THE ALGEBRAIC GEOMETRY OF GAMES AND THE TRACING PROCEDURE

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1. INTRODUCTION

This paper has two purposes. The immediate purpose is to point out some difficulties with the tracing procedure of Harsanyi and Selten, and show how they can be dealt with. The other purpose is to describe the theory of semi-algebraic sets and a few of its applications in game theory.

The *tracing procedure* is the heart of the extensive theory of equilibrium selection in games which has been developed by Harsanyi and Selten (Harsanyi [1975], Harsanyi-Selten [1988]). For each (normal or extensive form) game, the theory of Harsanyi and Selten prescribes (on the basis of Bayesian and risk analysis) a prior probability distribution over strategies for this game. Given this prior probability distribution, the *logarithmic tracing procedure* identifies a unique Nash equilibrium (the

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logarithmic solution) from the set of all Nash equilibria. In part because the logarithmic solution is difficult to compute in practice, Harsanyi and Selten also use another procedure, which they call the *linear tracing procedure* . In contrast to the logarithmic tracing procedure, the linear tracing procedure is relatively easy to compute, but does not always lead to a unique solution. However, the linear tracing procedure leads to a unique solution for "most" games, and when it does lead to a unique solution, it leads to the logarithmic solution. These two properties of the linear tracing procedure make it useful in applications.

Unfortunately, there are some difficulties with the descriptions and constructions that Harsanyi and Selten give for the tracing procedures. In particular, the arguments for the crucial properties of the tracing procedure (that the logarithmic tracing procedure always leads to a unique solution, and that the linear solution - when it is unique - coincides with the logarithmic solution) are not rigorous. (We discuss the tracing procedure and the difficulties in Section 4.)

In this paper we clarify the definitions of the logarithmic and linear tracing procedures and give rigorous proofs for the crucial properties. Our methods also show that, viewed as a function of the the initial data (i.e., the payoffs of the game and the prior probability distribution), the logarithmic solution is continuous on a dense open set of full measure. (It could not possibly be continuous everywhere.)

We believe that our methods are of interest in themselves, and will have wide applicability in game theory. Primarily, we make use of *real algebraic geometry* , which is the study of algebraic and semi-algebraic sets. (An *algebraic set* is defined by polynomial equalities; a *semi-algebraic* set is defined by (conjunctions and disjunctions of) polynomial inequalities.) It has been observed by Kohlberg and Mertens [1986] (and perhaps by many others) that certain of the constructions of game theory lead to semi-algebraic sets, and that semi-algebraic sets have a very special structure which is relevant for game theory. In particular, the set of Nash equilibria of any game is a semi-algebraic set, and hence has a finite number of connected components. This fact plays a significant role in the theory of stable equilibrium.

In this paper we show that virtually *all* of the constructions of game theory give rise to semi-algebraic sets. This is a consequence of a deep and remarkable result from mathematical logic, the Tarski-Seidenberg theorem (Tarski [1931], Seidenberg [1954]). The Tarski-Seidenberg theorem asserts that any first order formula in the language of the real numbers (i.e., a formula which does not involve quantification over sets) is equivalent to a first order formula which involves no quantifiers at all. (Indeed, the Tarski-Seidenberg theorem actually gives an explicit procedure for this "elimination of quantifiers.") A first order formula which involves no quantifiers is simply a conjunction and disjunction of polynomial inequalities, and hence defines a semi-algebraic set. The import of the Tarski-Seidenberg theorem for game theory is that virtually all of the usual game-theoretic constructions are (or are equivalent to) first order constructions, and hence give rise to semi-algebraic sets. (The Kohlberg-Mertens [1986] notion of *stability* may be an exception here. Since stability is a set-valued notion, it does not seem clear whether it has a first order formulation.)

As a consequence of the Tarski-Seidenberg theorem, we show (Theorem 1) that virtually all of the usual game-theoretic equilibrium correspondences (Nash, subgame perfect, sequential, perfect, etc.) have semi-algebraic graphs. It follows (Corollary 1.1) that each of these correspondences is continuous at every point of a dense open set of full measure (previously, it had not been known that the perfect equilibrium correspondence had any points of continuity at all) and admits a selection which is continuous at every point of the same set. In fact, the logarithmic solution provides such a selection (previously, the generic continuity properties of the logarithmic solution were unknown). It also follows (Corollary 1.2) that for every game, the set of Nash (respectively, subgame perfect, sequential, perfect) equilibria has a finite number of connected components, and that there is a bound for this number which depends only on the game form. Finally (Corollary 1.3), for every game, the set of Nash (respectively, subgame perfect, sequential, perfect) equilibria is the finite union of connected real-analytic manifolds (of various dimensions).

We stress that the above conclusions are immediate applications of the definitions and known facts about semi-algebraic sets. For other, less immediate, applications, see Simon [1987] and Blume and Zame [1989]. Blume and Zame use the theory of semi-algrebraic sets to show that, for generic games, all sequential equilibria are (trembling hand) perfect. Simon uses a generalization (due to van den Dries [1986]) of the theory of semi-algebraic sets to establish the existence of mixed-strategy equilibria for continuous time games. The only other application (of which we are aware) of the theory of semi-algebraic sets in game theory is the previously mentioned observation of Kohlberg-Mertens [1986] that the set of Nash equilibria of a game is a semi-algebraic set, and hence has only a finite number of connected components. A different aspect of the Tarski-Seidenberg theorem (that first order formulas true in one real closed field are true in all real closed fields) was used by Bewley and Kohlberg [1976] in their work on stochastic games.

The remainder of the paper is organized in the following way. Section 2 describes the basics of the theory of semi-algebraic sets and the Tarski-Seidenberg theorem, and gives some simple examples. Section 3 details the general applications to game theory, and in particular gives the results about equilibrium correspondences described above. Section 4 briefly reviews the tracing procedure, isolates what appear to be the crucial difficulties, and gives rigorous arguments to circumvent them. Finally, Section 5 briefly explicates the relationship of the tracing procedure to the entire Harsanyi-Selten equilibrium selection procedure.

2. SEMI-ALGEBRAIC SETS

In this section, we describe the basics of the theory of algebraic and semi-algebraic sets and the Tarski-Seidenberg theorem. Excellent general references are Bochnak-Coste-Roy [1988] and Delfs-Knebusch [1981a, b]. For a very brief, but readable, synopsis, we recommend van den Dries [1986].

By definition, an *algebraic set* in $\mathbb{R}^N$ is defined by (a finite number of) polynomial *equalities*. More precisely, an algebraic set is a set of the form:

$$A = \{ x = (x_1,\ldots,x_N) \in \mathbb{R}^N : p_1(x) = \ldots = p_n(x) = 0 \},$$

where $p_1, \ldots p_n$ are polynomials (with real coefficients). Note that the vanishing of an arbitrary collection of polynomials is equivalent to the vanishing of a finite number of polynomials because the ring of real polynomials in N variables is *Noetherian*. Note also that the vanishing of at least one of the polynomials $q_1, \ldots q_n$ is equivalent to the vanishing of the product $\prod q_i$, and that the simultaneous vanishing of the polynomials $p_1, \ldots p_n$ is equivalent to the vanishing of the sum $\sum p_i^2$. (The reader might note that this elementary fact already draws a sharp distinction between real polynomials and complex polynomials.) Hence, any arbitrary conjunction and/or finite disjunction of polynomial equalities is equivalent to a single polynomial equality. In particular, the family of algebraic sets is closed under arbitrary intersections and finite unions. Note that, aside from $\mathbb{R}$ itself, the algebraic subsets of $\mathbb{R}$ are precisely the finite sets.

By definition, a *semi-algebraic set* in $\mathbb{R}^N$ is the union of a finite number of sets, each defined by a finite number of polynomial *inequalities* . More precisely, a semi-algebraic set is a finite union of sets of the form:

$$B = \{x \in \mathbb{R}^N : p(x) = 0 \, , \, q_1(x) < 0 \, , \, . \, . \, . \, , \, q_n(x) < 0\} \, .$$

where p , $q_1, \ldots q_n$ are polynomials (with real coefficients). (In contrast with the case of polynomial equalities, a finite conjunction and/or disjunction of polynomial inequalities need not be equivalent to a single polynomial inequality.) Note that the complement of a semi-algebraic set is a semi-algebraic set and that the union and intersection of (a finite number of) semi-algebraic sets is a semi-algebraic set. In other words, the family of semi-algebraic sets forms a *Boolean algebra* of sets. We may also describe the semi-algebraic sets as the smallest Boolean algebra of sets containing all those defined by a single polynomial inequality $q(x) \leq 0$. The semi-algebraic subsets of $\mathbb{R}$ consist of the all finite unions of intervals (closed or open or half open, finite or infinite).

Every algebraic set is semi-algebraic, and that every semi-algebraic set is the union of (relatively) open subsets of algebraic sets.

If A , B are semi-algebraic sets, a function $f : A \to B$ (or more generally, a correspondence $F : A \twoheadrightarrow B$) is *semi-algebraic* if its graph is a semi-algebraic set. Note that we do not require a semi-algebraic function to be continuous, nor do we require a semi-algebraic correspondence to be upper or lower hemi-continuous, or have closed values, or even to have non-empty values.

Two aspects of the theory of semi-algebraic sets are of primary interest to us. The first is that semi-algebraic sets admit an alternate description, which allows us to recognize as semi-algebraic many sets which are not presented as the solution sets of polynomial inequalities. The second is that semi-algebraic sets (and hence semi-algebraic functions and correspondences) have a very special structure.

To explain the first aspect, it is convenient to discuss in a very informal way the first order theory of the real numbers. We begin with the first order language, which is built up from the usual logical symbols (and, or, not, such that, implies, $\forall$, $\exists$), the real numbers (as constants), (real) variables, the algebraic operations (+, ·), equality (=), and the order relation (<). A *first order formula* is any formula in this language in which all quantifiers are extended only over elements of $\mathbb{R}$ and not over *sets* . (Keep in mind that many substructures of $\mathbb{R}$ do not have names in this language. In particular, there is no name for the integers, and there is no predicate for "is an integer".) The following are examples of first order formulas:

(F1) $\quad x > 0$

(F2) $\quad v^2 - 4uw > 0$

(F3) $\quad \exists y$ such that $x = y^2$ and $y \neq 0$

(F4) $\quad \exists y$ and $\exists z$ such that $y \neq z$ and $uy^2 + vy + w = 0$ and $uz^2 + vz + w = 0$

In these formulas we have followed the usual convention that "unbound variables are free." If a formula contains n free variables $x_1,\ldots,x_n$, substituting a particular real number r_i for each free variable x_i yields a *sentence* (i.e., a formula with no free variables) which may be true or false. If this sentence is true, we say that $r_1,\ldots,r_n$ *satisfy* the formula. The set of all n-tuples $(r_1,\ldots,r_n)$ satisfying the formula is the set *defined* by the formula.

Thus, formulas (F1) and (F3) each define a set of real numbers, while (F2) and (F4) each define a subset of $\mathbb{R}^3$. Clearly (F1) and (F3) define the same set, namely the set of positive real numbers. (F4) defines the set of triples $(u,v,w) \in \mathbb{R}^3$ such that the polynomial $ut^2 + vt + w$ has two distinct real roots. Since a quadratic polynomial has two distinct real roots exactly when its discriminant is positive, we see that the formulas (F2) and (F4) define the same set.

If $\Phi(x_1,..x_n,y)$ is a first order formula involving the free variables $x_1,\ldots,x_n$ and y , then we obtain first order formulas whenever we specialize or bound the variable y ; i.e., if r is any real number then the following are also first order formulas in which only the variables $x_1,\ldots,x_n$ are free:

(F5) $\Phi(x_1,..,x_n,r)$

(F6) $\exists y\ ,\ \Phi(x_1,..x_n,y)$

(F7) $\forall y\ ,\ \Phi(x_1,..x_n,y)$

As we have noted, this language contains no name for the set of integers and no predicate for "is an integer" and there is no first order formula (in this language) which is satisfied precisely by the integers (or the positive integers, or the rational numbers). The restriction to first order formulas is crucial here; the formula

(F8) $\forall X \subset \mathbb{R}\ ,\ (\ 0 \in X \text{ and } (y \in X \Rightarrow y + 1 \in X\ )\ ) \Rightarrow y \in X$

(with the single free variable y) is satisfied precisely by the positive integers. Of course it is not a first order formula, since it involves quantification over a set.

Note that a first order formula cannot involve a polynomial of unspecified degree. However, it may certainly involve a polynomial of a particular, pre-specified degree (in a particular, pre-specified number of variables), since a polynomial of pre-specified degree in a pre-specified number of variables is determined entirely by its (finite, pre-specified mumber of) coefficients.

As we have noted, the two formulas (F1) and (F3) above are satisfied by the same values of the variable x , and define the same subset of $\mathbb{R}$; i.e., they are *equivalent* . (In view of this simple observation, it might seem that the order relation $<$ is redundant, since it can be expressed in terms of multiplication. However, doing so *requires* the use of quantifiers, while it is precisely the *elimination* of quantifiers which provides the powerful tool we shall use.) Similarly, the two formulas (F2) and (F4) are equivalent; they are satisfied by the same values of the variables u , v , w , and define the same subset of $\mathbb{R}^3$. The forms of these formulas are notable; (F3) and (F4) involve quantifiers, while (F1) and (F2) do not. That is, (F3) and (F4) are equivalent to formulas from which *the quantifiers have been eliminated* . The following theorem of Tarski and Seidenberg (Tarski [1931], Seidenberg [1954]) says that this is always possible. (The Tarski-Seidenberg theorem is frequently phrased as a statement about real closed fields, and it is in this form that it was used by Bewley-Kohlberg [1976]. We have phrased it as a statement about the real numbers because that is more convenient for our purposes. For a more formal discussion of the relationship between the two formulations, see Blume-Zame [1989].)

TARSKI-SEIDENBERG THEOREM (version I): Every first order formula is equivalent to a first order formula with no quantifiers.

A first order formula with no quantifiers is just a conjunction and disjunction of polynomial inequalities, and hence defines a semi-algebraic set. Thus, the Tarski-Seidenberg theorem can be phrased in the following (equivalent) way, which is more convenient for us:

TARSKI-SEIDENBERG THEOREM (version II): A subset of $\mathbb{R}^N$ is semi-algebraic if and only if it can be defined by a first order formula.

As we shall see, the Tarski-Seidenberg theorem is remarkably powerful and at the same time remarkably easy to apply. A few examples may help to suggest the kind of logical manipulations involved.

PROPOSITION 1: The image of a semi-algebraic set under a semi-algebraic map is a semi-algebraic set.

PROOF: If $f : A \to B$ is a semi-algebraic map between semi-algebraic sets, then graph(f) is a semi-algebraic subset of $A \times B$. The image of f is defined by the formula:

(*) $b \in B$ and $\exists\, a \in A$ such that $(a,b) \in \text{graph}(f)$

Since A, B and graph(f) are semi-algebraic sets and hence are defined by first order formulas, (*) "is" also a first order formula. A little more precisely: if Φ_A, Φ_B, Ψ are the first order formulas which define A, B, graph(f), then (*) is shorthand for the first order formula

(*') $\Phi_B(b)$ and $\exists\, a$ such that $\Phi_A(a)$ and $\Psi(a,b)$

Hence, the Tarski-Seidenberg theorem implies that image(f) is semi-algebraic. ∎

PROPOSITION 2: The closure of a semi-algebraic set is a semi-algebraic set.

PROOF: If A is a semi-algebraic set in $\mathbb{R}^N$, then its closure $\bar{A}$ is just the set of points $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ satisfying the formula:

$$(**) \qquad \forall \varepsilon > 0, \exists y \in A \text{ such that } |x - y|^2 < \varepsilon$$

Since $|x - y|^2$ is a polynomial and A is defined by a first order formula, (**) is (shorthand for) a first order formula. Hence, the Tarski-Seidenberg theorem implies that $\bar{A}$ is semi-algebraic. ■

As a final example of the sort of manipulations which are sometimes useful, we give an extension of Proposition 1 which will be needed in Section 4. By Proposition 1, if A is a semi-algebraic subset of $\mathbb{R}^n$ and $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is a semi-algebraic mapping, then $\varphi(A)$ is a semi-algebraic set. Hence $\varphi(A)$ can be defined by polynomial inequalities. We would like to have bounds for the number of polynomials required, and their degrees. One suspects that it should be possible to find such bounds if we constrain A to lie in a "semi-algebraic family of sets" and constrain φ to lie in a "semi-algebraic family of mappings" and this is what we shall demonstrate. (Precise statements of this sort can be formulated in many different ways; the formulation we choose is simply one that is convenient for the application we need.) We formalize the idea of a semi-algebraic family of sets by beginning with a semi-algebraic subset B of $\mathbb{R}^{n+1}$ and considering the family sets B_η obtained by intersecting B with the hyperplane $x_{n+1} = \eta$. Rather than formalizing the idea of a semi-algebraic family of mappings, we simply consider the family of all linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$; identifying a linear transformation with its matrix yields a natural "semi-algebraic parametrization" of this family.

To state our result precisely, we first collect some notation. A family of subsets of $\mathbb{R}^N$ is a *Boolean algebra of sets* if it is closed under the formation of complements, finite unions and finite intersections. If E is a family of subsets of $\mathbb{R}^N$, then by $B(E)$ we mean the Boolean algebra generated by E; i.e., the smallest Boolean algebra of sets containing E. Now, $B(E)$ can be constructed from E closing under complements, then finite intersections, and finally, finite unions. In particular, if E is a finite family, then $B(E)$ is also a finite family of sets. (It is not difficult to see that the cardinality of $B(E)$ is at most 2^k, where

$k = 2^{\text{cardinality}(E)}$.) If $f_1,\dots,f_n$ are polynomials, then we shall abuse notation to write $B(f_1,\dots,f_n)$ for the Boolean algebra generated by the sets $\{x : f_i(x) < 0\}$ and $\{x : f_i(x) \le 0\}$.

The following proposition contains the precise result we need in Section 4. The reader should keep in mind that it is merely a particular example of the way in which the Tarski-Seidenberg theorem can be "bootstrapped" to obtain bounds on the number and degrees of polynomials required to express certain sets.

PROPOSITION 3: Let B be a semi-algebraic subset of $\mathbb{R}^{n+1}$, and write $\pi : \mathbb{R}^{n+1} \to \mathbb{R}$ for the projection on the last coordinate. For each $\eta \in \mathbb{R}$, set $B_\eta = B \cap \{x : \pi(x) = \eta\}$. Let m be a fixed integer, and let $L = L(\mathbb{R}^{n+1},\mathbb{R}^m)$ be the family of all linear transformations $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^m$. Then there are integers K and D such that for every linear transformation $\varphi \in L$ and every $\eta \in \mathbb{R}$, there are polynomials $f_1,\dots,f_K$ (where the degree of each f_k is bounded by D), such that $\varphi(B_\eta) \in B(f_1,\dots,f_K)$.

PROOF: We identify a linear transformation $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^m$ with its matrix (in the standard bases), and hence view L as $\mathbb{R}^{m(n+1)}$. Define the function $\Lambda : \mathbb{R}^{n+1} \times L \to \mathbb{R}^m \times L \times \mathbb{R}$ by $\Lambda(x,\varphi) = (\varphi\cdot x,\varphi,\pi(x))$; this is a semi-algebraic (in fact, polynomial) mapping, and $B \times L$ is clearly a semi-algebraic subset of $\mathbb{R}^{n+1} \times L$. Hence by Proposition 1, there are polynomials $F_1,\dots,F_r$ in $m + m(n+1) + 1$ variables such that $\Lambda(B \times L) \in B(F_1,\dots,F_r)$. Writing y for the variables in $\mathbb{R}^m$, φ for the variables in $L = \mathbb{R}^{m(n+1)}$ and η for the last variable, observe that for fixed $\varphi^* \in L$, and fixed $\eta^* \in \mathbb{R}$, each $F_i(y,\varphi^*,\eta^*)$ is a polynomial in m variables. It is easily seen that

$$\varphi^*(B_{\eta^*}) \in B(F_1(y,\varphi^*,\eta^*),\dots,F_r(y,\varphi^*,\eta^*))$$

which yields the desired result. ■

We turn now to the second aspect of the theory of semi-algebraic sets which is of importance to us: semi-algebraic sets (and semi-algebraic functions and correspondences) have a very special structure. The most important consequences of this special structure (at least for our purposes) are given below.

Recall that a *finite simplicial complex* in $\mathbb{R}^N$ is a finite disjoint collection $\{K_j\}$ of open simplices (of various dimensions), having the property that if K_ℓ meets the closure of K_i, then K_ℓ is an open lower dimensional face of K_i.

TRIANGULABILITY (Lojasiewicz [1964, 1965], Hironaka [1975]): Every semi-algebraic set can be semi-algebraically triangulated. Indeed, for every semi-algebraic subset A of $\mathbb{R}^N$, there is a finite simplicial complex $\{K_j\}$ in $\mathbb{R}^N$ and a semi-algebraic homeomorphism $h : \mathbb{R}^N \to \mathbb{R}^N$ such that $h(K) = A$.

Since simplices are connected, it of course follows immediately that semi-algebraic sets have only a finite number of connected components.

STRATIFIABILITY (Whitney [1957], Bochnak-Coste-Roy [1988, pp. 188-189]): Every semi-algebraic set is the disjoint union of a finite number of semi-algebraic subsets, each of which is a real-analytic manifold.

In view of this, we may speak unambiguously of the *dimension* of a semi-algebraic set ($\dim A$ = maximum of the dimension of smooth submanifolds of A), and the *dimension* of a semi-algebraic set *at a point* ($\dim_p A$ = maximum of the dimension of smooth submanifolds of A whose closures contain p). Note that $\dim A = \max\{\dim_p A : p \in A\}$. (By convention, the dimension of the empty set is -1.) Note that the map $p \to \dim_p A$ is a semi-algebraic function.

If A is any set, we write $\overline{A}$ for its topological closure. By its *Zariski closure* $\mathrm{Zar}(A)$ we mean the smallest algebraic set containing A; equivalently, $\mathrm{Zar}(A)$ is the set of common zeroes of all polynomials which vanish on A. Note that $\mathrm{Zar}(A) \supset \overline{A}$. As we have noted before, the set of common zeroes of all polynomials which vanish on A is in fact the set of common zeroes of a single polynomial. In order that A be Zariski closed (i.e., that $A = \mathrm{Zar}(A)$) it is necessary and sufficient that there be a polynomial vanishing at every point of A and nowhere else; i.e., that A be an algebraic set.

DIMENSION (Bochnak-Coste-Roy [1988, pp. 47, 237]): If A is a non-empty semi-algebraic set, then $\dim \mathrm{Zar}(A) = \dim \overline{A} = \dim A$ and $\dim(\overline{A} - A) < \dim A$. If $f : A \to \mathbb{R}^k$ is a semi-algebraic mapping, then $\dim f(A) \le \dim A$; if f is one-to-one, then $\dim f(A) = \dim A$.

It should of course be kept in mind that no such result is true for arbitrary sets A (even for sets defined by smooth inequalities) or for arbitrary mappings f (even for mappings that are continuous and differentiable almost everywhere.) For example, set $A = \{(m/n, 1/n) \in \mathbb{R}^2 : m, n \text{ positive integers}\}$. It is easily seen that A is a discrete subset of the upper half plane, and in particular is a differentiable submanifold of dimension 0. On the other hand, $\overline{A}$ contains the line $\{(r,0) : r \in \mathbb{R}\}$, and so has dimension 1.

PIECEWISE MONOTONICITY (van den Dries [1986]): Every semi-algebraic function $f : (a,b) \to \mathbb{R}$ is piecewise monotone. That is, there exist points $c_i \in (a,b)$ with $a = c_0 < c_1 < \ldots < c_k = b$ such that the restriction of f to the subinterval (c_i, c_{i+1}) is either constant, or continuous and strictly monotone. In particular, the one-sided limits:

$$\lim_{x \to a+} f(x) \quad \text{and} \quad \lim_{x \to b-} f(x)$$

both exist (as extended real numbers).

Note that the analogous statement is false for functions of two variables; the function $f(x,y) = xy/(x^2 + y^2)$ has no limit as (x,y) approaches $(0,0)$.

GENERIC LOCAL TRIVIALITY (Hardt [1980], Bochnak-Coste-Roy [1988, p. 195]): Let A, B be semi-algebraic sets, and let $\varphi : A \to B$ be a continuous semi-algebraic function. Then there are is a (relatively) closed semi-algebraic subset $B' \subset B$ with $\dim B' < \dim B$ such that for each of the (finite number of) connected components B_i of $B \setminus B'$ there is a semi-algebraic set C_i and a semi-algebraic homeomorphism $h_i : B_i \times C_i \to \varphi^{-1}(B_i)$ with the property that $\varphi \circ h_i(b,c) = b$ for each $b \in B_i$, $c \in C_i$.

Informally: except for a small subset of the range, every continuous semi-algebraic function is locally a product. (Caution: this is false without the requirement that φ be continuous; see Bochnak-Coste-Roy [1988, p. 196].)

Generic local triviality has many striking consequences. For our purposes, the most important are that semi-algebraic correspondences and semi-algebraic functions are generically continuous, that semi-algebraic correspondences admit generically continuous semi-algebraic selections, and that semi-algebraic functions are generically real-analytic.

GENERIC CONTINUITY: If X , Y are semi-algebraic sets and $F : X \twoheadrightarrow Y$ is a semi-algebraic correspondence with closed values , then there is a closed semi-algebraic set $X' \subset X$ of lower dimension such that F is continuous at each point of $X \setminus X'$. In particular, a semi-algebraic function is continuous at each point of the complement of a lower dimensional sem-algebraic set.

PROOF: Set $B = X$, $A = \text{graph}(F) \subset X \times Y$, and let $\varphi : A \to B$ be the projection onto the first factor and $\psi : A \to Y$ the projection onto the second factor. Generic triviality yields a semi-algebraic set $B' \subset B$, and connected components B_i of $B \setminus B'$, semi-algebraic sets C_i , and semi-algebraic homeomorphisms $h_i : B_i \times C_i \to \varphi^{-1}(B_i)$ as above. It is evident that the restriction of F to B_i is continuous. Moreover, each B_i is a relatively open subset of B (being one of the finite number of connected components of the open set $B \setminus B'$), and $\cup B_i = B \setminus B'$. We conclude that F is continuous at each point of $B \setminus B'$. The second statement follows from the observation that a single-valued correspondence is a function. ∎

SELECTIONS: If X , Y are semi-algebraic sets and $F : X \twoheadrightarrow Y$ is a semi-algebraic correspondence with non-empty (not necessarily closed) values, then there is a semi-algebraic function $f : X \to Y$ and a closed semi-algebraic set $X' \subset X$ of lower dimension such that $f(x) \in F(x)$ for each $x \in X$ and f is continuous at each point of $X \setminus X'$.

PROOF: Set $B = X$, $A = \text{graph}(F) \subset X \times Y$, and let $\varphi : A \to B$ be the projection onto the first factor and $\psi : A \to Y$ the projection onto the second factor. Generic triviality yields a semi-algebraic set $B' \subset B$, and connected components B_i of $B \setminus B'$, semi-algebraic sets C_i , and semi-algebraic homeomorphisms $h_i : B_i \times C_i \to \psi^{-1}(B_i)$ as above. Choose (for each i) a point $c_i \in C_i$ and define f_1 on $\cup B_i$ by $f_1(b) = \psi \circ h_i(b,c_i)$; f_1 is evidently a continuous, semi-algebraic selection, defined on $X \setminus X_1$. We may now apply this procedure to the restriction of F to X_1 , obtaining a lower dimensional semi-aklgebraic subset $X_2 \subset X_1$ and a continuous semi-algebraic selection f_2 defined on $X_1 \setminus X_2$. Since the dimensions of the sets X_i are strictly decreasing, this is a finite process. Finally, we define the selection $f : X \to Y$ by $f(x) = f_1(x)$ for $x \in X \setminus X_1$ and $f(x) = f_{i+1}(x)$ for $x \in X_i \setminus X_{i+1}$. ∎

We have already noted that semi-algebraic functions are generically continuous; in fact we can say much more.

GENERIC REAL ANALYTICITY: If X , Y are semi-algebraic sets, X is a real-analytic manifold, and $f : X \to Y$ is a semi-algebraic function, then there is a closed semi-algebraic set $X' \subset X$ of lower dimension such that f is real analytic at each point of $X \setminus X'$.

PROOF: As noted above, there is a closed semi-algebraic subset $X'' \subset X$ such that f is continuous at each point of $X \setminus X''$. Let $Z \subset (X \setminus X'') \times Y$ be the graph of the restriction $f|(X \setminus X'')$, and let $\pi_1 : (X \setminus X'') \times Y \to (X \setminus X'')$ and $\pi_2 : (X \setminus X'') \times Y \to Y$ be the projections. Since Z is a semi-algebraic set, it is the union of real-analytic manifolds M_i . By the semi-algebraic version of Sard's theorem (Bochnak-Coste-Roy [1988, p. 205]), the set C_i of critical values of $\pi_1|M_i$ is a semi-algebraic subset of $\pi_1(M_i)$ of lower dimension. Since $f|(\pi_1(M_i) \setminus C_i) = \pi_2 \circ \pi_1^{-1}$, f is real-analytic on $\pi_1(M_i) \setminus C_i$. Taking the union over all M_i and replacing the lower dimensional semi-algebraic set $\cup C_i$ by its closure yields the desired result. ∎

3. EQUILIBRIUM

As we have noted in the Introduction, the general relevance of the Tarski-Seidenberg theorem to game theory is that virtually all of the constructions of game theory have - or can be given - first order descriptions, and hence define semi-algebraic sets. In particular, this is the case for almost all of the usual equilibrium correspondences, so the results of the preceeding section lead very easily to extremely strong conclusions about these equilibrium correspondences.

To make this precise, fix an extensive form Γ ; i.e., a finite set of players, a game tree, and information sets for each player). An extensive form game is obtained from Γ by specifying a probability distribution over initial nodes, and payoffs (for each player) at each terminal node. If there are N players, I initial nodes, and Z terminal nodes, then we may parametrize the set of all such games by $\Pi(I) \times \mathbb{R}^{NZ}$, where $\Pi(I)$ is the probability simplex of dimension $I-1$; the dimension of this set is $NZ+I-1$. (Alternatively, we could view the probability distribution π_0 over initial nodes as fixed and the payoffs u as variable, or vice versa.) If $\pi \in \Pi(I)$ and $u \in \mathbb{R}^{NZ}$, we denote the corresponding game by $\Gamma_{\pi,u}$. A (behavioral) strategy for a player in this game is a function from his information sets to probability distributions on available actions at these information sets. We write Δ^i for the set of behavioral strategies of player i , and $\Delta = \Delta^1 \times \ldots \times \Delta^N$ for the set of behavioral strategy profiles. Note that $\Pi(I) \times \mathbb{R}^{NZ}$ and Δ are naturally identified with subsets of Euclidean space, defined by the appropriate linear equalities and inequalities; in particular, these sets are semi-algebraic. For each $\pi \in \Pi(I)$ and $u \in \mathbb{R}^{NZ}$, a Nash (or subgame perfect or sequential or (trembling hand) perfect) equilibrium for the corresponding game $\Gamma_{\pi,u}$ is an element of Δ , so each of these equilibrium notions yields a correspondence $\Pi(I) \times \mathbb{R}^{NZ} \twoheadrightarrow \Delta$. (If we view the probability distribution π_0 over initial nodes as fixed, and the payoffs u as variable, each of these equilibrium notions yields a correspondence $\{\pi_0\} \times \mathbb{R}^{NZ} \twoheadrightarrow \Delta$.)

THEOREM 1: For every game form Γ , the Nash, subgame perfect, sequential, and (trembling hand) perfect equilibrium correspondences are all semi-algebraic.

PROOF: To see that the Nash equilibrium correspondence $NE : \Pi(I) \times \mathbb{R}^{NZ} \twoheadrightarrow \Delta$ is semi-algebraic, we write the set

$$\text{graph}(NE) = \{(\pi,u,\sigma) \in \Pi(I) \times \mathbb{R}^{NZ} \times \Delta : \sigma \text{ is a Nash equilibrium for } \Gamma_{\pi,u}\}$$

in terms of polynomial inequalities. This is elementary. Write $E_i(\sigma_i,\sigma_{-i})$ for the expected payoff to player i if he follows the strategy σ_i and everyone else follows follows the strategy profile σ_{-i} . For σ to be a

Nash equilibrium it is necessary and sufficient that $E_i(\sigma_i,\sigma_{-i}) \geq E_i(s_i,\sigma_{-i})$ for every i and for every pure strategy s_i of player i . Hence

$$\text{graph(NE)} = \{(\pi,u,\sigma) : \forall i\ ,\ \forall s_i\ ,\ E_i(\sigma_i,\sigma_{-i}) \geq E_i(s_i,\sigma_{-i})\ \}$$

Since each $E_i(\sigma_i,\sigma_{-i})$ and $E_i(s_i,\sigma_{-i})$ is a polynomial in σ_i , σ_{-i} , π , and u , we conclude that graph(NE) is a semi-algebraic set, as desired.

The argument for the subgame perfect equilibrium correspondence is similar, and equally elementary.

The argument that the sequential equilibrium correspondence is semi-algebraic is no longer elementary. We must show that

$$\text{graph(SE)} = \{(\pi,u,\sigma) : \sigma \text{ is a sequential equilibrium for } \Gamma_{\pi,u}\}$$

can be written in terms of polynomial inequalities. In view of the Tarski-Seidenberg theorem, it will suffice to show that graph(SE) can be defined by a first order formula. By definition, σ is a sequential equilibrium for the game $\Gamma_{\pi,u}$ if there exist beliefs θ such that the assessment (σ,θ) is consistent and sequentially rational. Since beliefs are probability distributions over actions, they can be represented as elements of $\mathbb{R}^{\ell}$ for sufficiently large ℓ . Hence assessments lie in $\Delta \times \mathbb{R}^{\ell}$. Following Kreps and Wilson [1982], write $\Psi^0 \subset \Delta \times \mathbb{R}^{\ell}$ for the set of assessments (σ,θ) with the property that each action of each player is given positive probability, and beliefs are derived by Bayes' rule. It is evident that Ψ^0 is defined by a polynomial equalities and inequalities. Since the set of consistent assessments Ψ is the closure of Ψ^0 , it follows from Proposition 1 that Ψ is a semi-algebraic set. Sequential rationality means that, for each player i and each information set h for that player, the strategy σ_i is optimal against σ_{-i} , starting from h , given the beliefs θ . Write $E_i{}^h(\sigma_i,\sigma_{-i}\,|\,\theta)$ for the expected payoff to player i starting from the information set h , if he has the beliefs θ , follows the strategy σ_i , and everyone else follows the strategy profile σ_{-i} ; note that this is a polynomial in σ_i , σ_{-i} , π , and u . Putting everything together, we obtain:

$$\begin{aligned}\text{graph(SE)} = \{(\pi,u,\sigma) : \exists\, \theta \in \mathbb{R}^{\ell} \text{ such that } (\sigma,\theta) \in \Psi \text{ and} \\ \forall i\ ,\ \forall \tau_i\ ,\ E_i{}^h(\sigma_i,\sigma_{-i}\,|\,\theta) \geq E_i{}^h(\sigma_i,\sigma_{-i}\,|\,\theta)\}\end{aligned}$$

The Tarski-Seidenberg theorem now guarantees that graph(SE) is a semi-algebraic set, as desired.

The proof that the perfect equilibrium correspondence is semi-algebraic follows the same outline.

Finally, to see that each of these correspondences remains semi-algebraic when we view the probability distribution π_0 over initial nodes as fixed, we need only observe that $\{\pi_0\} \times \mathbb{R}^{NZ}$ is a semi-algebraic subset of $\Pi(I) \times \mathbb{R}^{NZ}$, and that the restriction of a semi-algebraic correspondence to a semi-algebraic subset of its domain is again a semi-algebraic correspondence. ∎

With this observation in hand, we can derive some striking consequences. The first follows immediately from Theorem 1, Generic Continuity and Selection.

COROLLARY 1.1: For every game form Γ, there is a closed, semi-algebraic subset $X \subset \Pi(I) \times \mathbb{R}^{NZ}$ of dimension $NZ + I - 1$ such that the Nash, subgame perfect, sequential, and perfect equilibrium correspondences are continuous at every point of $(\Pi(I) \times \mathbb{R}^{NZ}) \setminus X$. Moreover, each of these correspondences admits a semi-algebraic selection which is continuous at each point of $(\Pi(I) \times \mathbb{R}^{NZ}) \setminus X$.

If we view the probability distribution π_0 over initial nodes as fixed, then we obtain a closed, semi-algebraic subset $X_{\pi_0} \subset \mathbb{R}^{NZ}$ of dimension $NZ - 1$ such that the Nash, subgame perfect, sequential, and perfect equilibrium correspondences are continuous at every point of $(\{\pi_0\} \times \mathbb{R}^{NZ}) \setminus X_{\pi_0}$. Moreover, each of these correspondences admits a semi-algebraic selection which is continuous at each point of $(\{\pi_0\} \times \mathbb{R}^{NZ}) \setminus X_{\pi_0}$.

It is instructive to compare the conclusions of Corollary 1.1 with those available from general facts about correspondences. Recall (Hildenbrand [1974]) that if A, B are complete metric spaces, and $F : A \twoheadrightarrow B$ is an upper hemi-continuous correspondence with compact values, then F is continuous at each point of a residual set (and hence admits a selection which is continuous at each point of the same residual set). Applying this result to the correspondences above leads to the conclusion that the Nash, subgame perfect, and sequential equilibrium correspondences are continuous at each point of a residual set (and admits a selection which is continuous at each point of the same residual set). It should be kept in mind, however, that a residual set might fail to be open (indeed, its complement could be dense) and might have (Lebesgue) measure zero. On the other hand, from Corollary 1.1 and the fact that a closed lower dimensional semi-algebraic subset of $\mathbb{R}^{\ell}$ has no interior and is of (Lebesgue) measure zero, we conclude that the Nash, subgame perfect, and sequential equilibrium correspondences are continuous at each point of a dense open set whose complement has measure zero (and admit selections which are continuous at each point of the same dense open set). This is clearly a substantial improvement over the conclusions available from general facts about correspondences. In the case of the perfect equilibrium correspondence, the improvement is much more dramatic, since the perfect equilibrium correspondence is not upper hemi-continuous, so might *a priori* have no points of continuity whatsoever.

Recall that Kohlberg and Mertens [1986] have used the fact that the set of Nash equilibria of any game is semi-algebraic to conclude that the set of Nash equilibria has only a finite number of connected components. (This fact plays an important role in the theor of stable equilibria.) The following Corollary sharpens this observation: the finiteness statement remains valid for various refinements, and there are uniform bounds on the number of connected componenets. The proof involves a useful theme: generic local triviality gives information off a lower dimensional set; restricting to that lower dimensional set yields a situation to which generic local triviality can be applied again, yielding finer information, etc.

COROLLARY 1.2: For any game form Γ, there is an integer r (depending only on Γ) such that the number of connected components of the set of Nash equilibria (respectively, subgame perfect equilibria, sequential equilibria, perfect equilibria) of any game $\Gamma_{\pi,u}$ is finite and bounded by r.

PROOF: We give the argument only for Nash equilibria. Applying generic local triviality to the projection

$$\text{proj} : \text{graph}(NE) \to \Pi(I) \times \mathbb{R}^{NZ}$$

implies that there is a lower dimensional set $X_1 \subset \Pi(I) \times \mathbb{R}^{NZ}$ and a finite number of semi-algebraic sets $Y_{11}, Y_{12}, \ldots, Y_{1k_1}$ with the property that, for every $(\pi,u) \in (\Pi(I) \times \mathbb{R}^{NZ} \setminus X_1)$, the set of Nash equilibria of the game $\Gamma_{\pi,u}$ is (semi-algebraically) homeomorphic with one of the sets Y_{1i}. Applying generic local triviality to the restriction

$$\mathrm{proj}\,|\,\mathrm{proj}^{-1}(X_1) \to X_1$$

implies that there is a lower dimensional set $X_2 \subset X_1$ and a finite number of semi-algebraic sets $Y_{21}, Y_{22}, \ldots, Y_{2k_2}$ with the property that, for every $(\pi,u) \in (X_1 \setminus X_2)$, the set of Nash equilibria of the game $\Gamma_{\pi,u}$ is (semi-algebraically) homeomorphic with one of the sets Y_{2j}. Continuing in this way, we obtain a finite collection $\{Y_{ij}\}$ of semi-algebraic sets with the property that for every $(\pi,u) \in (\Pi(I) \times \mathbb{R}^{NZ}$, the set of Nash equilibria of the game $\Gamma_{\pi,u}$ is (semi-algebraically) homeomorphic with one of the sets Y_{ij}. Triangulability now yields the desired conclusion. ∎

The last Corollary follows immediately from Theorem 1 and Stratifiability.

COROLLARY 1.3: For every game, the set of Nash equilibria (respectively, subgame perfect equilibria, sequential equilibria, perfect equilibria) is a finite disjoint union of connected real-analytic manifolds.

(Compare the discussion in Kreps and Wilson [1982] about the set of sequential equilibria.)

4. THE TRACING PROCEDURE

In this section, we use the general theory of semi-algebraic sets (as described above) and some other ideas from real algebraic geometry to repair some gaps in the description of the tracing procedures and the verifications of the crucial properties. We begin with a review of the tracing procedure, and a discussion of the difficulties.

We first collect some notation. In what follows, we fix an N-player normal form game Γ (the argument for extensive form games requires only minor modifications). Let $\Delta \subset \mathbb{R}^{\ell}$ be the set of profiles of mixed strategies and let Δ^* be the set of profiles of completely mixed strategies. Write $W = (0,1) \times (0,1) \times \Delta^*$, $W^* = (0,1] \times (0,1] \times \Delta^*$ and

$\overline{W} = [0,1] \times [0,1] \times \Delta$; these are subsets of $\mathbb{R}^{2+\ell} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\ell}$; W is open and $\overline{W}$ is its closure. Let $\pi_1, \pi_2 : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\ell} \to [0,1]$, be the projections into the first two variables, and let $\sigma : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\ell} \to \Delta$ be the projection into the last variable. We usually write $w = (\eta,t,q)$ for a typical element of $\mathbb{R}^{2+\ell}$, so that $\eta = \pi_1(w)$, $t = \pi_2(w)$ and $q = \sigma(w)$; we frequently refer to q as the *strategy part* of w . For $A \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\ell}$ and $\eta \in [0,1]$, write $A(\eta) = \{w \in A : \pi_1(w) = \eta\}$.

Now, let H_i be the payoff function for player i in the game Γ ; it is convenient to view H_i as a (affine) function from Δ to $\mathbb{R}$. Fix a probability distribution $p \in \Delta^*$. For $\eta \in [0,1]$, $t \in [0,1]$, we define a game $\Gamma(\eta,t)$ with the same set of players and the same strategies as Γ , but with payoffs $H_{i,t,\eta}$ defined by:

$$H_{i,t,\eta}(q_i,q_{-i}) = (1-t)H_i(q_i,q_{-i}) + tH_i(q_i,p_{-i})$$

$$+ \eta t \alpha_i \sum \log(q_{ik})$$

where α_i is a suitable positive constant. (Here we have written q_{ik} for the k-th component of the strategy vector q_i for player i . Note that if $t = 0$ the logarithmic term disappears and the game $\Gamma(\eta,0)$ coincides with Γ . However, when $0 < t \leq 1$ and $0 < \eta$ the logarithmic term is very important. Indeed, as Harsanyi and Selten show, for these values of t and η , every equilibrium of the game $\Gamma(\eta,t)$ is in completely mixed strategies. (A little care must be exercised here. If $0 < t \leq 1$ and $0 < \eta$, the payoff functions $H_{i,t,\eta}(q_i,q_{-i})$ are, strictly speaking, only defined if q_i is a completely mixed strategy; otherwise, $H_{i,t,\eta}(q_i,q_{-i}) = -\infty$. However, as Harsanyi and Selten show, this causes no difficulties. In particular, each of the games $\Gamma(\eta,t)$ does indeed have an equilibrium, in completely mixed strategies.) As a consequence, for these values of t and η , a completely mixed strategy profile $q \in \Delta^*$ is an equilibrium exactly when it satisfies the first order conditions. Because the derivative of $\log(t)$ is $1/t$ (a rational function), these first order conditions can be written as the conjunction of a finite number of polynomial identities, and hence as the vanishing of a single polynomial $P(\eta,t,q)$.

Write

$$L = \{(\eta,t,q) \in W^* : q \text{ is an equilibrium of } \Gamma(\eta,t)\} ,$$

the graph of the (Nash) equilibrium correspondence. In view of the above, L is the zero set of the polynomial $P(\eta,t,q)$, and so is an algebraic set. For each $\eta > 0$, the game $\Gamma(\eta,1)$ is separable and has a unique equilibrium $w(\eta,1)$. It follows that there is a t_0 sufficiently close to 1 such that the game $\Gamma(\eta,t)$ has a unique equilibrium for $t_0 < t < 1$. Equivalently, the restriction $\pi_2 | L(\eta) \cap \pi_2^{-1}((t_0,1))$ is a homeomorphism of the smoth curve $L(\eta) \cap \pi_2^{-1}((t_0,1))$ onto the interval $(t_0,1)$.

The logarithmic tracing procedure may now be described in the following way: For each $\eta > 0$, $L(\eta)$ is an algebraic set which contains the point $w(\eta,1)$, and near this point, $L(\eta)$ is a smooth one dimensional curve. Hence, by Puiseaux's theorem, we may analytically continue this curve until it first leaves W^* . Harsanyi and Selten prove that every limit point w of $L(\eta)$ in $\overline{W}$ has the property that the strategy part $\sigma(w)$ is a Nash equilibrium of the game $\Gamma(\pi_1(w),\pi_2(w))$. Since all equilibria of the games $\Gamma(\eta,t)$ are in completely mixed strategies when $\eta > 0$ and $t > 0$, this means that $L(\eta)$ can only leave W^* through a point $w(\eta)$ such that $\pi_2(w(\eta)) = 0$, so the strategy profile $\sigma(w(\eta))$ is an equilibrium of the game $\Gamma(\eta,0)$. Taking limits as $\eta \to 0$ yields the logarithmic solution.

Unfortunately, there are some difficulties with this construction. The first of these is that Puiseaux's theorem applies only to algebraic *curves* (i.e., one dimensional algebraic sets), and the algebraic set $L(\eta)$ is generally *not* a curve. (It is an algebraic set and a portion of it is one dimensional, but it may also have higher dimensional portions.) Hence Puiseaux's theorem does *not* imply that $L(\eta)$ can be analytically continued *as a curve* , and in such a case the recipe of following $L(\eta)$ is not well-defined. (This difficulty cannot be remedied by the "obvious" means of passing to the "one dimensional branch" of $L(\eta)$, since it is entirely possible that $L(\eta)$ has both one dimensional portions and higher dimensional portions, but only one branch. See Bochnak-Coste-Roy [1988, pp. 53-54].) The second difficulty is more subtle (and more difficult to deal with). Harsanyi and Selten assert that $\sigma(w(\eta))$ has a limit as $\eta \to 0$ because $L(\eta)$ is a curve and depends algebraically on η . As we have already noted, $L(\eta)$ need not be a curve. Even if $L(\eta)$ is a curve, however, it may branch, in which case the procedure of following $L(\eta)$ will be well-defined, but *not* an algebraic procedure. (See Figure 1.) To put it another way: If $L(\eta)$ is an algebraic curve, then following $L(\eta)$ until it first leaves W^* amounts to finding the intersection with $\{t = 0\}$ of the irreducible branch $L^*(\eta)$ of $L(\eta)$ that contains the portion of $L(\eta)$ near $\{t = 1\}$; but the irreducible branch $L^*(\eta)$ does *not* depend algebraically on η . Thus, there is no reason to suppose that the limit point $w(\eta)$ depends nicely on η , and in particular, there is no reason to suppose that $\sigma(w(\eta))$ has a limit as $\eta \to 0$.

Our approach has the same intuition but avoids the pitfalls identified above. After a preliminary construction (Lemma 1), we show (Lemma 2) that there is a finite set $E \in (0,1)$ with the property that for all $\eta \notin E$ there is a unique irreducible analytic curve $C(\eta) \subset W$ containing the portion of $L(\eta)$ near $\{t = 1\}$. (Recall that an *analytic subset* of W is a

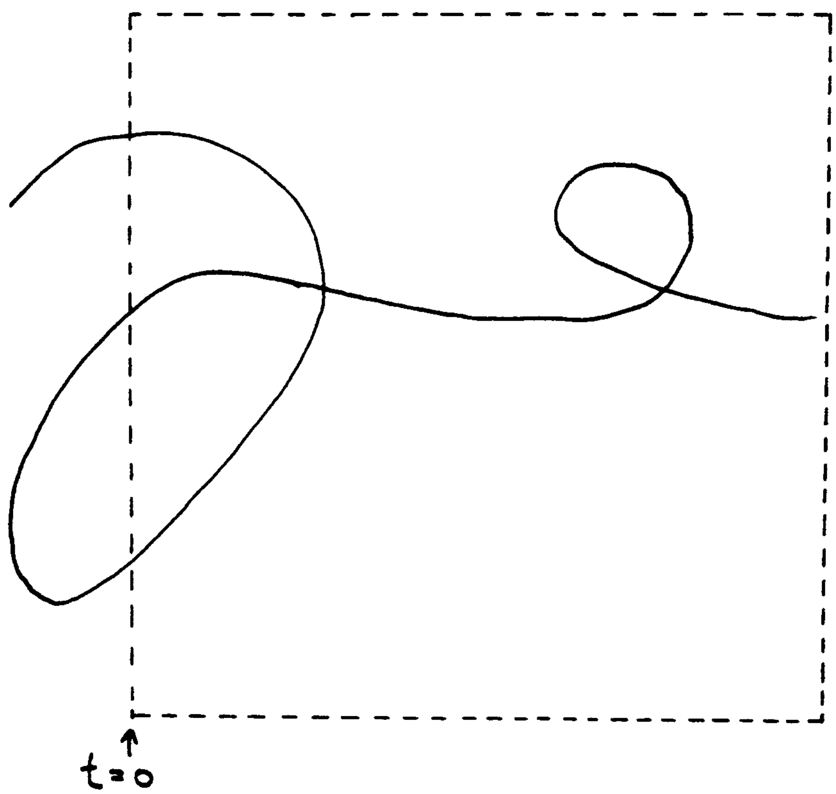

FIGURE 1

closed set which is locally the set of zeroes of a finite number of real analytic functions; an analytic curve is a one dimensional analytic set. See Milnor [1968].) Although $C(\eta)$ is not an algebraic curve, it is a semi-algebraic curve, and depends semi-algebraically on η . We then show (Lemma 3) that for each $\eta \notin E$, and for t sufficiently close to 0 , there is a unique point $z(\eta,t) \in C(\eta)$ with $\pi_2(z(\eta,t)) = t$. Because $C(\eta)$ depends semi-algebraically on η , The point $z(\eta,t)$ depends semi-algebraically on η and t . Hence, for each fixed η , $z(\eta,t)$ has a limit $z(\eta)$ as t approaches 0 , and $z(\eta)$ in turn depends semi-algebraically on η . Hence $z(\eta)$ in turn has has a limit as η approaches 0 (Lemma 4). This limit is the logarithmic solution.

To make these ideas precise, fix $\eta > 0$. As noted above, for t sufficiently close to 1 , there is a unique point $z(\eta,t) \in L(\eta)$ such that $\pi_2(z(\eta,t)) = t$. Hence there is a $t \in [0,1)$ such that the restriction $\pi_2 | L(\eta) \cap \pi_2^{-1}((t,1))$ is one to one. We write t_η for the smallest such t . Evidently, t_η is defined by a first order formula and hence is a semi-algebraic function of η . Set $L(\eta,t) = L(\eta) \cap \pi_2^{-1}((t_\eta,1))$, and note that $L(\eta,t)$ is a connected one dimensional manifold (since it is homeomorphic to the interval $(t_\eta,1)$) . Our first task is isolate the relevant portion of $L(\eta)$.

LEMMA 1: There is a two dimensional algebraic surface K and a finite subset E of $(0,1]$ such that:

(1) $K^* = K \cap W^* \subset L$;

(2) for all η , $K(\eta) \cap \pi_2^{-1}((t_\eta,1)) = L(\eta,t)$;

(3) if $\eta \notin E$ then $K(\eta)$ is a one dimensional algebraic curve.

PROOF: Set $L^0 = \{w \in L : \pi_2(w) > t_{\pi_1(w)}\}$. Since t_η is a semi-algebraic function of η , L^0 is a semi-algebraic set. Note that for each η ,

$$L^0(\eta) = \{w \in L^0 : \pi_1(w) = \eta\} = L(\eta,t)$$

and that this is a one dimensional curve. In particular, L^0 is a two dimensional semi-algebraic set, and its Zariski closure $K = \mathrm{Zar}(L^0)$ is a two dimensional algebraic surface. Since $L^0 \subset L$ and L is the intersection of W^* with an algebraic set, we conclude that $K^* = K \cap W^* \subset L$. In particular, $K(\eta) \cap \pi_2^{-1}((t_\eta,1)) = L(\eta,t)$.

Since K is two dimensional and $K(\eta) \cap \pi_2^{-1}((t_\eta,1)) = L(\eta,t)$, which is one dimensional, we conclude in particular that $K(\eta)$ has dimension at least one and at most two (for every η). Moreover, the set of η for which $K(\eta)$ has dimension two is a semi-algebraic set; hence either it is finite or it contains an interval. However, if it contains an interval, K would necessarily have dimension at least three, a contradiction. Hence, for all but a finite number of η the set $K(\eta)$ is an algebraic curve. We write E for the (possibly empty) set of η for which $K(\eta)$ is not an algebraic curve. ■

For each $\eta \notin E$, $K(\eta)$ is an algebraic curve so $K(\eta) \cap W$ is a one dimensional analytic curve. Let $C(\eta)$ be the analytically irreducible branch of $K(\eta) \cap W$ that contains $L(\eta,t)$. (That is, $C(\eta)$ is the smallest analytic subset of $K(\eta) \cap W$ containing $L(\eta,t)$.) Set $C = \{w \in W : \pi_1(w) \notin E$ and $w \in C(\pi_1(w))\}$.

It is important to note that the set $C(\eta)$ is uniquely defined, and independent of the choices made in this construction. Indeed, it follows immediately from uniqueness of analytic continuation that for $\eta \notin E$ and $t_\eta < t < 1$, $C(\eta)$ is the unique irreducible analytic curve in W that contains $L(\eta) \cap \pi_2^{-1}((t,1))$.

LEMMA 2: C is a semi-algebraic set, and, for each $\eta \notin E$, $C(\eta)$ is a semi-algebraic set.

PROOF: We are going to provide a first order description of $C(\eta)$; to do so, we use some of the structure theory of algebraic curves. (See Bochnak-Coste-Roy [1988] and Milnor [1968].) By a *local algebraic curve* (in $\mathbb{R}^n$) we mean the intersection of an algebraic curve with an open semi-algebraic subset of $\mathbb{R}^n$. For J a local algebraic curve and $z \in J$, we say that z is a *(topological) regular point* if some neighborhood of z in J is an arc; otherwise, z is a *branch point*. (A word of caution: this is a topological notion of regularity, not a smooth notion. A point may be be regular in this sense and still be a cusp.) An alternate description of regular and branch points can be given in the following way. Write $B(z,\varepsilon)$ for the open ball with center z and radius ε, and $\partial B(z,\varepsilon)$ for its boundary. For any point z in the algebraic curve J, and for all $\varepsilon > 0$, we consider the number of points in the intersection $\partial B(z,\varepsilon) \cap J$. If ε is sufficiently small, then this number of points becomes independent of ε, and is an even integer $2\beta(z)$. The integer $\beta(z)$ is called the *branching order* of J at z; z is a branch point exactly if the branching order is greater than 1.

We write J_b for the (necessarily finite) set of branch points of J and $J_r = J \setminus J_b$ for the set of regular points; note that J_b and J_r are semi-algebraic sets. Let $J_r = \cup A_i$ be the decomposition of J_r into (a finite number of) connected components; A_i is an arc and is a semi-algebraic set. We usually refer to the sets A_i as the *regular branches* of J.

We say that connected components A_i, A_j of J_r (or more generally, any pair of disjoint connected subsets of J) are *continuations of each other at the branch point* z if z is a limit point of both A_i and A_j and there are a neighborhood U of z, and connected components $\hat{A}_i$ and $\hat{A}_j$ of $A_i \cap U$ and $A_j \cap U$ respectively such that $\hat{A}_i \cup \hat{A}_j \cup \{z\}$ is an arc and is an analytic subset of U. (We cannot insist that $A_i \cup A_j \cup \{z\}$ itself be an analytic set, because of the possibility of loops. Note that this definition is entirely local in nature. See Figure 2.)

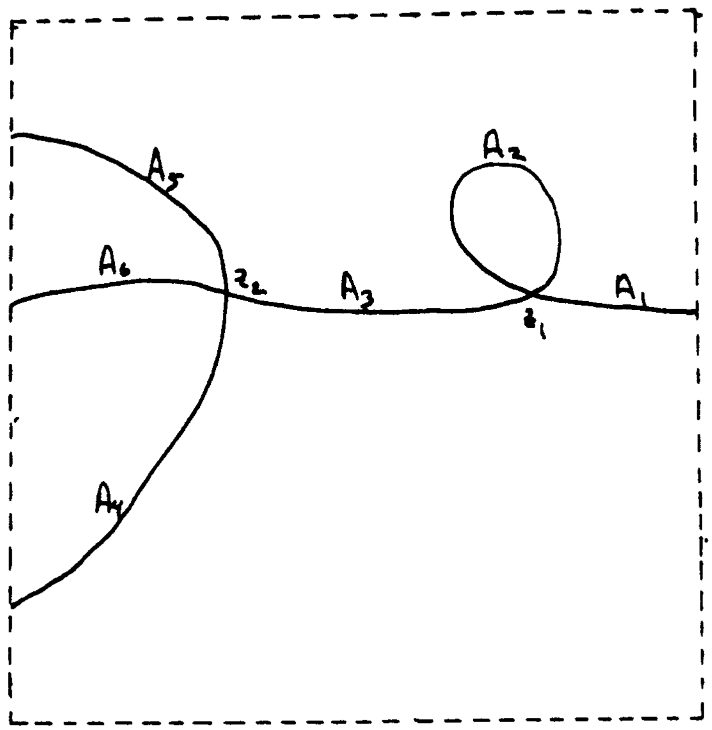

FIGURE 2

A_1, A_2 and A_2, A_3 are continuations of each other at z_1.
A_3, A_6 and A_5, A_6 are continuations of each other at z_2.
A_3, A_5 are not continuations of each other at z_2.

We will show that $C(\eta)$ is the union of regular branches of $K(\eta) \cap W$, together with some branch points. Hence to show that $C(\eta)$ is semi-algebraic (and depends semi-algebraically on η), we must deal with the problem of recognizing when two regular branches of a local algebraic curve are continuations of each othe at a point z. We show that this analytic problem can in fact be reduced to a semi-algebraic problem. In essence, this is because the algebraic nature of the curve implies that two regular branches are continuations of each other at z if they agree to sufficiently high order at z. The first step in this program is to reduce to a planar problem (i.e., a problem in $\mathbb{R}^2$.)

Consider a linear mapping $\theta : \mathbb{R}^n \to \mathbb{R}^2$ with the property that there is a finite set $J' \subset J$ such that $\theta|(J \setminus J')$ is one to one. (Such linear maps always exist (Bochnak-Coste-Roy [1988, p. 203]). The image of J is a semi-algebraic set and so belongs to the Boolean algebra generated by a finite number of polynomials; without loss, we may assume that none of these polynomials is identically zero. The set M of common zeroes of these polynomials is a plane algebraic curve contining $\theta(J)$. If the connected components A_i, A_j of J_r are continuations of each other at the branch point z, then the images $\theta(A_i)$, $\theta(A_j)$ are continuations of each other at $\theta(z)$. On the other hand, if z is a branch point of J which is in the closure of both A_i and A_j and A_i, A_j are not continuations of each other at z, then there is another component A_k which is a continuation of A_i at z. It follows that $\theta(A_i)$ and $\theta(A_k)$ are continuations of each other at $\theta(z)$. Since $\theta|J$ is one to one on the complement of a finite set, uniqueness of analytic continuation implies that $\theta(A_j)$ cannot also be a continuation of $\theta(A_i)$ at $\theta(z)$. Conclusion: A_i and A_j are continuations of each other at z if and only if $\theta(A_i)$ and $\theta(A_j)$ are continuations of each other at $\theta(z)$.

We have now reduced our original problem to a two dimensional problem; the next step is to reduce it to the solution of a power series equation. The image $\theta(J)$ is a semi-algebraic set and is contained in the algebraic curve M, which is the set of common zeroes of a finite number of polynomials, and hence of a single polynomial f. In view of Proposition 3, there is a bound on the degree d of this polynomial that depends only on J, and not on the choice of linear mapping θ. Fix a point $p = (x_0, y_0) \in M$. If V is a sufficiently small neighborhood of p, then $M \cap V$ is the union of a finite number of analytic arcs, each containing p; these are the *local analytic branches* of M at p. Puiseaux's theorem (see Milnor [1968] for example), implies that, (for V sufficiently small) each local analytic branch of M at p can be parametrized by a pair of functions:

$$x(t) = x_0 + t^\mu \quad , \quad y(t) = y_0 + \sum_{i=1}^{\infty} c_i t^i$$

where μ is an integer not exceeding the degree of f and the power series $\sum c_i t^i$ is convergent on some interval $(-\varepsilon, +\varepsilon)$. (Or else such a parametrization exists with the roles of x, y interchanged, a possibility we shall henceforth not mention.) Given disjoint connected subsets M_1, M_2 of M, it follows (using uniqueness of analytic continuation) that M_1 and M_2 are continuations of each other at p if and only if there exist an integer μ and a convergent power series $\sum c_i t^i$ such that for arbitarily small $\varepsilon > 0$, the image of the interval $(-\varepsilon,+\varepsilon)$ under the mapping $x(t) = x_0 + t^\mu$, $y(t) = y_0 + \sum c_i t^i$ is contained in M and meets both $M_1 \cup M_2$. Note that, if this is the case, then the image of $(-\varepsilon,0)$ must be contained in one of M_1, M_2 and the image of $(0,+\varepsilon)$ must be contained in the other.

We now show that this power series problem can be reduced to a semi-algebraic problem. To do this we make the following observation. Suppose we are given an integer μ and a polynomial $g(t)$ of degree ℓ and no constant term; we ask: when it is possible to find a convergent power series $\sum c_i t^i$ whose first ℓ terms agree with $g(t)$, such that the functions $x(t) = x_0 + t^\mu$, $y(t) = y_0 + \sum c_i t^i$ parametrize one of the local analytic branches of M at p ? Since M is the zero set of the polynomial f, this will be possible if and only if it is possible simply to find a convergent power series $\sum c_i t^i$ whose first ℓ terms agree with $g(t)$ such that $f(x(t),y(t)) = 0$. An elementary calculation with coefficients (or an appeal to a result of E. Cartan) shows that this will be possible if $\ell \geq \mu d + 1$ (where d is the degree of the polynomial f) and (1) $f(x_0 + t^\mu, y_0 + g(t))$, which is a polynomial in t, has no terms of order less than $\mu d + 1$. Given such a polynomial $g(t)$, Taylor's theorem and the observation at the end of the last paragraph imply that, in order that the parametrization $(x(t),y(t))$ meet both M_1 and M_2 and be contained in $M_1 \cup M_2$, we must also have, in addition to the above (reversing the roles of M_1 and M_2 if necessary): (2) for all sufficiently small $\varepsilon > 0$, and all $t \in (-\varepsilon,0)$, there is a point $(x_1,y_1) \in M_1$ such that:

$$|x_1 - x_0 - t^\mu| < |t|^{(\mu d+(1/2))} \text{ and } |y_1 - y_0 - g(t)| < |t|^{(\mu d+(1/2))}$$

(3) for all sufficiently small $\varepsilon > 0$, and all $t \in (0,+\varepsilon)$, there is a point $(x_2,y_2) \in M_2$ such that

$$|x_2 - x_0 - t^\mu| < |t|^{(\mu d+(1/2))}$$

$$|y_2 - y_0 - g(t)| < |t|^{(\mu d+(1/2))}$$

That is, M_1 and M_2 are continuations of each other at $p = (x_0,y_0)$ if and only if there is an integer $\mu \leq d$ and a polynomial $g(t)$ of degree $\mu d + 1$ and no constant term satisfying the conditions (1), (2), (3). The existence of such an integer and polynomial is a first order statement. (Because it amounts to the the assertion that there is a polynomial of degree $d + 1$ or a polynomial of degree $2d + 1$, . . . , or a polynomial of degree $d^2 + 1$, and because asserting the existence of a polynomial of degree ℓ with no constant term really amounts to asserting the existence of ℓ real numbers - the coefficents of the polynomial.)

Putting all of this together, we see that we have reduced the analytic problem of determining when two regular branches of the local algebraic curve J are continuations of each other to the existence of a linear mapping θ, an integer $\mu \leq d$ and a polynomial $g(t)$ of degree at most d. This is a first order - hence semi-algebraic - problem, as desired.

Specializing to the local algebraic curve $K(\eta) \cap W$, we write $K_b(\eta)$ for the set of branch points, $K_r(\eta)$ for the set of regular points, and $K_r(\eta) = \cup A_i(\eta)$ for the decomposition into regular branches. We may number these connected components so that $A_1(\eta)$ is the regular branch containing $L(\eta,t)$. We can describe $C(\eta)$ as the closure (in W) of the union of all those connected components $A_i(\eta)$ of $K_r(\eta)$ which are either equal to $A_1(\eta)$, or are continuations of $A_1(\eta)$ at some branch point, or are continuations of a regular branch which is a continuation of $A_1(\eta)$, etc. Since each of the regular branches $A_i(\eta)$ is semi-algebraic, the set $C(\eta)$ is the closure of the union of (a finite number of) semi-algebraic sets and hence is semi-algebraic.

What remains is to show that the set C is semi-algebraic. To do this, we first appeal to Proposition 3 to guarantee the existence of an integer D such that, for every $\eta \notin E$ and every linear mapping $\theta : \mathbb{R}^{2+\ell} \to \mathbb{R}^2$, the image $\theta(K(\eta) \cap W)$ is contained in an algebraic curve defined by a single polynomial of degree at most D. We then appeal to generic triviality to conclude that there are a finite number of $\eta_i \in (0,1)$, with $0 = \eta_1 < \eta_2 \ldots < \eta_k = 1$, semi-algebraic curves J_i and semi-algebraic homeomorphisms

$$\Phi_i : (\eta_i, \eta_{i+1}) \times J_i \to C \cap \pi_1^{-1}((\eta_i, \eta_{i+1}))$$

which map $\{\eta\} \times J_i$ onto $C(\eta)$ for each $\eta \in (\eta_i, \eta_{i+1})$. In particular, for each i and each $\eta \in (\eta_i, \eta_{i+1})$, the curves J_i and $C(\eta)$ have the same number of regular branches; we let $J_{i1}, \ldots, J_{im}$ be the regular branches of J_i. Since $A_1(\eta)$ is the regular branch of $C(\eta)$ that contains

$L(\eta,t)$, we may number so that $\Phi_i(\eta,J_{i1}) = A_1(\eta)$. We can now write a first order description of the set C: it is the set of all points $z = (\eta,t,q)$ in $K(\eta) \cap W$ satisfying one of the following statements:

(a) $\eta = \eta_i$ for some i, and $z \in C(\eta)$;

(b) $\eta \in (\eta_i,\eta_{i+1})$, and $z \in \Phi_i(\eta,J_{i1})$ or there is an index j such that $z \in \Phi_i(\eta,J_{ij})$ and $\Phi_i(\eta,J_{ij})$ and $\Phi_i(\eta,J_{i1})$ are continuations of each other, or there is an index j and an index k such that $z \in \Phi_i(\eta,J_{ik})$ and $\Phi_i(\eta,J_{ik})$ and $\Phi_i(\eta,J_{ij})$ are continuations of each other and $\Phi_i(\eta,J_{ij})$ and $\Phi_i(\eta,J_{i1})$ are continuations of each other, or . . . ;

(c) $\eta \in (\eta_i,\eta_{i+1})$, and $z \in K_b(\eta)$ and for every $\delta > 0$ there is a point $z' \in K(\eta)$ such that $|z - z'| < \delta$ and z' satisfies (b) above.

Since we have reduced the problem of determining when regular branches of $C(\eta)$ are continuations of each other to a first order problem this provides a first order description of C. The Tarski-Seidenberg theorem therefore guarantees that C is a semi-algebraic set. ■

With this hard work out of the way, the remainder is relatively straightforward.

LEMMA 3: For each $\eta \notin E$, there is an $s \in (0,1)$ such that the restriction $\pi_2 | C(\eta) \cap \pi_2^{-1}((0,s))$ is a one to one map onto the interval $(0,s)$.

PROOF: Since π_2 is a semi-algebraic map and $C(\eta)$ is a semi-algebraic set, it follows that for each $t \in (0,1)$ the set $C(\eta) \cap \pi_2^{-1}(t)$ is semi-algebraic and of dimension zero or one. If $C(\eta) \cap \pi_2^{-1}(t)$ were of dimension one then it would contain a relatively open subset of $C(\eta)$, which would imply that all of $C(\eta)$ lies entirely in the hyperplane $\pi_2^{-1}(t)$, an absurdity. We conclude that for each $t \in (0,1)$, $C(\eta) \cap \pi_2^{-1}(t)$ is a finite set.

Generic triviality implies that there is a $\delta > 0$ and an integer r (possibly 0) such that $C(\eta) \cap \pi_2^{-1}((0,\delta))$ is the union of arcs $A_1,\ldots,A_r$ with the following properties: (1) each A_i is a relatively open subset of $C(\eta) \cap \pi_2^{-1}((0,\delta))$; (2) $A_i \cap A_j = \emptyset$ if $i \neq j$; (3) the restriction $\pi_2 | A_i$ is a homeomorphism of A_i onto the interval $(0,\delta)$. The desired conclusion is that $r = 1$.

That $r \neq 0$ follows as in Harsanyi-Selten [1988]. Because $C(\eta)$ is an analytic curve in W , every point of $C(\eta)$ has a neighborhood in $C(\eta)$ which is the union of a finite number of open (analytic) arcs. Hence if we delete from $C(\eta)$ a small neighborhhood of $z(0,1)$, what remains must have a limit point ζ on the boundary of W ; i.e., $\zeta \in \overline{W} \setminus W$. Moreover, the strategy part $\sigma(\zeta)$ must be an equilibrium for the game $\Gamma(\pi_1(\zeta),\pi_2(\zeta))$. Since $\pi_1(\zeta) = \eta \in (0,1)$, this can only be the case if $\pi_2(\zeta) = 0$, or $\pi_2(\zeta) = 1$, or $0 < \pi_2(\zeta) < 1$ and $\sigma(\zeta) \in \Delta \setminus \Delta^*$. The last of these is impossible, since for $0 < \eta < 1$ and $0 < t < 1$, every equilibrium of the game $\Gamma(\eta,t)$ is in completely mixed strategies. The second of these is impossible because it would imply that, for t arbitrarily close to 1 , the game $\Gamma(\eta,t)$ had two distinct equilibria. We conclude that $\pi_2(\zeta) = 0$ and hence that $r \neq 0$.

It remains to show that r cannot exceed 1 . Since $C(\eta)$ is an irreducible analytic curve, there is a real analytic function $\varphi : (0,1) \to C(\eta)$ and a finite set $B \subset (0,1)$ such that $\varphi | (0,1) \setminus B$ is a homeomorphism onto $C(\eta) \setminus \varphi(B)$. Shrinking δ if necessary, there is no loss of generality in assuming that $\varphi^{-1}(A_i) \cap B = \emptyset$ for each i . It follows that the sets $\varphi^{-1}(A_i)$ are disjoint open intervals in $(0,1)$; say that $\varphi^{-1}(A_i) = (\alpha_i,\beta_i)$. For each i , $\varphi | (\alpha_i,\beta_i)$ is a homeomorphism onto A_i and $\pi_2 | A_i$ is a homeomorphism onto $(0,\delta)$, so it follows that $\pi_2 \circ \varphi | (\alpha_i,\beta_i)$ is a homeomorphism onto $(0,\delta)$. Now if $r \geq 2$ then we can find indices j , k such that $\beta_j < \alpha_k$. Since $\pi_2 \circ \varphi$ maps (α_k,β_k) homeomorphically onto the interval $(0,\delta)$, and $\pi_2 \circ \varphi$ maps all of $(0,1)$ onto $(0,1)$, it can only be that $\pi_2 \circ \varphi(\alpha_k) = \delta$ and $\beta_k = 1$. Similarly, we conclude that $\pi_2 \circ \varphi(\beta_j) = \delta$ and that $\alpha_j = 0$. But this means that $(0,1)$ is the union of the sets $\pi_2 \circ \varphi((0,\beta_j))$, $\pi_2 \circ \varphi((\alpha_k,1))$ and $\pi_2 \circ \varphi([\beta_j,\alpha_k])$. However, the first two of these coincide with the interval $(0,\delta)$, and the third is compact,which contradicts the fact that the image of $\pi_2 \circ \varphi$ is the entire interval $(0,1)$. We conclude that $r = 1$, as desired. ∎

Now, for each $\eta \notin E$, we define s_η to be the largest s so that the restriction $\pi_2 | C(\eta) \cap \pi_2^{-1}((0,s))$ is one to one. Since s_η is defined by a first order formula, it is a semi-algebraic function of η . For $\eta \notin E$ and $0 < t < s_\eta$, we write $z(\eta,t)$ for the unique point in $\pi_2 | C(\eta) \cap \pi_2^{-1}((0,s_\eta))$.

We have shown that $C(\eta)$ depends semi-algebraically (not algebraically!) on η ; it follows that $z(\eta,t)$ depends semi-algebraically on η and t .

LEMMA 4: (1) $z(\cdot,\cdot)$ is a semi-algebraic function of η and t .

(2) The limit

$$z(\eta) = \lim_{t \to 0} z(\eta,t)$$

exists and is a semi-algebraic function of η .

(3) The limit $\lim_{\eta \to 0} z(\eta)$ exists.

PROOF: To see that $z(\cdot,\cdot)$ is a semi-algebraic function of η and t , note that its domain is $D = \{(\eta,t) \in (0,1) \times (0,1) : t < s_{\eta}\}$ (which is a semi-algebraic set because $s(\cdot)$ is a semi-algebraic function) and that its graph is $\{w \in C : (\pi_1(w), \pi_2(w)) \in D\}$ (which is a semi-algebraic set because π_1 and π_2 are semi-algebraic - indeed, linear - functions). The existence of the limit $z(\eta)$ follows immediately from piecewise monotonicity of the semi-algebraic functions which are the components of $z(\eta,\cdot)$. To see that $z(\cdot)$ is itself a semi-algebraic function, write its graph as:

$$\text{graph}(z(\cdot)) = \{(\eta,w) : \forall \varepsilon > 0 , \exists t \text{ such that } 0 < t < \varepsilon \text{ and } |w - z(\eta,t)|^2 < \varepsilon^2 \}$$

and apply the Tarski-Seidenberg theorem. Finally, the existence of the limit $\lim z(\eta)$ follows immediately from piecewise monotonicity. ∎

We define the *logarithmic solution for the game* Γ to be the strategy part of the limit $\lim z(\eta)$. That is, the logarithmic solution for the game Γ is $\lambda = \sigma(\lim z(\eta)) = \lim \sigma(z(\eta))$.

This procedure depends of course on the prior probability distribution p and on the game Γ . If we fix the game form and consider the game as parametrized by the payoffs u , we may view the logarithmic solution λ as a function of the prior probability distribution p and the payoffs u . Routine modifications of Lemmas 1 - 4 show that $\lambda(p,u)$ depends semi-algebraically on p and u . Summarizing:

THEOREM 2: The logarithmic solution $\lambda(p,u)$ is well-defined, and is a semi-algebraic function of the prior probability distribution p and the payoffs u .

In particular, if we view the prior probability distribution as fixed, and view the logarithmic solution as a function solely of the payoffs u , we conclude that it is continuous at every point of a dense open set whose complement has (Lebesgue) measure zero.

We turn now to the linear tracing procedure. Write

$$G = \{(0,t,q) : q \text{ is an equilibrium of the game } \Gamma(0,t)\}$$

A *linear trace* is by definition a curve in G from $z(0,1)$ (the strategy part of which is the unique equilibrium of the game $\Gamma(0,1)$) to a point of the form $(0,0,q)$ (so that in particular, q is an equilibrium of the game Γ). If a linear trace exists and is unique, we say the *linear tracing procedure is well-defined* and the strategy q is the *linear solution* of the game Γ . Harsanyi and Selten assert that there is always a linear trace from $z(0,1)$ to $(0,0,\lambda)$, where λ is the logarithmic solution. (In particular, if the linear tracing procedure is well-defined then the the linear solution and the logarithmic solution coincide.) To deduce this, Harsanyi and Selten give the following argument. For each η , parametrize the curve $L(\eta)$ by arc length, normalized so that the total length of $L(\eta)$ is 1 . For $s \in [0,1]$, let $\omega(\eta,s)$ be the point in $L(\eta)$ whose distance from $z(\eta,1)$ is s . Define $\omega(s) = \lim \omega(\eta,s)$, so that the function ω defines a curve in G , beginning at $z(0,1)$ and ending at $(0,0,\lambda)$.

Unfortunately this argument is not correct. The first difficulty is that, just as above, the set $L(\eta)$ need not be a curve. Even if it is a curve, the function $\omega(\eta,s)$ need not depend on the parameter η in any sort of algebraic or semi-algebraic way, so that $\omega(s) = \lim \omega(\eta,s)$ need not be defined. (Keep in mind that arclength is not an algebraic- or even a semi-algebraic - function!) And even if $\omega(s) = \lim \omega(\eta,s)$ is defined, it will not in general be a continuous function of s , so that we will not obtain the desired curve beginning at $z(0,1)$ and ending at $(0,0,\lambda)$.

· However, our construction provides a straighforward route around these difficulties.

THEOREM 3: There is a linear trace from $z(0,1)$ to $(0,0,\lambda)$, where λ is the logarithmic solution. In particular, the logarithmic solution and linear solution coincide whenever the latter is unique.

PROOF: Let Z be the limiting set, as $\eta \to 0$, of $C(\eta)$; i.e.,

$$Z = \{z : \forall\varepsilon > 0, \exists\eta, \exists w \in C(\eta) \text{ such that } 0 < \eta < \varepsilon \text{ and } |w - z|^2 < \varepsilon^2\}$$

It is evident that Z is a semi-algebraic set. As before, we see that every point $z \in Z$ is an equilibrium for the game $\Gamma(\pi_1(z),\pi_2(z))$. On the other hand, $\pi_2(z) = 0$ for every $z \in Z$, so $Z \subset G$. Since $z(\eta,1)$, $z(\eta) \in C(\eta)$ for each η, we see that $z(0,1)$ and $(0,0,\lambda)$ belong to Z. Since each $C(\eta)$ is connected, so is Z. Finally, since Z is a connected semi-algebraic set, it follows from triangulability that every two points of Z lie on a curve in Z. ∎

5. EQUILIBRIUM SELECTION

As we have shown, the tracing procedure yields, for each game, a well-defined logarithmic solution, but the tracing procedure itself constitutes only a part of the entire Harsanyi-Selten equilibrium selection procedure. However, as the sketch below shows, the results of Section 4 yield rather easily that the entire Harsanyi-Selten equilibrium selection procedure is well-defined. (For details on the selection procedure we refer to Harsanyi-Selten [1988], and especially to the Summary of Procedures, Section 5.7.)

To apply the equilibrium selection procedure to a game Γ (in standard form), we begin by constructing, for all sufficiently small $\varepsilon > 0$, the uniformly perturbed game Γ_ε. For each of these games, we apply the appropriate decompositions and reductions, and - if necessary - apply the tracing procedure to find the solution $\zeta(\varepsilon)$ for the game Γ_ε. It is easy to see that the decompositions and reductions are semi-algebraic; thus, in view of Theorem 2, above, the solution $\zeta(\varepsilon)$ depends semi-algebraically on the parameter ε. Piecewise monotonicity assures us that the solutions $\zeta(\varepsilon)$ approach a limit ζ as ε approaches 0, and this limit is the solution for the original game Γ.

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