) guarantees that if f is locally Lipschitz then almost every element of $\mathbf{R}^{m-1}$ is a regular value of f. Because σ is locally Lipschitz, it follows that almost every $\gamma \in \mathbf{R}^{m-1}$ is a regular value of σ. Clearly α_{-m} is a regular value of σ if and only if 0 is a regular value of $\widehat{S}(\cdot, e^{\alpha})$, so the set

$$A_r(e^*, v) \equiv \left\{ \alpha \in A(e^*, v) : 0 \text{ is a regular value of } \widehat{S}(\cdot, e^{\alpha}) \right\}$$

has full $(m-1)$-dimensional Lebesgue measure.

To complete the proof it remains only to show that $A_r(e^*, v) \subset A_d(e^*, v)$; that is, if 0 is a regular value of $\widehat{S}(\cdot, e^{\alpha})$ then $\mathcal{E}(e^{\alpha})$ is a determinate economy. To see this, fix an $\alpha \in A_r(e^*, v)$. We must show that $\mathcal{E}(e^{\alpha})$ has only finitely many equilibria and that E_{Λ} is continuous at e^{α}.

To see that $\mathcal{E}(e^{\alpha})$ has finitely many equilibria, note that each equilibrium corresponds to a vector of weights $\lambda \in \Lambda^0$ by individual rationality. Each equilibrium vector of weights is locally unique because 0 is a regular value of $\widehat{S}(\cdot, e^{\alpha})$ (see Shannon (1994)). Then to show there are only finitely many equilibria it suffices to show that

$$\Lambda^{IR} \equiv \{\lambda \in \Lambda : U_i(x_i(\lambda)) \geq U_i(e_i^{\alpha}) \text{ for all } i\}$$

is a compact subset of Λ^o. To that end, first note that because $x(\cdot)$ is weakly continuous on Λ and $U_i(\cdot)$ is weakly upper semi-continuous, Λ^{IR} is a compact set. Moreover, because utilities are strictly monotone, $x_i(\lambda) = 0$ if $\lambda_i = 0$. Since $e_i^{\alpha} > 0$ for each i, $U_i(e_i^{\alpha}) > U_i(0)$ for each i, again using the strict monotonicity of utilities, which implies that $\Lambda^{IR} \subset \Lambda^0$. Now each equilibrium vector of welfare weights is locally unique and contained in the compact set Λ^{IR}, so there are only finitely many equilibria in $\mathcal{E}(e^{\alpha})$.

Upper hemi-continuity of E_{Λ} at e^{α} (and indeed, at every $e \in D^0(\bar{e})$) follows immediately from the joint continuity of S on $\Lambda^0 \times D^0(\bar{e})$. To see that E_{Λ} is lower hemi-continuous at e^{α}, fix $\lambda^* \in E_{\Lambda}(e^{\alpha})$ and a neighborhood V^* of λ^* in Λ^0. We must find a neighborhood W of e^{α} in $D^0(\bar{e})$ such that if $e \in W$ then $\widehat{S}(\lambda, e) = 0$ for some $\lambda \in V^*$. To accomplish this, we use the invariance of Brouwer degree under small perturbations. If $N \subset \Lambda^0$ is an open set and $f : N \to \mathbf{R}^{m-1}$ is a continuous mapping, write $\deg(f, N, 0)$ for the Brouwer degree of f on N at 0.

Let $\lambda^* \in E_\Lambda(e^\alpha)$. Choose a neighborhood V of λ^* in Λ^0 such that $E_\Lambda(e^\alpha) \cap V = \{\lambda^*\}$. Because 0 is a regular value of $\widehat{S}(\cdot, e^\alpha)$, $|\deg(\widehat{S}(\cdot, e^\alpha), V', 0)| = 1$ for every neighborhood $V' \subset V$ of λ^* (see Shannon (1994), Theorem 9). Then for each such neighborhood $V' \subset V$ of λ^* there exists a neighborhood B' of graph $\widehat{S}(\cdot, e^\alpha)|_{V'}$ such that $|\deg(f, V', 0)| = 1$ for any continuous function $f : V' \to \mathbf{R}^{m-1}$ for which graph $f \subset B'$. In particular, for any such function f there exists $\lambda \in V'$ such that $f(\lambda) = 0$. To establish our result, it thus suffices to show that given a neighborhood B' of graph $\widehat{S}(\cdot, e^\alpha)|_{V'}$ there exists a neighborhood W of e^α in $D^0(\bar{e})$ such that graph $\widehat{S}(\cdot, e)|_{V'} \subset B'$ for each $e \in W$.

To see this, note that for any $e \in D^0(\bar{e})$ and $\lambda \in V'$,

$$\widehat{S}_i(\lambda, e) - \widehat{S}_i(\lambda, e^\alpha) = \hat{p}(\lambda) \cdot (e_i - e_i^\alpha)$$

Now let $\varepsilon > 0$ be given. The map $(\lambda, v) \mapsto p(\lambda) \cdot v$ is jointly continuous on $\Lambda^0 \times [0, \bar{e}]$ by Lemma 6.1. Thus for each $\lambda \in \overline{V}'$ there exists a neighborhood V_λ of λ and a neighborhood W_λ of e^α in $D^0(\bar{e})$ such that for each $e \in W_\lambda$ and $\tilde{\lambda} \in V_\lambda$ we have $|\hat{p}(\tilde{\lambda}) \cdot (e_i - e_i^\alpha)| < \varepsilon$. Since $\{V_\lambda\}$ is an open cover of $\overline{V}'$ and $\overline{V}'$ is compact, there is a finite subcover $\{V_{\lambda^1}, \ldots, V_{\lambda^N}\}$. Set $W = \cap W_{\lambda^j}$. Then for $e \in W$, $|\widehat{S}_i(\lambda, e) - \widehat{S}_i(\lambda, e^\alpha)| < \varepsilon$ for each $\lambda \in V'$. Thus given a neighborhood B' of graph $\widehat{S}(\cdot, e^\alpha)|_{V'}$ there exists a neighborhood W of e^α in $D^0(\bar{e})$ such that graph $\widehat{S}(\cdot, e)|_{V'} \subset B'$ for each $e \in W$. Hence for each $e \in W$ there exists $\lambda \in V'$ such that $\widehat{S}(\lambda, e) = 0$, that is, such that $\lambda \in E_\Lambda(e)$. We conclude that E_Λ is lower-hemi-continuous at e^α, so the proof is complete. □

Because of the finite-dimensional nature of this parameterization, this result is not entirely satisfactory. To obtain a more satisfactory result, we would like to parameterize over the (infinite-dimensional) set of all possible endowment distributions, thus we need a notion of genericity suitable for use in an infinite-dimensional setting. Unfortunately, there is no natural analogue of Lebesgue measure in an infinite-dimensional space. A frequently used alternative topological notion of genericity would require that the set of endowment distributions leading to determinate economies be the intersection of a countable family of open sets that is dense in the space of all endowment distributions. The set of endowment distributions leading to determinate economies is dense, as a consequence of Theorem 7.2 below, but because the nature of regularity for Lipschitz functions is weaker than for smooth economies, the set of regular economies is not open nor is it the intersection of a countable family of open sets — even in the finite-dimensional setting. Instead we turn to a measure-theoretic analogue of "full Lebesgue measure" developed by Christensen (1974), Hunt, Sauer, and Yorke (1992), and Anderson and Zame (1997). Christensen (1974) and Hunt, Sauer, and Yorke (1992) have developed analogues of Lebesgue measure 0 and full Lebesgue measure for infinite-dimensional spaces, called shyness and prevalence. Their notions are not directly applicable in our problem, however, since our parameters are drawn not from the whole space but from the set of distributions of a fixed social endowment, itself a shy subset of the ambient space. Anderson and Zame (1997) have extended the work of Hunt, Sauer and Yorke and Christensen to notions of prevalence and shyness relative to a convex subset that may be a small subset of the ambient space. Their notion of relative prevalence, given below, is the concept of infinite-dimensional determinacy we use.

Definition Let Y be a topological vector space and let $C \subset Y$ be a convex Borel subset which is completely metrizable in the relative topology. Let $c \in C$. A universally measurable subset $E \subset Y$ is *shy in C at c* if for each $\delta > 0$ and each neighborhood W of 0 in Y, there is a regular Borel probability measure μ on Y with compact support such that $\operatorname{supp} \mu \subset (\delta(C - c) + c) \cap (W + c)$ and $\mu(E + y) = 0$ for every $y \in Y$.[17] The set E is *shy in C* if it is shy at each point $c \in C$. A (not necessarily universally measurable) subset $F \subset C$ is *shy in C* if it is contained in a shy universally measurable set. A subset $E \subset C$ is *prevalent in C* if its complement $C \setminus E$ is shy in C.

Anderson and Zame (1997) show that, like shyness and prevalence, relative shyness and prevalence have the properties we ought to require of measure-theoretic notions of "smallness" and "largeness." In particular, the countable union of shy sets is shy, no relatively open subset is shy, prevalent sets are dense, and a subset of $\mathbf{R}^n$ is shy in $\mathbf{R}^n$ if and only if it has Lebesgue measure 0. They also provide the following simple sufficient conditions for relative shyness and prevalence.

[17] Recall that a set $E \subset Y$ is universally measurable if for every Borel measure η on Y, E belongs to the completion with respect to η of the sigma algebra of Borel sets.

Definition Let Y be a topological vector space and let $C \subset Y$ be a convex Borel subset which is completely metrizable in the relative topology. A universally measurable set $E \subset C$ is *finitely shy in* C if there is a finite dimensional subspace $V \subset Y$ such that $(E+y) \cap V$ has Lebesgue measure 0 in V for every $y \in Y$. A universally measurable set $E \subset C$ is *finitely prevalent in* C if its complement $C \setminus E$ is finitely shy.

Sets that are finitely shy are shy, hence sets that are finitely prevalent are prevalent (see Anderson and Zame (1997)). Using these facts leads to more satisfactory infinite-dimensional determinacy results.

Theorem 7.2 *If $\mathcal{E}$ is a basic economy and for each i there is a norm $\|\cdot\|_i$ such that*

(a) $\|\cdot\|_i$ is adapted to U_i on weakly compact subsets of $P_i^0(\bar{e})$

(b) U_i is quadratically concave with respect to $\|\cdot\|_i$ on weakly compact subsets of $P_i^0(\bar{e})$

then almost all endowment distributions lead to a determinate economy; i.e.,

$$D_d^0(\bar{e}) = \left\{ e \in D^0(\bar{e}) : \mathcal{E}(e) \text{ is determinate } \right\}$$

is prevalent in $D^0(\bar{e})$.

Proof: We will show that $D_d^0(\bar{e})$ is finitely prevalent in $D^0(\bar{e})$. As before, we use the fact that $\mathcal{E}(e)$ is determinate exactly if $E_\Lambda(e)$ is finite and E_Λ is continuous at e.

It is evident that $D^0(\bar{e})$ is a Borel set. To see that it is completely metrizable, define a norm on X^m by $\|(x_1, \ldots, x_m)\| = \max_i \|x_i\|_i$. Adaptedness of $\|\cdot\|_i$ implies that the topology induced by $\|\cdot\|_i$ agrees with the topology τ on the order interval $[0, \bar{e}]$, so the topology induced by $\|\cdot\|$ agrees with the product topology τ^m on the set $D(\bar{e}) = \{e \in X_+^m : \sum e_i = \bar{e}\}$. Because order intervals are weakly compact in X, $D(\bar{e})$ is weakly compact in X^m. It follows that $D(\bar{e})$ is complete in the metric induced by the norm $\|\cdot\|_{max}$.[18] Because

[18] Let $\widetilde{X}^m$ be the completion of X^m with respect to the topology τ. Note that X^m and $\widetilde{X}^m$ have the same dual spaces. Hence $D(\bar{e})$ is weakly closed in $\widetilde{X}^m$. Since τ is a stronger topology, $D(\bar{e})$ is also τ-closed in $\widetilde{X}^m$. Now because $\widetilde{X}^m$ is complete, $D(\bar{e})$ is τ-complete as well.

$D^0(\bar{e})$ is a relatively open subset of $D(\bar{e})$, there is a complete metric on $D^0(\bar{e})$ having the property that the metric topology coincides with the topology τ^m.

We next show that $D^0_d(\bar{e})$ is a Borel set. Toward this end, write $D^0_f(\bar{e})$ for the endowment distributions e for which $E(e)$ is finite (equivalently, for which $E_\Lambda(e)$ is finite), and $D^0_c(\bar{e})$ for the endowment distributions e at which the equilibrium correspondence E is continuous (equivalently, at which the equilibrium weight correspondence E_Λ is continuous). As $D^0_d(\bar{e}) = D^0_f(\bar{e}) \cap D^0_c(\bar{e})$, it suffices to show that these are Borel sets.

To see that $D^0_f(\bar{e})$ is a Borel set, write $\mathbf{Q}_+$ for the set of strictly positive rational numbers. For each positive integer n, let $\mathcal{R}^n = (\mathbf{Q}^m_+ \cap \Lambda^0)^n$ be the set of n-tuples of points in Λ^0 with rational coordinates. For $r_j \in \mathbf{Q}^m_+ \cap \Lambda^0$ and $\beta_j \in \mathbf{Q}_+$, let $B(r_j, \beta_j)$ be the open ball in $\mathbf{R}^m$ with center r_j and radius β_j. An endowment distribution $e \in D^0(\bar{e})$ leads to an economy with at most n equilibria exactly if the set of equilibrium weights is contained in the union of n balls with rational centers and arbitrarily small rational radii. Hence

$$D^0_f(\bar{e}) = \bigcup_{n=1}^{\infty} \bigcap_{r\in\mathcal{R}^n} \bigcup_{\beta\in\mathbf{Q}^n_+} \left\{ e \in D^0(\bar{e}) : E_\Lambda(e) \subset \bigcup_{j=1}^{n} B(r_j, \beta_j) \right\}$$

Because E_Λ is upper hemi-continuous (see the proof of Theorem 7.1), each of the sets in curly brackets is is open, so $D^0_f(\bar{e})$ is a Borel set.

To see that $D^0_c(\bar{e})$ is a Borel set, let h denote the Hausdorff distance between compact subsets of Λ^0. The correspondence E_Λ is continuous at e exactly if for each integer n there is a neighborhood W of e with the property that $h(E_\Lambda(e'), E_\Lambda(e'')) < 1/n$ for $e', e'' \in W$. Hence

$$\begin{aligned} D^0_c(\bar{e}) \quad = \quad & \bigcap_{n=1}^{\infty} \{ e \in D^0(\bar{e}) : \exists \text{ open } W \subset D^0(\bar{e}) \text{ s.t. } e \in W \\ & \qquad \text{and } h(E_\Lambda(e'), E_\Lambda(e'')) < 1/n \text{ for all } e', e'' \in W \} \end{aligned}$$

Thus, $D^0_c(\bar{e})$ is the countable intersection of open sets, and in particular is a Borel set.

Now let $D^0_{nd}(\bar{e}) = D^0(\bar{e}) \setminus D^0_d(\bar{e})$. To show that $D^0_{nd}(\bar{e})$ is finitely shy, set $v = \frac{1}{m}\bar{e}$, and let $V \subset X^m$ be the $(m-1)$-dimensional subspace

$$V = \{(\alpha_1 v, \ldots, \alpha_m v) : \sum \alpha_i = 0\}$$

If $\eta^* = (v, \ldots, v)$ then

$$V \cap [D^0(\bar{e}) - \eta^*] = \{(\alpha_1 v, \ldots, \alpha_m v) : \sum \alpha_i = 0 \text{ and } \alpha_i > -1 \text{ all } i\}$$

so $V \cap [D^0(\bar{e}) - \eta^*]$ certainly has positive measure in V. Now let $\eta \in X^m$ and consider $V \cap [D^0_{nd}(\bar{e}) - \eta]$. If $y \in V \cap [D^0_{nd}(\bar{e}) - \eta]$, then there exists $e \in D^0_{nd}(\bar{e})$ and α such that $\sum \alpha_i = 0$ for which $y_i = \alpha_i v = e_i - \eta_i$ for each i. In particular, $e_i = \eta_i + \alpha_i v > 0$ for each i and $\mathcal{E}(e)$ is not determinate. Thus $\alpha \in A_{nd}(\eta, v)$, which has $(m-1)$-dimensional Lebesgue measure 0 by Theorem 7.1. Thus

$$V \cap [D^0(\bar{e}) - \eta^*] = \{(\alpha_1 v, \ldots, \alpha_m v) : \alpha \in A_{nd}(\eta, v)\}$$

has $(m-1)$-dimensional measure 0. We conclude that $D^0_{nd}(\bar{e})$ is finitely shy, and thus that $D^0_d(\bar{e})$ is finitely prevalent, in $D^0(\bar{e})$ as asserted. □

Since prevalent sets are dense, this result allows us to conclude immediately that the set of endowment distributions that lead to a determinate economy is dense in $D^0(\bar{e})$ as well.

Stronger assumptions on consumers' utility functions lead to a stronger conclusion about local comparative statics. To make this statement precise we need two additional notions. The first is a stronger notion of determinacy.

Definition The economy $\mathcal{E}(e)$ is *Lipschitz determinate* with respect to $\|\cdot\|$ if it is determinate and for every equilibrium $x \in E(e)$ there exist neighborhoods O of x and W of e such that on W every selection from $O \cap E$ is pointwise Lipschitz at e with respect to $\|\cdot\|$.[19]

For this result we will need to specify a single norm on the commodity space. To this end, for each $x \in X$ define

$$\|x\|_{max} = \max_i \|x\|_i.$$

If each individual norm $\|\cdot\|_i$ is *absolute*, that is if $\||x|\| = \|x\|$ for every $x \in X$, then Lipschitz determinacy with respect to $\|\cdot\|_{max}$ holds for a prevalent set

[19]Recall that if X, Y are normed spaces, then $f : X \to Y$ is *pointwise Lipschitz* on X at $\bar{z} \in X$ if there exists $K > 0$ such that $\|f(z) - f(\bar{z})\| \leq K\|z - \bar{z}\|$ for all $z \in X$.

of endowment distributions.[20]

Theorem 7.3 *If $\mathcal{E}$ is a basic economy and for each i there is an absolute norm $\|\cdot\|_i$ such that*

(a) $\|\cdot\|_i$ is adapted to U_i on weakly compact subsets of $P_i^0(\bar{e})$

(b) U_i is quadratically concave with respect to $\|\cdot\|_i$ on weakly compact subsets of $P_i^0(\bar{e})$

then almost all endowment distributions lead to an economy that is Lipschitz determinate with respect to the norm $\|\cdot\|_{max}$ on X. That is,

$$D_{ld}^0(\bar{e}) = \left\{ e \in D^0(\bar{e}) : \mathcal{E}(e) \text{ is Lipschitz determinate with respect to } \|\cdot\|_{max} \right\}$$

is prevalent in $D^0(\bar{e})$.

Proof: Fix a compact subset $\Lambda^* \subset \Lambda^0$. We first show that, for $\lambda \in \Lambda^*$, supporting prices $p(\lambda)$ are uniformly bounded in the $\|\cdot\|_{max}$ norm. To this end, note that $\|\cdot\|_{max}$ is an absolute norm, as

$$\||x|\|_{max} = \max_i \||x|\|_i = \max_i \|x\|_i = \|x\|_{max}$$

Moreover, by definition,

$$\|p(\lambda)\|_{max} = \sup_{\|z\|_{max} \leq 1} |p(\lambda) \cdot z|$$

Because $p(\lambda)$ is positive and $\|\cdot\|_{max}$ is absolute,

$$\|p(\lambda)\|_{max} = \sup_{\|z\|_{max} \leq 1} |p(\lambda) \cdot z| = \sup_{\substack{\|z\|_{max} \leq 1 \\ z \in X_+}} p(\lambda) \cdot z$$

[20] Requiring that a norm be absolute is not innocuous. For instance, there are many norms on $M[0,1]$ (the space of signed measures on the unit interval) for which the topology induced by the norm coincides with the weak star topology (viewing $M[0,1]$ as the dual of the space $C[0,1]$ of continuous functions) on order intervals, but there is no absolute norm with this property. See **?** for further discussion.

By definition, $p(\lambda) = \bigvee \lambda_i DU_i(x_i(\lambda))$ and each $\lambda_i DU_i(x_i(\lambda))$ is a positive linear functional, so for $z \in X_+$ we have

$$0 \leq p(\lambda) \cdot z \leq \left[\sum \lambda_i DU_i(x_i(\lambda))\right] \cdot z = \sum \left[\lambda_i DU_i(x_i(\lambda)) \cdot z\right]$$

Using the adaptedness of $\|\cdot\|_i$ on $x(\Lambda^*)$ and the definition of $\|\cdot\|_{max}$ we obtain

$$\begin{aligned} \|p(\lambda)\|_{max} &= \sup_{\substack{\|z\|_{max} \leq 1 \\ z \in X_+}} p(\lambda) \cdot z \\ &\leq \sup_{\substack{\|z\|_{max} \leq 1 \\ z \in X_+}} \sum \left[\lambda_i DU_i(x_i(\lambda)) \cdot z\right] \\ &\leq \sum \lambda_i B_i \|z\|_i \leq \sum \lambda_i B_i \leq \sum B_i \end{aligned}$$

for some constants $B_i > 0$.

We now show that the excess spending map S is jointly locally Lipschitz on $\Lambda^0 \times D^0(\bar{e})$. Fix a consumer i. For $\lambda, \lambda' \in \Lambda^0$ and $e, e' \in D^0(\bar{e})$

$$\begin{aligned} |S_i(\lambda, e) - S_i(\lambda', e')| &= |p(\lambda) \cdot [x_i(\lambda) - e_i] - p(\lambda') \cdot [x_i(\lambda') - e_i']| \\ &\leq |p(\lambda) \cdot x_i(\lambda) - p(\lambda') \cdot x_i(\lambda')| + |p(\lambda') \cdot e_i' - p(\lambda) \cdot e_i| \\ &\leq |p(\lambda) \cdot x_i(\lambda) - p(\lambda') \cdot x_i(\lambda')| \\ &\quad + |p(\lambda') \cdot e_i' - p(\lambda') \cdot e_i| + |p(\lambda') \cdot e_i - p(\lambda) \cdot e_i| \\ &\leq |p(\lambda) \cdot x_i(\lambda) - p(\lambda') \cdot x_i(\lambda')| \\ &\quad + \|p(\lambda')\|_{max} \|e_i' - e_i\|_{max} + |[p(\lambda') - p(\lambda)] \cdot e_i| \end{aligned}$$

Consider the last three terms. Lemma 6.1 guarantees that there is a constant C_1 such that

$$|p(\lambda) \cdot x_i(\lambda) - p(\lambda') \cdot x_i(\lambda')| \leq C_1 |\lambda - \lambda'|$$

The bound obtained in the previous paragraph guarantees that

$$\|p(\lambda')\|_{max} \|e_i' - e_i\|_{max} \leq \sum B_i \|e_i' - e_i\|_{max}$$

Lemma 6.1 guarantees there is a constant C_2 such that

$$|[p(\lambda') - p(\lambda)] \cdot e_i| \leq C_2 |\lambda - \lambda'|$$

Putting these together, we conclude that S is Lipschitz on $\Lambda^* \times D^0(\bar{e})$, and in particular, is locally Lipschitz on $\Lambda^0 \times D^0(\bar{e})$, as asserted. The result now follows from the transversality results in Theorem 2.2 and Theorem 3.7 in Shannon (1998b). □

8 Examples

In this section we develop examples illustrating our results in the setting of continuous time trading in financial markets and of trade over an infinite horizon. For examples in the setting of commodity differentiation, see **?**.

Example 8.1 Continuous-Time Trading in Financial Markets. The standard model[21] of continuous time trading begins with a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t : 0 \leq t \leq T\}$ of sub-sigma-algebras of $\mathcal{F}$ such that $\mathcal{F}_0 = \{\emptyset, \Omega\}, \mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_t \subset \mathcal{F}_{t'}$ if $t < t'$. The filtration $\{\mathcal{F}_t\}$ represents revelation of information over the time interval $[0, T]$; $\mathcal{F}_t$ is the sigma-algebra of events observable at time t. Commodity bundles are square integrable predictable stochastic processes, thus the commodity space is $L^2(\Omega \times [0, T], \mathcal{P}, \nu)$, where $\mathcal{P}$ is the predictable sigma algebra and $\nu = P \times \mu$ is the product of P with (normalized) Lebesgue measure on $[0, T]$.

Each consumer i is characterized by an initial endowment e_i and a utility function, usually assumed to have an expected utility representation

$$U^i(x) = E\left[\int_0^T u^i(x_t, t)dt\right] = \int_\Omega \left[\int_0^T u^i(x_t(\omega), t)dt\right] dP(\omega)$$

where $u^i(\cdot, t) : \mathbf{R}_+ \to \mathbf{R}$ is strictly increasing and strictly concave for each t. It is usually assumed that utility functions satisfy Inada conditions, so that the partial derivatives $u^i_c(x, t) \to \infty$ as $x \to 0$, uniformly in t, and that the social endowment $\bar{e}$ is bounded above and uniformly bounded away from 0.

Under these assumptions for each consumer, it is easily verified that every Pareto optimal allocation in $P^0(\bar{e})$ is uniformly bounded away from 0. A straightforward computation then shows that for each i the L^2 norm is adapted to U_i on weakly compact subsets of $P^0_i(\bar{e})$.

If in addition $u^i(\cdot, t)$ is C^2 and differentiably strictly concave, uniformly in t, then U^i is quadratically concave on weakly compact subsets of $P^0_i(\bar{e})$ with respect to the L^2 norm. To see this, note that for each $b^* \in \mathbf{R}_+$ there is a constant $K_i > 0$ such that

$$u^i(b, t) \leq u^i(a, t) - u^i_c(a, t)(b - a) - K_i\|b - a\|^2$$

[21]See Duffie and Zame (1989) or Breeden (1979) for instance.

for each $t \in [0,T]$ and $a, b \in [0, b^*]$.[22] Then for $x, y \in [0, \bar{e}]$

$$\begin{aligned} U^i(y) - U^i(x) &= E\left[\int_0^T u^i(y_t,t)dt\right] - E\left[\int_0^T u^i(x_t,t)dt\right] \\ &= E\left[\int_0^T [u^i(y_t,t) - u^i(x_t,t)]dt\right] \\ &\leq E\left[\int_0^T [u^i_c(x_t,t)(y_t - x_t) - K_i|y_t - x_t|^2]dt\right] \\ &= E\left[\int_0^T u^i_c(x_t,t)(y_t - x_t)dt\right] - K_i E\left[\int_0^T |y_t - x_t|^2 dt\right] \\ &= DU^i(x)\cdot(y-x) - K_i\|y-x\|^2 \end{aligned}$$

which is the required inequality.

Because the L^2 norm is absolute, Theorem 7.3 guarantees that almost all endowment distributions lead to economies which are Lipschitz determinate with respect to the L^2 norm.

In the framework above, a commodity bundle x represents a rate of consumption. Hindy, Huang, and Kreps (1992) (see also Hindy and Huang (1992, 1993)) argue that intertemporal consumption patterns should admit the possibility of consumption in discrete lumps ("gulps"), as well as in rates ("sips"). For consumption over the time interval $[0,1]$, they suggest that commodities should be represented by positive, increasing, right continuous functions $\varphi : [0,1] \to \mathbf{R}_+$, where $\varphi(t)$ gives total consumption at or before time t. In this formulation, consumption occurs in "gulps" at points where φ has an upward jump and in "sips" at points of continuity of φ. For our purposes it is convenient to adopt an equivalent formulation in which commodity bundles are non-negative measures x on $[0,1]$, so that $x[0,t]$ represents total consumption on the interval $[0,t]$. In our formulation, consumption occurs in "gulps" at atoms of x and in "sips" elsewhere. Equivalently, note that the functions that represent commodity bundles in the Hindy, Huang and Kreps formulation are just the cumulative distribution functions of the measures that represent commodity bundles in our formulation. This alternative formulation leads to the commodity space $M[0,1]$, the space of signed measures

[22] Smoothness and quadratic concavity of $u^i(\cdot,t)$ would suffice.

on $[0,1]$. As the following example shows, models such as those developed by Hindy, Huang and Kreps also satisfy our requirements.

Example 8.2 Lumpy Consumption. The commodity space is $M[0,1]$, endowed with the weak star topology when viewed as the dual of the space of continuous functions $C[0,1]$. To capture the idea that consumptions at nearby dates should be nearly perfect substitutes at the margin, Hindy, Huang, and Kreps (1992) assume that preferences are continuous in the weak star topology and uniformly proper with respect to one of a particular family of norms of the form

$$\|x\|_p = \left[\int_0^1 |x[0,t]|^p dt + |x[0,1]|^p\right]^{1/p}$$

for $p \geq 1$. A typical utility function satisfying their assumptions is:

$$U(x) = \int_0^1 u(x[0,t],t)dt + v(x[0,1])$$

where $u(\cdot,t) : \mathbf{R}_+ \to \mathbf{R}$ is C^2, strictly increasing and strictly concave for each t and $v : \mathbf{R}_+ \to \mathbf{R}$ is C^2, strictly increasing, and strictly concave. Suppose in addition that $v''(c) < 0$ for each c and that $u_{cc}(c,t) < 0$ for each c, t.[23] We assert that the norm $\|\cdot\|_1$ is adapted to U on every order interval $[0,\bar{e}]$, and that U is quadratically concave on every order interval $[0,\bar{e}]$ with respect to this norm.[24]

Hindy, Huang, and Kreps (1992) show that the topology induced by $\|\cdot\|_1$ (or indeed any of their norms) coincides with the weak star topology on order intervals. To verify that the norm $\|\cdot\|_1$ is adapted to U, therefore, we must only verify the relevant properties of derivatives. To this end, note that our assumptions provide a constant C such that $v'(c) \leq C$ and $u_c(\cdot,t) \leq C$ for every $c \leq \bar{e}[0,1]$. Thus

$$|DU(x)\cdot y| \quad = \quad \left|\int_0^1 u_c(x[0,t],t)y[0,t]dt + v'(x[0,1])y[0,1]\right|$$

[23] Again, smoothness and quadratic concavity of u and v would suffice.

[24] Note that $\|\cdot\|_1$ is not an absolute norm: if $x \in M[0,1]$ then $\|x\|_1 \geq \||x|\|_1$, but equality holds exactly when $x \geq 0$ or $x \leq 0$. Indeed, as we have noted in footnote 20, no absolute norm on $M[0,1]$ has the property that the norm topology coincides with the weak star topology on order intervals. The total variation norm *is* an absolute norm on $M[0,1]$ — but these utility functions are not quadratically concave with respect to the total variation norm.

$$
\begin{aligned}
&= \leq \int_0^1 C|y[0,t]|dt + C|y[0,1]| \\
&= C\|y\|_1
\end{aligned}
$$

for $x \in [0, \bar{e}]$ and $y \in M[0, 1]$. Similarly, there exists $C' > 0$ such that

$$
\begin{aligned}
|DU(y) \cdot z - DU(y') \cdot z| &\leq \left| \int_0^1 [u_c(y[0,t],t) - u_c(y'[0,t],t)]\, z[0,t]dt \right| \\
&\quad + |[v'(y[0,1]) - v'(y'[0,1])]z[0,1]| \\
&\leq C'\|y - y'\|_1
\end{aligned}
$$

for $y, y', z \in [0, \bar{e}]$, so $\|\cdot\|_1$ is adapted to U on $[0, \bar{e}]$.

To see that U is quadratically concave, fix $x, y \in [0, \bar{e}]$. Differential strict concavity of v and u provides a constant $C'' > 0$ such that:

$$
\begin{aligned}
u(c', t) - u(c, t) &\leq u_c(c, t)(c' - c) - C''|c' - c|^2 \\
v(c') - v(c) &\leq v'(c)(c' - c) - C''|c' - c|^2
\end{aligned}
$$

for each $c, c' \leq \bar{e}[0, 1]$ and for each t. Hence

$$
\begin{aligned}
U(y) - U(x) &= \int_0^1 [u(y[0,t],t) - u(x[0,t],t)]\, dt \; + \; v(y[0,1]) - v(x[0,1]) \\
&\leq \int_0^1 \left[u_c(x[0,t],t)(y[0,t] - x[0,t]) \; - \; C''|y[0,t] - x[0,t]|^2 \right] dt \\
&\quad + v'(x[0,1])(y[0,1] - x[0,1]) \; - \; C''|y[0,1] - x[0,1]|^2 \\
&= DU(x) \cdot (y - x) \\
&\quad - C'' \left[\int_0^1 |y[0,t] - x[0,t]|^2 dt + |y[0,1] - x[0,1]|^2 \right] \\
&\leq DU(x) \cdot (y - x) - C''\|y - x\|_1^2
\end{aligned}
$$

where the last inequality is a consequence of Jensen's inequality.

Infinite horizon economies are perhaps the most familiar examples of models with infinitely many commodities. In the following example, we consider such a model with non-separable habit formation preferences.

Example 8.3 Infinite-Horizon Economies. Consider a discrete time infinite horizon economy in which the commodity space is ℓ_∞ endowed with the Mackey topology. Consumer i is characterized by an endowment e_i and a utility function that displays habit formation of the form:

$$U_i(x) = v_i(x_0) + \sum_{t=1}^{\infty} \beta_i^t u_i(x_{t-1}, x_t)$$

where $\beta_i \in (0,1)$ is a discount factor. We assume that $v_i : \mathbf{R}_+ \to \mathbf{R}$ and $u_i : \mathbf{R}_+^2 \to \mathbf{R}$ are C^2, strictly increasing and differentiably strictly concave.

As in Example 3.1, such utility functions are not quadratically concave on any bounded set with respect to the ℓ_∞ norm. However, it is quadratically concave on bounded sets with respect to the weighted norm:

$$\|x\|_{\beta_i} = \sum_{t=1}^{\infty} \beta_i^t |x_t|$$

To see this, let $\bar{e} \in \ell_{\infty+}$ be given and let $x, y \in [0, \bar{e}]$. To simplify notation, for each $z \in \ell_\infty$ and for each t let $z(t) = (z_{t-1}, z_t)$. Then

$$\begin{aligned}
U_i(y) - U_i(x) &= v_i(y_0) - v_i(x_0) + \sum_{t=1}^{\infty} \beta_i^t \left[u_i(y(t)) - u_i(x(t))\right] \\
&\leq v_i'(x_0)(y_0 - x_0) - c|y_0 - x_0|^2 \\
&\quad + \sum_{t=1}^{\infty} \beta_i^t Du_i(x(t)) \cdot [y(t) - x(t)] - c\sum_{t=1}^{\infty} \beta_i^t \|y(t) - x(t)\|^2 \\
&= DU_i(x) \cdot (y - x) - c\left[|y_0 - x_0|^2 + \sum_{t=1}^{\infty} \beta_i^t \|y(t) - x(t)\|^2\right] \\
&\leq DU_i(x) \cdot (y - x) - c(1 + \beta_i) \sum_{t=0}^{\infty} \beta_i^t |y_t - x_t|^2 \\
&\leq DU_i(x) \cdot (y - x) - c(1 + \beta_i) b \left(\sum_{t=0}^{\infty} \beta_i^t |y_t - x_t|\right)^2 \\
&= DU_i(x) \cdot (y - x) - c(1 + \beta_i) b \|y - x\|_{\beta_i}^2
\end{aligned}$$

for some $c, b > 0$. Here the first inequality follows from the quadratic concavity of u, and the last inequality uses the fact that in a finite measure space, the L^2 norm dominates a multiple of the L^1 norm; i.e., there is a constant $B > 0$ such that $\|f\|_2 \geq B\|f\|_1$ for all f.

Similarly, it is straightforward to verify that this weighted norm is adapted to the utility function U_i on bounded sets. Note, however, that if different individuals discount future consumption at different rates then we must use different norms for each consumer. Because each of these weighted norms is absolute, Theorem 7.3 guarantees that almost all endowment distributions lead to Lipschitz determinate economies.

Appendix

Proof of Lemma 4.2: (i) $\Rightarrow$ (ii): Let x be a Pareto optimal allocation and let q be a supporting price. We first establish the desired representation of q.

For each i, set

$$\beta_i = \frac{q \cdot x_i}{DU_i(x_i) \cdot x_i}$$

The fact that utility functions are strictly monotone guarantees that the denominator is strictly positive. The fact that q is a supporting price guarantees that the numerator, and hence β_i, is strictly positive. Our goal is to show that

$$q \cdot y = \left[\bigvee_i \beta_i DU_i(x_i) \right] \cdot y$$

for every $y \in X$, from which we will easily obtain the desired representation. We proceed by verifying this equality first when $0 \leq y \leq x_i$ for some i, then when $0 \leq y \leq \bar{e}$, then when y is in the order ideal generated by $\bar{e}$, and finally for arbitrary $y \in X$.

Fix a consumer i. Supporting prices equate marginal rates of substitution so if $y \in X_+$ then

$$\frac{DU(x_i) \cdot x_i}{q \cdot x_i} \geq \frac{DU(x_i) \cdot y}{q \cdot y}$$

with equality if $y \leq x_i$. Rearranging yields

$$q \cdot y \geq \left(\frac{q \cdot x_i}{DU_i(x_i) \cdot x_i} \right) DU_i(y) \cdot y$$

for every $y \in X_+$, with equality if $y \leq x_i$. Using the definition of β_i and substituting, we have

$$q \cdot y \geq \beta_i DU_i(x_i) \cdot y \quad \text{for all } y \in X_+, \text{with equality if } y \leq x_i \tag{7}$$

If $y \in X_+$ and $y \leq x_j$ for $i \neq j$, then two applications of (7) imply that

$$\beta_j DU_j(x_j) \cdot y = q \cdot y \geq \beta_i DU_i(x_i) \cdot y$$

In particular

$$\beta_j DU_j(x_j) \cdot y \geq \beta_i DU_i(x_i) \cdot y \quad \text{if } y \in X_+, y \leq x_j \tag{8}$$

For $y \in X_+$,

$$\left[\bigvee_i \beta_i DU_i(x_i)\right] \cdot y = \sup\left\{\sum \beta_i DU_i(x_i) \cdot a_i : a_i \geq 0, \sum a_i = y\right\} \tag{9}$$

by the definition of the supremum of linear functionals. Thus

$$\left[\bigvee_i \beta_i DU_i(x_i)\right] \cdot y = \beta_j DU_j(x_j) \cdot y = q \cdot y \quad \text{if } y \in X_+, y \leq x_j \tag{10}$$

Next consider any $y \in X_+$ for which $0 \leq y \leq \bar{e}$. The Riesz Decomposition Property of vector lattices guarantees that we can find vectors $y_j \in X_+$ such that $y = \sum y_j$ and $0 \leq y_j \leq x_j$ for each j. Repeated applications of (10) yield

$$\begin{aligned}
\left[\bigvee_i \beta_i DU_i(x_i)\right] \cdot y &= \left[\bigvee_i \beta_i DU_i(x_i)\right] \cdot \left[\sum_j y_j\right] \\
&= \sum_j \left\{\left[\bigvee_i \beta_i DU_i(x_i)\right] \cdot y_j\right\} \\
&= \sum_j \beta_j DU_j(x_j) \cdot y_j \\
&= \sum_j q \cdot y_j \\
&= q \cdot \sum_j y_j \\
&= q \cdot y
\end{aligned}$$

Now consider any y in the order ideal generated by $\bar{e}$; that is, $y \in X$ such that $|y| \leq k\bar{e}$ for some $k > 0$. Write $z = (1/k)y$ and decompose $z = z^+ - z^-$ as the sum of positive and negative parts. Then $0 \leq z^+ \leq \bar{e}$ and $0 \leq z^- \leq \bar{e}$, so the previous paragraph implies that

$$\begin{aligned}\left[\bigvee_i \beta_i DU_i(x_i)\right] \cdot z^+ &= q \cdot z^+ \\ \left[\bigvee_i \beta_i DU_i(x_i)\right] \cdot z^- &= q \cdot z^-\end{aligned}$$

It follows from linearity that

$$\left[\bigvee_i \beta_i DU_i(x_i)\right] \cdot y = q \cdot y$$

Strict positivity of the social endowment $\bar{e}$ means that the order ideal generated by $\bar{e}$ is dense in X, so continuity entails that

$$\left[\bigvee_i \beta_i DU_i(x_i)\right] \cdot y = q \cdot y$$

for every $y \in X$, which was our goal.

Write $\beta = \sum \beta_i$ and $\lambda_i = \beta_i/\beta$ for each i, and note that $\lambda_i > 0$ because $\beta_i > 0$. Then

$$q = \beta \bigvee_i \lambda_i DU_i(x_i) \tag{11}$$

which is the desired representation of q.

It remains only to show that x solves the planner's problem for these weights λ. To see this, suppose that x' is an allocation. Then $x'_i \geq 0$ for each i and $\sum x'_i = \bar{e}$, so

$$q \cdot \bar{e} \geq \beta \sum \lambda_i DU_i(x_i) \cdot x'_i$$

Moreover, since utilities are concave,

$$\begin{aligned}\sum \lambda_i U_i(x'_i) - \sum \lambda_i U_i(x_i) &= \sum \lambda_i \left[U_i(x'_i) - U_i(x_i)\right] \\ &\leq \sum \lambda_i \left[DU_i(x_i) \cdot (x'_i - x_i)\right]\end{aligned}$$

Thus

$$\begin{aligned}\sum \lambda_i U_i(x'_i) - \sum \lambda_i U_i(x_i) &\leq \sum \lambda_i \left[DU_i(x_i) \cdot (x'_i - x_i)\right] \\ &= \sum \lambda_i DU_i(x_i) \cdot x'_i - \sum \lambda_i DU_i(x_i) \cdot x_i\end{aligned}$$

$$\begin{aligned}
&= \sum \lambda_i DU_i(x_i) \cdot x_i' - \frac{1}{\beta} \sum q \cdot x_i \\
&\le \frac{1}{\beta} q \cdot \bar{e} - \frac{1}{\beta} \sum q \cdot x_i \\
&= \frac{1}{\beta} q \cdot \bar{e} - \frac{1}{\beta} q \cdot \sum x_i \\
&= \frac{1}{\beta} q \cdot \bar{e} - \frac{1}{\beta} q \cdot \bar{e} \\
&= 0
\end{aligned}$$

Thus x solves the planner's problem for the weights λ.

(ii) $\Rightarrow$ (i): Solutions to the planner's problem are Pareto optima, so we need only show that $\bigvee_i \lambda_i DU_i(x_i)$ is a supporting price. Note first that for every i, j the first order condition for Pareto optimality implies

$$\lambda_i DU_i(x_i) \cdot (-y) + \lambda_j DU_j(x_j) \cdot y \le 0$$

if $y \in X_+, y \le x_i$. Rewriting yields that

$$\lambda_i DU_i(x_i) \cdot y \ge \lambda_j DU_j(x_j) \cdot y$$

if $y \in X_+, y \le x_i$. It follows as above that

$$q \cdot z \ge \lambda_i DU_i(x_i) \cdot z$$

for $z \in X_+$ with equality if $z \le x_i$.

Now fix i. To see that q supports U_i at x_i, let $z \ge 0$. Then

$$\begin{aligned}
U_i(z) - U_i(x_i) &\le DU_i(x_i) \cdot (z - x_i) \\
&= DU_i(x_i) \cdot z - DU_i(x_i) \cdot x_i \\
&= DU_i(x_i) \cdot z - \frac{1}{\lambda_i} q \cdot x_i \\
&\le \frac{1}{\lambda_i} q \cdot z - \frac{1}{\lambda_i} q \cdot x_i \\
&= \frac{1}{\lambda_i} q \cdot (z - x_i)
\end{aligned}$$

Thus if $z \ge 0$ and $U_i(z) \ge U_i(x_i)$, then $q \cdot z \ge q \cdot x_i$. It follows that q is a supporting price, so the proof is complete. □

Proof of Lemma 6.1: To establish (i), fix $\delta > 0$. Recall that $\Lambda^\delta = \{\lambda \in \Lambda : \lambda_i \ge \delta \text{ for each } i\}$ and $P^\delta(\bar{e}) = x(\Lambda^\delta)$. Because Λ^δ is a compact set and x is weakly continuous, $P_i^\delta(\bar{e})$ is a weakly compact subset of $P_i^0(\bar{e})$ for each i. Because each of the norms $\|\cdot\|_i$ is adapted to U_i on weakly compact subsets of $P_i^0(\bar{e})$, there are constants $B_i, C_i > 0$ such that

$$|DU_i(x_i) \cdot y| \leq B_i \|y\|_i \tag{12}$$

for all $x_i \in P_i^\delta(\bar{e})$ and $y \in X$, and

$$|DU_i(x_i) \cdot z - DU_i(x_i') \cdot z| \leq C_i \|x_i - x_i'\|_i \tag{13}$$

for all $x_i, x_i' \in P_i^\delta(\bar{e})$ and $z \in [0, \bar{e}]$. By Lemma 5.1, the solution to the planner's problem is Lipschitz on Λ^δ, so there is a constant $K_i > 0$ such that

$$\|x_i(\lambda) - x_i(\lambda')\|_i \leq K_i \sum |\lambda_i - \lambda_i'| \tag{14}$$

for $\lambda, \lambda' \in \Lambda^\delta$.

Now fix a consumer j and weights $\lambda, \lambda' \in \Lambda^\delta$. To simplify notation, write $x = x(\lambda), x' = x(\lambda')$, $p = p(\lambda), p' = p(\lambda')$, $p_i = \lambda_i DU_i(x_i)$ and $p_i' = \lambda_i' DU_i(x_i')$ for each i. If $0 \leq z \leq x_j(\lambda)$ then, as in the proof of Lemma 4.2, the first order conditions imply that

$$\lambda_j DU_j(x_j) \cdot z \geq \lambda_k DU_k(x_k) \cdot z$$

for each k. Hence $p(\lambda) \cdot x_j = \lambda_j DU_j(x_j) \cdot x_j = p_j \cdot x_j$. Similarly, $p(\lambda') \cdot x_j' = \lambda_j' DU_j(x_j') \cdot x_j' = p_j' \cdot x_j'$. Thus

$$\begin{aligned} |p \cdot x_j - p' \cdot x_j'| &= |p_j \cdot x_j - p_j \cdot x_j' + p_j \cdot x_j' - p_j' \cdot x_j'| \\ &\leq |p_j \cdot x_j - p_j \cdot x_j'| + |p_j \cdot x_j' - p_j' \cdot x_j'| \\ &= |p_j \cdot (x_j - x_j')| + |(p_j - p_j') \cdot x_j'| \end{aligned} \tag{15}$$

Because the norm $\|\cdot\|_j$ is adapted to U_j on $P_j^\delta(\bar{e})$ and the planner's problem is Lipschitz on Λ^δ, we conclude that

$$|p_j \cdot (x_j - x_j')| \leq B_j \|x_j - x_j'\|_j \leq B_j K_j \sum |\lambda_i - \lambda_i'| \tag{16}$$

$$|(p_j - p_j') \cdot x_j'| \leq C_j \|x_j - x_j'\|_j \leq C_j K_j \sum |\lambda_i - \lambda_i'| \tag{17}$$

Combining (15), (16), (17) yields (i).

To establish (ii), again fix $\delta > 0$ and let $\lambda, \lambda' \in \Lambda^\delta$. Again write $x = x(\lambda), x' = x(\lambda')$, $p = p(\lambda), p' = p(\lambda')$, $p_i = \lambda_i DU_i(x_i)$ and $p_i' = \lambda_i' DU_i(x_i')$ for each i. Fix an arbitrary $w \in [0, \bar{e}]$. By definition,

$$p \cdot w = \left(\bigvee p_i\right) \cdot w = \sup \left\{ \sum p_i \cdot a_i : a_i \geq 0, \sum a_i = w \right\}.$$

Fix $\varepsilon > 0$ and choose (a_i) so that $\sum a_i = w$ and

$$p \cdot w \leq \varepsilon + \sum p_i \cdot a_i.$$

As in the proof of (i), we use quadratic concavity and adaptedness of the norm $\|\cdot\|_i$ to U_i on $P_i^\delta(\bar{e})$ to choose constants B_i, C_i, K_i so that (12)-(14) obtain. Thus

$$\begin{aligned} p \cdot w - p' \cdot w &\leq \varepsilon + \sum p_i \cdot a_i - \sum p'_i \cdot a_i \\ &\leq \varepsilon + \sum (p_i - p'_i) \cdot a_i \\ &\leq \varepsilon + \sum C_i \|x_i - x'_i\|_i \\ &\leq \varepsilon + \sum_i C_i K_i \left(\sum_k |\lambda_k - \lambda'_k| \right) \\ &= \varepsilon + \left(\sum_i C_i K_i \right) \left(\sum_k |\lambda_k - \lambda'_k| \right). \end{aligned}$$

Reversing the roles of p, p' and keeping in mind that $\varepsilon > 0$ was arbitrary, we obtain

$$|p \cdot w - p' \cdot w| \leq \left(\sum_i C_i K_i \right) \left(\sum_k |\lambda_k - \lambda'_k| \right)$$

Because $w \in [0, \bar{e}]$ was arbitrary, this is the desired uniform Lipschitz estimate (ii).

(iii) is immediate from (i) and (ii). To establish (iv), fix a consumer i and fix an arbitrary $(\lambda, e) \in \Lambda^0 \times D^0(\bar{e})$ and let $(\lambda', e') \in \Lambda^0 \times D^0(\bar{e})$ be another point. To see that S_i is continuous at (λ, e), note that

$$\begin{aligned} |S_i(\lambda, e) - S_i(\lambda', e')| &= |p(\lambda) \cdot x_i(\lambda) - p(\lambda') \cdot x_i(\lambda') - p(\lambda) \cdot e_i + p(\lambda') \cdot e'_i| \\ &\leq |p(\lambda) \cdot x_i(\lambda) - p(\lambda') \cdot x_i(\lambda')| \\ &\quad + |p(\lambda) \cdot e_i - p(\lambda) \cdot e'_i + p(\lambda) \cdot e'_i - p(\lambda') \cdot e'_i| \\ &\leq |p(\lambda) \cdot x_i(\lambda) - p(\lambda') \cdot x_i(\lambda')| \\ &\quad + |p(\lambda) \cdot e_i - p(\lambda) \cdot e'_i| + |p(\lambda) \cdot e'_i - p(\lambda') \cdot e'_i| \end{aligned}$$

Let $\varepsilon > 0$ be given. By (i) and (ii), there is a neighborhood V of λ such that if $\lambda' \in V$ then

$$|p(\lambda) \cdot x_i(\lambda) - p(\lambda') \cdot x_i(\lambda')| < \varepsilon/3 \quad \text{and} \quad |p(\lambda) \cdot e'_i - p(\lambda') \cdot e'_i| < \varepsilon/3$$

Continuity of the linear functional $p(\lambda)$ guarantees that there exists a neighborhood W of e such that if $e' \in W$ then

$$|p(\lambda) \cdot e_i - p(\lambda) \cdot e'_i| < \varepsilon/3$$

Thus if $(\lambda', e') \in V \times W$, then $|S_i(\lambda, e) - S_i(\lambda', e')| < \varepsilon$. It follows that S_i is continuous at (λ, e). Since (λ, e) is arbitrary, we conclude that S_i is continuous on $\Lambda^0 \times D^0(\bar{e})$, which is (iv). □

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