Collateral and the Enforcement of Intertemporal Contracts*

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Abstract

This paper argues that the reliance on collateral to secure loans has a profound impact on the allocation of commodities, on the efficiency of markets, on the prices of commodities and assets, and on the volatility of these prices. These effects can be clearly seen even in situations in which there is no default.

Among the reasons for these effects are: i) collateral is scarce; ii) the necessity for sellers to purchase collateral imposes an endogenous borrowing constraint; iii) the necessity for sellers to purchase collateral distorts consumption decisions; iv) the possibility of surrendering collateral limits deliveries on promises and hence inhibits lending; v) collateral may entail a dead-weight loss.

To make these points, this paper formulates an extension of intertemporal general equilibrium theory that incorporates durable goods, collateral and the possibility of default, and establishes the existence of collateral equilibrium under standard assumptions. A variety of examples and theorems demonstrate the impact of collateral on economic variables.

1 Introduction

What enforces intertemporal contracts? Although many different mechanisms have been in use in various times and places, a large fraction of all loans in modern economies are secured by some form of collateral that will be forfeit in the event of default. The variety of such collateralized lending is enormous: pawn shop loans, home mortgages, corporate stocks and bonds, margin purchases of stock, overnight repurchase agreements, derivatives, and many others. The collateral used to secure such loans may be a watch, a house, a factory, stock in a firm, government debt, or even other collateralized contracts. The collateral may be kept in a warehouse, or held for use by the borrower or by the lender. The total of all such collateralized loans is many trillions of dollars. (For example, the total value of the U.S. residential mortgage market exceeds \$5 trillion, and the total value of publically traded stock exceeds \$10 trillion.) Remarkably, however, most of the attention paid to collateralized borrowing has been in the financial literature, where the emphasis has been on the pricing of collateralized instruments, especially derivatives, and in the popular press, where the emphasis has been on the dangers of such derivatives.

In this paper we argue that the reliance on collateral to secure loans has a profound impact on the allocation of commodities, on the efficiency of markets, on the prices of commodities and assets, and on the volatility of these prices — in short, on every aspect of the economy. Moreover, these effects can be clearly seen even in situations in which there is no default.

Perhaps the most important reason for these effects is that collateral is scarce. Even in a world in which every conceivable asset is available for trade, the scarcity of collateral means that some assets will not be traded, and that smoothing of consumption and sharing of risk will be correspondingly inhibited. The scarcity of collateral creates incentives to find innovations that economize on collateral. Indeed, much of the financial engineering that has rapidly accelerated in the last decade is designed explicitly to cope with this problem of scarce collateral. Scarce collateral can be stretched by permitting the same physical collateral to be used many times: allowing the same collateral to back many different promises (tranching), or permitting assets to be collateralized by other assets (pyramiding). These two innovations are at the bottom of the securitization and derivatives boom on Wall Street, and have greatly expanded the scope of financial markets.

Even when collateral is not scarce, however, reliance on collateral may have enormous effects. Because sellers of assets (borrowers) must purchase the collateral which secures the promises they are making, the collateral requirement both imposes an endogenous borrowing constraint and distorts consumption decisions. Because buyers of assets (lenders) know that deliveries on these assets are limited by the value of the collateral that secures them, the reliance on collateral inhibits lending. And because collateral may not be held by the agents who value the collateral most, the collateral requirement entails a potential dead-weight loss to society.

To make these points, we formulate and analyze a straightforward extension of intertemporal general equilibrium theory that incorporates durable goods, collateral and the possibility of default. (We distinguish between goods such as gold, which are durable in the sense of being usable many times, and tobacco, which can be stored but can be used only once.) Although default is suggestive of disequilibrium, we show that imposing collateral requirements and allowing for default pose no special technical problems; under the hypotheses on agent behavior and foresight that are standard in the general equilibrium literature, collateral equilibrium always exists. Indeed, our model is better behaved in this regard than is the standard general equilibrium model with asset markets; see the discussion in Section 3.

As is usual in general equilibrium theory, we view individuals as anonymous price-takers. Anonymity in particular might appear strange in an environment in which individuals might default. In our context, however, individuals will default when the value of promises exceeds the value of collateral and not otherwise; thus lenders do not care about the identity of borrowers, but only about the collateral they bring. In particular, we do *not* require a pooling interpretation, as in Dubey, Geanakoplos, and Shubik (1990). For simplicity, we build a model with a finite number of agents; the assumptions of anonymity and price-taking could be made more transparent by building a model with a continuum of individuals, but that would complicate the model without adding anything of substance. Alternatively, the reader might wish to imagine a continuum of consumers of each type.

To focus the discussion, we assume here that agents may default on some promises and not on others, and that the only consequence of default is forfeiture of collateral. For pawn shop loans, overnight repurchase agreements, margin loans, and home mortgages — at least in California — these assumptions are a reasonably close approximation of reality. But our work should probably not be interpreted as providing a model of bankruptcy, which will typically involve default on all promises followed by a broad spectrum of consequences in addition to forfeiture of collateral. Hellwig (1981b) and Sabarwal (1999) provide theoretical models of bankruptcy that encompass various consequences, and Fay, Hurst, and White (forthcoming) and Lin and White (forthcoming) provide an interesting empirical treatment.

In most present-day Western societies, the consequences of default are primarily economic — seizure of possessions and partial or total exclusion from credit markets in addition to forfeiture of collateral — but it is useful to recall that in other times and places extra-economic consequences as extreme as hanging (12th Century France) or exile to the colonies (18th Century England) have played an important role. Indeed, debtor's prisons were in widespread use in Western societies into the middle of the 19th Century. See Dubey, Geanakoplos, and Shubik (1990) and Zame (1993) for a general equilibrium treatment of extra-economic penalties and Sabarwal (1999) for a general equilibrium treatment of exclusion. Kehoe and Levine (1993) treat a world in which the possibility of default constrains borrowing, although default does not occur at equilibrium. For a model driven by strategic considerations, see Hart and Moore (1995).

Because promises are collateralized, borrowers and lenders must take into account the future value of commodities. We assume here that durable goods held by the borrower and durable goods held by the lender will have the same value tomorrow, but it would be a simple matter (requiring only a notational change) to distinguish such goods. Importantly, doing so would not violate the anonymity that we wish to retain in our general equilibrium framework, nor create any problem of adverse selection, because the transaction itself would reveal the use to which the commodity is being put.

We take the set of assets available for trade as given exogenously; to avoid the difficulties inherent in treating infinite portfolios, we assume that this set of assets is finite. However, this set of assets might be *very large*; indeed, we might imagine that *many* assets, or even *all possible* assets are available for trade. Of course, not all assets need be traded at equilibrium, and the set of assets traded — as opposed to the set of assets potentially available for trade — will be determined *endogenously.* In particular, we may view the *market* as determining the asset structure — and collateral requirements in particular. For a different view of the endogenous determination of collateral requirements, see Araujo, Barbachan, and Pascoa (2001).

A few final remarks:

- It is important to keep in mind that, while asset promises are given exogenously, actual deliveries are determined endogenously in equilibrium. Of course, having solved for equilibrium, one could respecify asset promises to equal actual deliveries, so that there would be no default — but such a respecification would depend on the particular equilibrium.

- If one thinks of pawn shop loans as prototypical of collateralized borrowing, it might be tempting to think that a collateralized loan is equivalent to a sale and a future repurchase, and thus to think that the principle function of collateral is to economize on transaction costs. Even for pawn shop loans, however, this is quite wrong when — as is usually the case — there is uncertainty about asset promises or the future price of the collateral. Moreover, in many instances (home mortgages and consumer durables being the most obvious examples), the collateral that secures the loan is held and enjoyed by the borrower him/herself.

- Because borrowers can choose to surrender the collateral rather than keep the promise, a collateralized asset is an *option*, and our model shares a number of features with other models that incorporate options. In particular, the pricing of collateralized securities, like the pricing of options, is not generally linear. Since the celebrated Modigliani and Miller (1958) theorem on the irrelevance of capital structure depends on linear pricing, it need not hold in an environment in which assets are collateralized; the value of a firm may well depend on the way in which it chooses to finance its undertakings. For a slightly different take on this point, see Hellwig (1981a)

In the following sections, we address these various issues through several theorems and propositions and especially through a large collection of examples. Section 2 presents the formal model of an economy with durable goods and collateralized assets. In Section 3, Theorem 1 is our basic existence result, and Example 1 illustrates the way in which collateral requirements may limit borrowing and distort consumption choices. Section 4 explores the effects of collateral requirements by asking when a given Walrasian equilibrium can be supported as a collateral equilibrium for *some* family of assets. Theorem 2 gives a complete solution to this question for environments with no uncertainty; Example 2 presents a simple illustration. Example 3 shows that the situation is much more complicated in environments with uncertainty, and illustrates the usefulness of tranching. Example 4 in Section 5 illustrates the way in which the shortage of collateral determines the set of assets traded at equilibrium, and also that, in the presence of uncertainty, collateralized loans are not equivalent to sales and repurchases. Section 6 discusses the market choice of assets to be traded; Examples 5 and 6 illustrate the way in which the market chooses the assets that are traded, and Theorem 3 identifies a circumstance in which the market chooses the asset structure efficiently. As Example 7 shows, the question of whether, in general, the market chooses the asset structure efficiently is complicated by the possibility of multiple equilibria; a given economy may admit several collateral equilibria, of which one is efficient and one is not. Section 7 discusses the way in which margin requirements impact on the volatility of prices; Example 8 illustrates that lowering margin requirements can lead to a market crash. The Appendix records the rather messy proof of Theorem 1.

2 Durable Goods and Collateralized Assets

As in the canonical model of securities trading, we consider a world with two dates, where agents know the present but face an uncertain future. In date 0 (the present) a finite set of agents trade in a finite set of commodities and assets. Between date 0 and date 1 (the future) the state of nature is revealed. In date 1 assets pay off and commodities are traded once again. Our framework is formalized below.

2.1 Time and Uncertainty

There are two dates, 0 and 1, and a finite number S of exogenously given possible states of nature at date 1.

2.2 Durable Goods

There are $L \geq 1$ commodities available for trade and consumption at each date and state of nature, so the commodity space is $\mathbf{R}^L \times \mathbf{R}^{SL} = \mathbf{R}^{(S+1)L}$. For a commodity bundle $x \in \mathbf{R}^{(S+1)L}$ and indices s, ℓ, we write x_s for the vector of spot s consumption specified by x, and $x_{s\ell}$ for the quantity of commodity ℓ specified in spot s. We abuse notation and view $\mathbf{R}^L$ as the subspace of $\mathbf{R}^{(S+1)L}$ consisting those vectors which are 0 except in the first L coordinates; thus we identify a vector $x \in \mathbf{R}^L$ with $(x, 0, \ldots, 0) \in \mathbf{R}^{(S+1)L}$. Similarly we view $\mathbf{R}^{SL}$ as the subspace of $\mathbf{R}^{(S+1)L}$ consisting those vectors which are 0 in the first L coordinates. We write $\delta_{s\ell} \in \mathbf{R}^{(S+1)L}$ for the commodity bundle consisting of one unit of commodity ℓ in spot s and nothing else. We write $x \geq y$ to mean that $x_{s\ell} \geq y_{s\ell}$ for each s, ℓ, $x > y$ to mean that $x \geq y$ and $x \neq y$ and $x >> y$ to mean that $x_{s\ell} > y_{s\ell}$ for each s, ℓ.

Since the commodity space is $\mathbf{R}^{(S+1)L}$, so is the price space. In parallel with our notation for commodity bundles, we write p_s for the vector of prices in spot s and $p_{s\ell}$ for the price of commodity ℓ in spot s.

We depart from the usual intertemporal models by supposing that commodities may be durable and/or storable.[1] If the services of a (possibly durable) commodity bundle $x \in \mathbf{R}^L$ are used (consumed) at date 0, we write $F_s^U(x) \in \mathbf{R}^L$ for what remains in state s at date 1. If commodity bundle $x \in \mathbf{R}^L$ is warehoused (stored) at date 0, we write $F_s^W(x) \in \mathbf{R}^L$ for what remains in state s at date 1. We assume throughout that for each state s, the mappings $F_s^U, F_s^W : \mathbf{R}^L \to \mathbf{R}^L$ are linear and positive, so map $\mathbf{R}_+^L$ to $\mathbf{R}_+^L$.[2] The commodity 0ℓ is *durable* if $F_s^U(\delta_{0\ell}) \neq 0$ for some s, *storable* if $F_s^W(\delta_{0\ell}) \neq 0$ for some s, *non-perishable* if it is either durable or storable, and *perishable* otherwise.

2.3 Consumers

There are N consumers. Consumer i is described by a consumption set, which we take to be $\mathbf{R}_+^{(S+1)L}$, an endowment $e^i \in \mathbf{R}^{(S+1)L}$, and a utility function $u^i : \mathbf{R}^{(S+1)L} \to \mathbf{R}$. Standard assumptions about endowments and utility functions are collected below.

2.4 Assets

A *collateralized asset*, or just *asset* for short, is a pair (A, C); A is the *promise* or *face value*, C is the *collateral requirement*. If no confusion is likely, we abuse notation and write A for short. The promise A specifies units of account to be delivered in each state in date 1, as a function of the vector of all commodity prices, and so is a function $A : S \times \mathbf{R}_+^{(S+1)L} \to \mathbf{R}$.[3] We assume that asset promises are non-negative (so that $A(s,p) \geq 0$ for each s, p) and depend continuously on prices. This formulation is sufficiently general to encompass a wide variety of assets, including nominal assets, real assets, options and derivatives.

[1] Recall the distinction between durability and storability discussed in the Introduction.

[2] It would suffice to assume that F_s^U, F_s^W are concave; this would be particularly natural for an alternative interpretation of F^U, F^W as production functions.

[3] More generally, we could allow the face value of A_j to be a function of asset prices as well; this would be especially natural if we expanded the model to allow assets to be collateralized by other assets.

The collateral requirement C is a triple

$$C = \left(C^B, C^L, C^W\right) \in \mathbf{R}^L_+ \times \mathbf{R}^L_+ \times \mathbf{R}^L_+$$

which the seller of asset A must own. Of this requirement, C^B will be held by the borrower (the seller of the asset), C^L will be held by the lender (the buyer of the asset), and C^W will be warehoused (stored). In our framework, the collateral requirement is the only means for enforcing promises, so no agent will ever make positive deliveries on uncollateralized assets. Hence equilibrium prices for such assets will necessarily be 0 and trade in such assets will be irrelevant. It therefore involves no loss of generality to assume that collateral requirements are non-zero: $C^B + C^L + C^W > 0$. Moreover, optimization by agents implies that the *yield* (delivery) per unit of asset A_j in state s will not be the face value $A_j(s,p)$ but rather the minimum of the face value and the value of the collateral in state s:

$$Y_j(s,p) = \min\Big\{A_j(s,p)\ ,\ p_s \cdot [F^W_s(C^W_j) + F^U_s(C^L_j) + F^U_s(C^B_j)]\Big\}$$

We take as given a set of J assets (but recall from the discussion in the Introduction that J might be very large). Write $\varphi, \psi \in \mathbf{R}^J_+$ for the portfolios of asset purchases and sales, respectively, and $\theta = \varphi - \psi \in \mathbf{R}^J$ for the net portfolio of purchases minus sales. (In our model there is no advantage to buying and selling the same asset, so we could just as well take φ, ψ to be the positive and negative parts of θ, but keeping purchases separate from sales is notationally convenient.) We assume that buying and selling prices for assets are identical, so asset prices belong to $\mathbf{R}^J$. The buyer of φ_j units of asset j will obtain the use of the collateral bundle $\varphi_j C^L_j$. The seller of ψ_j units of asset j will have to own the collateral bundle $\psi_j(C^L_j + C^B_j + C^W_j)$ and will enjoy the use of $\psi_j C^B_j$.

2.5 The Economy

An *economy $\mathcal{E}$ with collateralized assets* consists of a finite family of consumers $\{(e^i, u^i) : i = 1, \ldots, N\}$, use and storage technologies F^U_s, F^W_s, and a finite family of assets $\{(A_j, C_j) : j = 1, \ldots J\}$. Write $\bar{e} = \sum_{i=1}^N e^i$ for the social endowment. The following assumptions are always in force:

A1 for each agent i: $e^i > 0$

A2 $\bar{e}_0 >> 0$

A3 for each state s: $\bar{e}_s + F_s^U(\bar{e}_0) + F_s^W(\bar{e}_0) >> 0$

A4 for each agent i: u^i is continuous and quasi-concave (for each i)

A5 for each agent i: if $x \geq y \geq 0$ then $u^i(x) \geq u^i(y)$

A6 for each agent i: if $x \geq y \geq 0$ and $x_{s\ell} > y_{s\ell}$ for some $s \neq 0$ and some ℓ, then $u^i(x) > u^i(y)$

A7 for each agent i: if $x \geq y \geq 0$, $x_{0\ell} > y_{0\ell}$, and commodity 0ℓ is perishable, then $u^i(x) > u^i(y)$

Assumptions **A1-A3** say that that individual endowments are non-zero and that all goods are represented in the aggregate. (Note that some goods — used cars or vintage wine, for instance — may come into being at date 1 only if they are used or stored at date 0.) Assumptions **A4-A6** say that utility functions are continuous, quasi-concave, weakly monotone, and strictly monotone in date 1 consumption; these are standard assumptions. The last assumption **A7** says that utility functions are strictly monotone in perishable date 0 goods. We do not require strict monotonicity in non-perishable date 0 goods because we allow for the possibility that claims to date 1 consumption are traded at date 0; of course, such claims would typically provide no utility at date 0.

2.6 Budget Sets

Given commodity prices, each agent makes plans for consumption, for storage, for asset purchases and sales, and for deliveries against promises. In view of our earlier comments, we assume that deliveries are precisely the minimum of promises and the value of collateral, so we henceforward suppress the choice of deliveries. We therefore define the budget set $B^i(p, q)$ for consumer i, given commodity prices p and asset prices q to be the set of *plans* $\pi^i = (x^i, y^i, \varphi^i, \psi^i)$ that satisfy the budget constraints at date 0 and in each state at date 1. This entails:

- At date 0:

$$p_0 \cdot x_0^i + p_0 \cdot y^i + p_0 \cdot \sum_j \psi_j^i \Big(C_j^B + C_j^L + C_j^W \Big) + q \cdot \varphi^i \ \leq \ p_0 \cdot e_0^i + q \cdot \psi^i$$

  That is, expenditures (for consumption, storage, collateral, and asset purchases) do not exceed income (from endowment and asset sales).

- In state s:

$$\begin{aligned} p_s \cdot x_s^i + \sum_j \psi_j^i Y_j(s,p) \quad \leq \quad & p_s \cdot e_s^i + \ p_s \cdot F_s^W(y^i) \\ & + p_s \cdot F_s^W(\sum_j \psi_j^i C_j^W) \\ & + p_s \cdot F_s^U(\sum_j \psi_j^i C_j^B) + p_s \cdot F_s^U(\sum_j \psi_j^i C_j^L) \\ & + p_s \cdot F_s^U(x_0^i) + \sum_j \varphi_j^i Y_j(s,p) \end{aligned}$$

  That is, expenditures (for consumption and deliveries on promises) do not exceed income (from endowment, goods removed from storage, return of warehoused collateral, return of used collateral, used goods held by the consumer, and collections on others' promises).

Of course agents *know* date 0 prices but must make *forecasts* about date 1 prices. Our equilibrium notion implicitly incorporates the requirement that price forecasts be correct, so we take the familiar shortcut of suppressing forecasts and treating all prices as known to agents at date 0. For a model in which forecasts might be incorrect, see Barrett (2000).

2.7 Utility and Optimization

Consumers choose plans to maximize their utility subject to the budget constraints. Given the plan $\pi^i = (x^i, y^i, \varphi^i, \psi^i)$, consumer i enjoys utility from consumption, from borrower-held collateral on assets he sells (loans he takes out) and from lender-held collateral on assets he buys (loans he makes).

Abusing notation to view u^i as a function of plans, this means that i's total utility will be:

$$u^i(\pi^i) = u^i\Big(x_0^i + \sum_j \psi_j^i C_j^B + \sum_j \varphi_j^h C_j^L \, , \, x_1^i, \ldots, x_s\Big) \tag{1}$$

2.8 Equilibrium

Given an economy $\mathcal{E}$ a *collateral equilibrium* consists of commodity prices $p \in \mathbf{R}_+^{(S+1)L}$, asset prices $q \in \mathbf{R}_+^J$ and consumer plans (π^h) satisfying the usual conditions:

- **Commodity Markets Clear in Date 0**

$$\sum_i \Big[x_0^i + y^i + \sum_j \psi^j (C_j^B + C_j^L + C_j^W)\Big] = \sum_i e_0^i$$

- **Commodity Markets Clear in State** s

$$\sum_i x_s^i = \sum_i \left[e_s^i + F_s^W \Big(y^i + \sum_j \psi_j^i C_j^W\Big) + F_s^U \Big(x_0^h + \sum_j \psi_j^i [C_j^B + C_j^L]\Big)\right]$$

- **Asset Markets Clear**

$$\sum_i \varphi^i = \sum_i \psi^i$$

- **Plans are Budget Feasible**

$$\pi^i \in B^i(p,q)$$

- **Consumers Optimize**

$$\eta^i \in B^i(p,q) \Rightarrow u^i(\pi^i) \geq u^i(\eta^i)$$

2.9 Rental Markets

Note that the services of a commodity at date 0 and at date 1 may not be marketed separately (as is usually assumed in the general equilibrium framework). Rental markets could be created by marketing date 0 and date 1 services separately, but the same effect can be achieved through the asset market, because a rental is precisely the purchase of a durable and the simultaneous sale of a promise to deliver the durable back again at the end of the rental period, collateralized by the durable itself. If $p_{0\ell}$ is the date 0 price of the durable and q is the date 0 price of this asset, then the cost of this transaction is $p_{0\ell} - q$, which may be interpreted as the rental price of the commodity 0ℓ.

2.10 Walrasian Equilibrium with Durable Goods

Because we are interested in the impact of collateral requirements, it will be useful to give a formal definition of Walrasian equilibrium in the present durable goods context; see Dubey, Geanakoplos, and Shubik (1992) for more details. We continue to treat a world with 2 dates $0, 1$, with S states of nature at date 1, with L commodities available for trade at each date and state, and with N consumers. A *durable goods economy* is a tuple

$$\mathbf{E} = \Big((e^i, u^i), (F_s^U, F_s^W)\Big)$$

where e^i, u^i are the endowment and utility function of agent i (consumption sets taken to be $\mathbf{R}_+^{(S+1)L}$) and F_s^U, F_s^W specify the durable/storable goods technologies, as above. If commodity prices are p, the budget set $B^i(p)$ for consumer i consists of plans x^i, y^i for consumption and storage such that

$$p \cdot x^i + p_0 \cdot y^i \leq p \cdot e^i$$

A *Walrasian equilibrium* consists of commodity prices $p \in \mathbf{R}_+^{(S+1)L}$ and plans (x^i, y^i) such that:

- **Commodity Markets Clear in Date 0**

$$\sum_i (x_0^i + y^i) = \sum_i e_0^i$$

- **Commodity Markets Clear in State** s

$$\sum_i x_s^i = \sum_i \left[e_s^i + F_s^U(x_0^i) + F_s^W(y^i) \right]$$

- **Plans are Budget Feasible**

$$(x^i, y^i) \in B^i(p)$$

- **Consumers Optimize**

$$(\bar{x}^i, \bar{y}^i) \in B^i(p) \Rightarrow u^i(\bar{x}^i) \leq u^i(x^i)$$

3 Equilibrium

Under the maintained assumptions discussed in Section 2, collateral equilibrium always exists. We defer the rather messy proof to the Appendix.

Theorem 1 *Every economy satisfying the assumptions* **A1 - A7** *set forth in Section 2 admits a collateral equilibrium.*

For those familiar with the incomplete markets literature, this may seem a surprising result, because we allow for real assets, options, derivatives, and even more complicated non-linear assets. In the standard framework, the presence of such assets implies that the space of feasible income transfers does not depend continuously on commodity prices, whence equilibrium may not exist. (See Hart (1975) for an example of non-existence of equilibrium, Duffie and Shafer (1985) and Duffie and Shafer (1986) for generic existence with real assets, and Ku and Polemarchakis (1996) for a robust example of non-existence of equilibrium with options.) In our framework, however, the requirement that asset sales be collateralized places an *endogenous* bound on short sales. As in the standard incomplete markets framework — see Radner (1972) for instance — a bound on short sales eliminates the discontinuity and guarantees the existence of equilibrium.

A simple home mortgage example contrasts collateral equilibrium with Walrasian equilibrium and illustrates the way in which collateral requirements limit borrowing and distort consumption choices.

Example 1 Consider a world with no uncertainty ($S = 1$), two consumers $1, 2$ and two goods F (food) and H (housing) at each date. Food is perishable (1 unit of food at date 0, when stored or used, yields nothing at date 1), housing is perfectly durable (1 unit of housing at date 0, when stored or used, yields 1 unit of housing at date 1).

Consumer 1 owns the housing stock; consumer 2 is poor at date 0 but wealthy at date 1:

$$\begin{aligned} e^1 &= (e^1_{0F}, e^1_{0H}, e^1_{1F}, e^1_{1H}) = (20, 1, 20, 0) \\ e^2 &= (e^2_{0F}, e^2_{0H}, e^2_{1F}, e^2_{1H}) = (4, 0, 50, 0) \end{aligned}$$

Consumer 1 finds housing and food to be perfect substitutes, and is perfectly patient; consumer 2 likes housing much more than consumer 1:

$$\begin{aligned} u^1(x_{0F}, x_{0H}, x_{1F}, x_{1H}) &= x_{0F} + x_{0H} + x_{1F} + x_{1H} \\ u^2(x_{0F}, x_{0H}, x_{1F}, x_{1H}) &= 9x_{0F} - 2(x_{0F})^2 + 15x_{0H} + x_{1F} + 15x_{1H} \end{aligned}$$

We compare equilibrium prices, consumptions, and utilities for the Walrasian equilibrium, the collateral equilibrium in which no assets are available for trade, and the collateral equilibrium in which a single asset A, promising the value of 15 units of food and collateralized by 1 house, is available for trade. We omit the straightforward but somewhat messy computations.

- **Walrasian Equilibrium**

$$\begin{aligned} p &= (p_{0F}, p_{0H}, p_{1F}, p_{1H}) &= (1, 30, 1, 15) \\ x^1 &= (x^1_{0F}, x^1_{0H}, x^1_{1F}, x^1_{1H}) &= (22, 0, 48, 0) \\ x^2 &= (x^2_{0F}, x^2_{0H}, x^2_{1F}, x^2_{1H}) &= (2, 1, 22, 1) \end{aligned}$$

Equilibrium utilities are $u^1 = 70, u^2 = 62$.

- **No Assets Equilibrium**

$$\begin{aligned} p &= (p_{0F}, p_{0H}, p_{1F}, p_{1H}) &= (1, 16, 1, 15) \\ x^1 &= (x^1_{0F}, x^1_{0H}, x^1_{1F}, x^1_{1H}) &= (20 + \tfrac{71}{32},\ 1 - \tfrac{71}{32\cdot 16},\ 35 - \tfrac{71\cdot 15}{32\cdot 16},\ 0) \\ x^2 &= (x^2_{0F}, x^2_{0H}, x^2_{1F}, x^2_{1H}) &= (\tfrac{57}{32},\ \tfrac{71}{32\cdot 16},\ 35 + \tfrac{71\cdot 15}{32\cdot 16},\ 1) \end{aligned}$$

Equilibrium utilities are (approximately) $u^1 \approx 56, u^2 \approx 64$.

- **Collateral Equilibrium**

$$\begin{aligned} p &= (p_{0F}, p_{0H}, p_{1F}, p_{1H}) &= (1, 18, 1, 15) \\ q &= 1 \\ x^1 &= (x^1_{0F}, x^1_{0H}, x^1_{1F}, x^1_{1H}) &= (23, 0, 35, 0) \\ x^2 &= (x^2_{0F}, x^2_{0H}, x^2_{1F}, x^2_{1H}) &= (1, 0, 35, 1) \end{aligned}$$

(In words: consumer 2 borrows the value of 15 units of date 0 food which he uses, together with 3 units he already owns, to buy a house which collateralizes the loan; at date 1, consumer 2 repays the loan.)

Equilibrium utilities are $u^1 = 58, u^2 = 72$.[4]

[4] Recall that our notation keeps collateral separate from consumption, but that consumer 2 enjoys the use of the housing at date 0.

Figure 1 (below) displays the Pareto frontier and equilibrium utilities. Several points are worth making:

- The collateral equilibrium is Pareto inefficient: consumer 2 would be glad to trade 2 units of date 1 food for 1 unit of date 0 food and consumer 1 would be glad to oblige. This inefficiency has two sources. The first is that the requirement that borrowing be collateralized *limits* consumer 2's borrowing power. The second is that the requirement that borrowing be collateralized *distorts* consumer 2's consumption decision. In the collateral equilibrium, consumer 2 borrows 15 units of account; this borrowing must be collateralized by 1 house, which consumer 2 must purchase. If consumer 2 could borrow the same 15 units of account but were free to spend her wealth in any way she chose (subject only to the constraint that she repay the loan in full at date 1), she would choose to purchase less than 1 house at date 0; her optimal consumption bundle would be $(\frac{147}{72}, \frac{68}{72}, 35, 1)$. At the collateral equilibrium, consumer 2's consumption decision is distorted, because *she can borrow to purchase housing but not to purchase food.*

- Another way to make the same point is to compare consumer 2's consumption in the collateral equilibrium with her consumption in a Walrasian environment with a borrowing limit. If the borrowing limit is $\lambda \geq 0$ and prices are p, the budget set $B^i_\lambda(p)$ for consumer i consists of plans x^i, y^i for consumption and storage satisfying the Walrasian budget constraint
$$p \cdot x^i + p_0 \cdot y^i \leq p \cdot e^i$$
and the borrowing constraint
$$p_0 \cdot x^i_0 + p_0 \cdot y^i \leq p_0 \cdot e^i_0 + \lambda$$
The easy but slightly messy calculation of equilibrium is left to the reader. If $\lambda < \frac{450}{32}$ then consumer 1 purchases some housing, if $\lambda \geq \frac{450}{32}$ then consumer 2 purchases all the housing — but in either case consumer 2's consumption of date 0 food is a weakly increasing function of the borrowing limit λ, and lies between her consumption in the no-assets equilibrium (equivalently, the equilibrium with $\lambda = 0$) and her consumption in the Walrasian equilibrium with no borrowing limit

Figure 1: The Mortgage Market

(equivalently, the equilibrium with $\lambda \geq 28$): $\frac{57}{32} \leq x^2_{0F} \leq 2$. In particular, consumer 2's consumption of date 0 food *always* exceeds her consumption in the collateral equilibrium.

- The collateral equilibrium dominates the no-asset equilibrium. The possibility to use housing as collateral for its own purchase drives up the price of housing — which benefits sellers — and enlarges the budget set — which benefits buyers. Thus, both sellers and buyers have an incentive to see that houses can be used as collateral.

- Neither the collateral equilibrium nor the no asset equilibrium is dominated by the Walrasian equilibrium. Thus, there is no reason to suppose that borrowers and lenders would agree on perfect, frictionless enforcement of contingent contracts, rather than collateralized borrowing.

It is instructive to compute Walrasian and collateral equilibrium from an alternative distribution of the social endowment. Suppose that consumer 2 is endowed with the housing but no food:

$$\begin{aligned} \hat{e}^1 &= (24, 0, 20, 0) \\ \hat{e}^2 &= (0, 1, 50, 0) \end{aligned}$$

Walrasian equilibrium and collateral equilibrium are unique, and yield the same consumptions and commodity prices:

$$\begin{aligned} \hat{x}^1 &= (22, 0, 22, 0) \\ \hat{x}^2 &= (2, 1, 48, 1) \\ \hat{p} &= (1, 30, 1, 15) \end{aligned}$$

In the collateral equilibrium, consumer 2 borrows the price of two units of date 0 food, using $\frac{2}{15}$ of his house as collateral, and repays the loan at 0 interest at date 1; consumer 1 takes the opposite position. $\diamondsuit$

4 Supporting Walrasian Equilibrium

One way to understand the effect of collateral requirements is to ask whether a given Walrasian equilibrium can be realized as a collateral equilibrium. Formally, say that the Walrasian equilibrium $\langle \bar{p}, (\bar{x}^i, \bar{y}^i) \rangle$ for the durable goods economy $\mathbf{E}$ can be *supported as a collateral equilibrium* if there is a family of assets $\{(A_j, C_j)\}$ and a collateral equilibrium $\langle p, q, (x^i, y^i, \varphi^i, \psi^i) \rangle$ for the economy $(\mathbf{E}, \{(A_j, C_j)\})$ such that

- spot prices coincide: $\bar{p} = p$
- date 1 consumptions coincide: $\bar{x}^i_s = x^h_s$ for each i, s
- date 0 usages coincide: $\bar{x}^i = x^i + \sum_j \varphi^i_j C^L_j + \sum_j \psi^i_j C^B_j$ for each i
- date 0 storages coincide: $\bar{y}^i = y^i + \sum_j \varphi^i_j C^W_j$ for each i

For environments in which there is no uncertainty, we can provide a simple characterization of Walrasian equilibria that can be supported as collateral equilibria.

Theorem 2 *Assume that $S = 1$ (so there is no uncertainty). Let $\mathbf{E}$ be a durable goods economy and let $\bar{p}, (\bar{x}^h, \bar{y}^h)$ be a Walrasian equilibrium for $\mathbf{E}$. In order that $\bar{p}, (\bar{x}^h, \bar{y}^h)$ be supportable as a collateral equilibrium it is necessary and sufficient that*

$$\bar{p}_1 \cdot x^i_1 \geq \bar{p}_1 \cdot e^i_1 \qquad \text{for each } i$$

It is convenient to isolate a part of the argument that does not depend on the absence of uncertainty.

Proposition *If $p, q, (\pi^i)$ is a collateral equilibrium for the economy $\mathcal{E}$ then $p_s \cdot x^i_s \geq p_s \cdot e^i_s$ for each consumer i and state s.*

Proof If $p_s \cdot x^i_s < p_s \cdot e^i_s$ for some consumer i and state s, then i could default on all his promises in state s, surrender all the collateral, and still afford *more* than x^i_s. Since this would contradict optimality of i's equilibrium plan, we obtain the desired conclusion. ■

We now turn to the proof of Theorem 2.

Proof of Theorem 2 Necessity follows immediately from the Proposition. To prove sufficiency, assume that $\bar{p}_1 \cdot x_1^i \geq \bar{p}_1 \cdot e_1^i$ for each i. For each i, write

$$\begin{aligned} m_0^i &= \bar{p}_0 \cdot (x_0^i + y^i) - \bar{p}_0 \cdot e_0 \\ m_1^i &= \bar{p}_1 \cdot [F^U(x_0^i) + F^W(y^i)] + \bar{p}_1 \cdot e_1^i - \bar{p}_1 \cdot x_1^i \end{aligned}$$

The first quantity is the excess of expenditure over income in date 0, the second quantity is excess of income over expenditure in date 0. Because the Walrasian budget constraint is satisfied with equality, it follows that $m_0^i + m_1^i = 0$ for each agent i. Market clearing for commodities entails that $\sum_i m_0^i = \sum_i m_1^i = 0$. Write $\mathcal{I} = \{i : m_0^i > 0\}$ for the set of consumers who borrow at date 0, $\mathcal{K} = \{k : m_0^k < 0\}$ for the set of consumers who lend at date 0, and $\mathcal{T} = \{t : m_0^t = 0\}$ for the set of consumers who do neither. For each $i \in \mathcal{I}$, define a "personalized" asset with promise $A_i = m_1^i$ and collateral requirement is $C_i^B = x_0^i, C_i^W = y^i, C_i^L = 0$.

Define the data of a collateral equilibrium in the following way:

- $p = \bar{p}$
- $q_i = m_0^i$
- consmers $i \in \mathcal{I}$ consumer x_0^i, store y^i, sell one unit of the asset A_i, secured by x_0^i, y^i, and make full delivery at date 1
- agents $k \in \mathcal{K}$ buy x^k, store y^k, and buy the portfolio

$$\varphi^k = \frac{m_1^k}{\sum_{i \in \mathcal{I}} m_1^i}(1, \ldots, 1)$$

- agents $t \in \mathcal{T}$ buy x^t, store y^t, and neither buy nor sell assets

It is easy to see that the budget sets with these collateralized assets, given the spot prices and asset prices, are no larger than the budget sets in the Walrasian equilibrium but still contain the Walrasian consumption plans — which are therefore optimal in these budget sets. Hence these data define a collateral equilibrium. (Note that the argument works equally well if more assets are available — indeed, if *all* assets are available.) We conclude that the given Walrasian equilibrium can be supported as a collateral equilibrium, as asserted. This completes the proof. ■

Theorem 2 anwers a natural question, but one might ask other questions as well. For instance, one might ignore the coincidence of prices and ask whether the allocations of a given Walrasian equilibrium are supportable as the allocations of a collateral equilibrium for some family of assets. If equilibrium allocations are at kinks or on the boundary, such questions seem difficult to answer, but when preferences are smooth and allocations are interior, supporting prices are unique, so coincidence of prices follows automatically.

A variation on Example 1 provides a useful illustration of Theorem 2.

Example 2 As in Example 1, consider a world with no uncertainty ($S = 1$), two consumers $1, 2$ and two goods F (food) and H (housing) at each date. Food is perishable, housing is perfectly durable. Utility functions are

$$\begin{aligned} u^1(x_{0F}, x_{0H}, x_{1F}, x_{1H}) &= x_{0F} + x_{0H} + x_{1F} + x_{1H} \\ u^2(x_{0F}, x_{0H}, x_{1F}, x_{1H}) &= 9x_{0F} - 2(x_{0F})^2 + 15x_{0H} + x_{1F} + 15x_{1H} \end{aligned}$$

Consumer 1 owns the housing stock; we treat the division of the social endowment of date 0 food as a parameter, so endowments are:.

$$\begin{aligned} e^1 &= (24 - t, 1; 20, 0) \\ e^2 &= (t, 0; 50, 0) \end{aligned}$$

It is easily checked that the unique Walrasian prices are $\bar{p} = (1, 30, 1, 15)$ and Walrasian consumptions are:

$$\begin{aligned} \bar{x}^1_t &= (22, 0; 52 - t, 0) \\ \bar{x}^2_t &= (2, 1; 18 + t, 1) \end{aligned}$$

In view of Theorem 2, this Walrasian equilibrium can be supported as a collateral equilibrium exactly when $t \geq 17$; the only asset required is the one promising the price of 15 units of food, collateralized by one house. ♢

In the presence of uncertainty, the condition identified in Theorem 2 is no longer sufficient to guarantee that a Walrasian equilibrium can be supported as a collateral equilibrium; the following example illustrates precisely the problem — and how tranching (allowing the same goods to collateralize several assets) solves it.

Example 3 Consider a world with two states, three consumers $1, 2, 3$, and two goods F (food) and H (housing) at each date. Food is perishable, housing is perfectly durable. Consumer 1 is endowed with the housing but poor at date 0; consumers 2, 3 are wealthy at different states at the terminal date:

$$\begin{aligned} e^1 &= (4, 0; 3, 0; 1, 0) \\ e^2 &= (4, 0; 1, 0; 3, 0) \\ e^3 &= (1, 1; 5, 0; 5, 0) \end{aligned}$$

All consumers are risk averse in food and risk neutral in housing; consumer 3 likes housing more than consumers 1, 2:

$$\begin{aligned} u^1(x_{0F}, x_{0H}; x_{1F}, x_{1H}; x_{2F}, x_{2H}) &= \sqrt{x_{0F}} + \frac{1}{2}x_{0H} + \frac{1}{2}(\sqrt{x_{1F}} + \frac{1}{2}x_{1H}) \\ &\quad + \frac{1}{2}(\sqrt{x_{1F}} + \frac{1}{2}x_{1H}) \\ u^2(x_{0F}, x_{0H}; x_{1F}, x_{1H}; x_{2F}, x_{2H}) &= \sqrt{x_{0F}} + \frac{1}{2}x_{0H} + \frac{1}{2}(\sqrt{x_{1F}} + \frac{1}{2}x_{1H}) \\ &\quad + \frac{1}{2}(\sqrt{x_{1F}} + \frac{1}{2}x_{1H}) \\ u^3(x_{0F}, x_{0H}; x_{1F}, x_{1H}; x_{2F}, x_{2H}) &= \sqrt{x_{0F}} + x_{0H} + \frac{1}{2}(\sqrt{x_{1F}} + x_{1H}) \\ &\quad + \frac{1}{2}(\sqrt{x_{1F}} + x_{1H}) \end{aligned}$$

It is easily checked that in the unique Walrasian equilibrium, consumer 1 owns the housing and all consumers perfectly smooth food consumption:

$$\begin{aligned} \bar{x}^1 &= (3, 1; 3, 1; 3, 1) \\ \bar{x}^2 &= (3, 0; 3, 0; 3, 0) \\ \bar{x}^3 &= (3, 0; 3, 0; 3, 0) \end{aligned}$$

Equilibrium prices are $\bar{p} = (1, 1; \frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \frac{1}{2})$.

At the Walrasian equilibrium, $\bar{p}_s \cdot x^i_s \geq \bar{p}_s \cdot e^i_s$ for each state s and consumer i. However, the Walrasian equilibrium cannot be supported as a collateral equilibrium (for any specification of assets). For to support the Walrasian equilibrium as a collateral equilibrium, consumer 3 would need to borrow from *both* consumers 1, 2, and in order to guarantee repayment, *each* of these loans would need to be collateralized by an entire house — but only one house is available.

Note that this problem would disappear if we allowed the house to collateralize *two* (Arrow) securities, one promising payment only in state 1, the other promising payment only in state 2. Of course, this is precisely what tranching accomplishes. $\diamondsuit$

5 The Shortage of Collateral

Here we present a simple example that illustrates two points:

- the shortage of collateral influences the choice of assets that are traded at equilibrium and hence influences risk sharing
- in an environment with uncertainty, collateralized borrowing is not equivalent is not equivalent to a sale and repurchase, even if collateral is warehoused

Example 4 Consider a world with two dates and three states of nature at the second date. There are two goods available at date 0, F (food) and G (government debt); only food F is available in any state at date 1. Food is perfectly perishable; government debt, if stored, yields one unit of food in each state at date 1. Government debt yields no utility to either consumer at date 0; for simplicity, we therefore omit government debt as an argument in utility functions.

The two consumers hold common priors, and assign equal probability to each state. Consumer 1 is risk neutral and indifferent to the timing of consumption; consumer 2 is risk averse:

$$
\begin{aligned}
u^1(x_0, x_1, x_2, x_3) &= x_0 + \frac{1}{3}[x_1 + x_2 + x_3] \\
u^2(x_0, x_1, x_2, x_3) &= \sqrt{x_0} + \frac{1}{3}[\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}]
\end{aligned}
$$

(Recall that we omit government debt as an argument in utility functions, so each x_s is a quantity of food.) Consumer 1 is endowed with all the government debt; consumer 2 has both a saving and and an insurance motive:

$$
\begin{aligned}
e^1 &= (10, \gamma; 10 - \gamma, 10 - \gamma, 10 - \gamma) \\
e^2 &= (9, 0; 1, 3, 5)
\end{aligned}
$$

We treat γ, the amount of government debt in the economy, as a parameter; for our purposes, we consider the range $0 \leq \gamma \leq 7$.

We first consider Walrasian equilibrium. Since government debt yields no utility at date 0 and yields consumption at date 1, "real wealth" in this economy is independent of the level of government debt. The economy admits a unique Walrasian equilibrium which is independent of γ. Normalizing so that the price of date 0 food $p_{0F} = 1$, equilibrium prices and consumptions are

$$\begin{aligned} p_{0G} &= 1 \ , \ p_{1F} \ = \ p_{2F} \ = \ p_{3F} \ = \ \frac{1}{3} \\ x^1 &= (13; 5, 7, 9) \\ x^2 &= (6; 6, 6, 6) \end{aligned}$$

Equilibrium utilities are $u^1 = 20, u^2 = 2\sqrt{6}$.

Now suppose Arrow securities are traded: for $j = 1, 2, 3$, A_j promises 1 unit of food in state j, collateralized by 1 unit of government debt; collateral is warehoused. (Since government debt yields no utility at date 0, it is irrelevant who holds the collateral.) It is easy to see that consumer 1 sells, and consumer 2 purchases, assets and/or government debt. Solving for the equilibrium as a function of γ is a simple but tedious exercise in manipulating first-order conditions, which we leave to the reader. We find 4 regimes, according to the supply of government debt; equilibrium asset trades and consumptions in each regime are described below.

- $0 \leq \gamma \leq 2$

  A_1 is traded; A_2, A_3 and government debt $0G$ are not traded.

  $$\begin{aligned} x^1 &= (10 + \frac{9\gamma}{3 + 4\gamma}; 10 - \gamma, 10, 10) \\ x^2 &= (9 - \frac{9\gamma}{3 + 4\gamma}; 1 + \gamma, 3, 5) \end{aligned}$$

  Equilibrium is inefficient.

- $2 < \gamma \leq 6$

  A_1, A_2 are traded; A_3 and government debt $0G$ are not traded.

  $$\begin{aligned} x^1 &= (10 + \frac{18\gamma}{12 + 5\gamma}; 9 - \frac{\gamma}{2}, 11 - \frac{\gamma}{2}, 10) \\ x^2 &= (9 - \frac{18\gamma}{12 + 5\gamma}; 2 + \frac{\gamma}{2}, 2 + \frac{\gamma}{2}, 5) \end{aligned}$$

Equilibrium is inefficient.

- $6 < \gamma < 7$

  A_1, A_2 and government debt $0G$ are traded; A_3 is not traded.

$$
\begin{aligned}
x^1 &= (10 + 6\frac{9}{6\gamma - 15} + (\gamma - 6)\frac{9}{2\gamma - 5}; 12 - \gamma, 14 - \gamma, 16 - \gamma) \\
x^2 &= (9 - 6\frac{9}{6\gamma - 15} - (\gamma - 6)\frac{9}{2\gamma - 5}; \gamma - 1, \gamma - 1, \gamma - 1)
\end{aligned}
$$

  Equilibrium is inefficient.

- $\gamma = 7$

  A_1, A_2 and government debt $0G$ are traded; A_3 is not traded.

$$
\begin{aligned}
x^1 &= (13; 5, 7, 9) \\
x^2 &= (6; 6, 6, 6)
\end{aligned}
$$

  Equilibrium coincides with Walrasian equilibrium and is efficient.

(For $\gamma > 7$ there are multiple equilibria, in which different quantities of A_3 are traded, but equilibrium prices and consumptions coincide with the equilibrium when $\gamma = 7$.) $\diamondsuit$

6 Which Assets are Traded?

If many assets are potentially available for trade, which ones will actually be traded at equilibrium? An example, built around Example 1 of Section 3, may help to illuminate the question.

Example 5 Consider a world with 2 states of nature $1, 2$ and two goods F (food) and H (housing) at each date and state. Food is perfectly perishable, housing is perfectly durable. The (objective) probability that state 1 occurs is $1 - \varepsilon$, the probability that state 2 occurs is ε; to make calculation easy, we take $0 \leq \varepsilon < \frac{1}{30}$. Endowments of the two consumers are state-independent:

$$\begin{array}{rcccl} e^1 & = & (e^1_{0F}, e^1_{0H}; e^1_{1F}, e^1_{1H}; e^1_{2F}, e^1_{2H}) & = & (20, 1; 20, 0; 20, 0) \\ e^2 & = & (e^2_{0F}, e^2_{0H}; e^2_{1F}, e^2_{1H}; e^2_{1F}, e^2_{1H}) & = & (\ 4, 0; 50, 0; 50, 0) \end{array}$$

Consumer 2 likes housing more than consumer 1 but less in state 2 than in state 1:

$$\begin{array}{rcl} u^1(x_{0F}, x_{0H}, x_{1F}, x_{1H}, x_{2F}, x_{2H}) & = & x_{0F} + x_{0H} \\ & & + (1-\varepsilon)x_{1F} + (1-\varepsilon)x_{1H} \\ & & + \varepsilon x_{2F} + \varepsilon x_{2H} \\ u^2(x_{0F}, x_{0H}, x_{1F}, x_{1H}, x_{2F}, x_{2H}) & = & 9x_{0F} - 2(x_{0F})^2 + 15x_{0H} \\ & & + (1-\varepsilon)x_{1F} + (1-\varepsilon)15x_{1H} \\ & & + \varepsilon x_{2F} + \varepsilon 5x_{2H} \end{array}$$

For each real number $t > 0$, let A_t be the asset whose promise in each state at date 1 is the value of t units of food, and which is collateralized by 1 unit of housing, held by the borrower. We suppose that *all* the assets A_t are available for trade. (Of course, this is not entirely consistent with our framework, since we have insisted that the number of assets available be *finite*. That assumption, however, was made only to avoid the inconvenience of infinite portfolios, which would not occur in the present example. Alternatively, we could insist that the promises t be chosen from a fine — but discrete — grid.)

This economy has a trivial multiplicity of equilibria, but they are all equivalent, in the sense of having the same prices and consumptions, to the unique equilibrium in which *only* the asset A_{15} is traded. To see this, fix an arbitrary equilibrium; normalize so that the price of food in each date and state is 1. It is easy to see that housing prices coincide with the marginal utility of consumer 2, so $p_{1H} = 15, p_{2H} = 5$. It follows that, for $t \geq 15$, the asset A_t, which promises t units of account in each state, will in fact yield only 15 units of account in state 1 and 5 units of account in state 2 (the value of the collateral). It is easy to see that asset prices are determined by the marginal utility of consumer 1, so it follows that for each $t \geq 15$, the price of A_t will be its expected yield, so $q_t = 15 - 10\varepsilon$. Thus, the assets $A_t, t \geq 15$ are perfect substitutes, so every equilibrium is equivalent to one in which, of the assets $A_t, t \geq 15$, only the asset A_{15} is traded. On the other hand, for $t < 15$, the asset A_t, which promises t units of account in each state, will in fact yield t units of account in state 1 and $\min\{t, 5\}$ units of account in state 2. The price of A_t will be its expected yield in date 1, so $q_t \leq (1-\varepsilon)t + (\varepsilon)5$. Suppose consumer 2's plan involves the sale of a strictly positive quantity of A_t for some $t < 15$. An alternate plan for consumer 2 would be to sell δ fewer units of A_t and δ more units of A_{15}, for some small $\delta > 0$; such a plan would require precisely the same collateral, increase wealth at date 0 by at least $\delta(1 - \varepsilon)$, and decrease wealth at each state in date 1 by δ. If the marginal utility for consumption of food in date 0 exceeds $1 - \varepsilon$ — which is easily seen to be the case at equilibrium — then this alternative plan would be preferred to the equilibrium plan, which would contradict optimality of the equilibrium plan. We conclude that for every $t < 15$, the asset A_t is not traded at equilibrium.

At the unique equilibrium in which only the asset A_{15} is traded, spot prices are $p = (1, 18-10\varepsilon; 1, 15; 1, 5)$. Equilibrium plans are easily described: consumer 2 borrows 1 unit of the asset and uses the loan, together with $3-10\varepsilon$ units of her date 0 food endowment, to buy all the housing at date 0, repays the face value of the loan in state 1, but defaults in state 2; consumer 1 takes the complementary positions. Equilibrium utilities are

$$u^1 = 58 - 10\varepsilon \ , \ u^2 = 72 + 50\varepsilon - 200\varepsilon^2$$

$\diamondsuit$

As this example illustrates, the market may choose levels of collateral which lead to default at equilibrium. One might suppose that lenders, left to their own devices, would prefer that the collateral requirement be sufficiently high that loans would be perfectly safe and there would be no default. The truth, however, is more complicated: *ceteris parabus*, lenders prefer to make *safer loans*, but they also prefer to make *more loans*, and there is a tension between these two objectives. As the following example (another variation on the same theme) illustrates, collateral requirements sufficiently high to guarantee the absence of default may discourage borrowing to such an extent that *both* borrowers and lenders are harmed.

Example 6 To illustrate this point, suppose that everything is as in Example 5, except that only the asset A_5 is available. An easy calculation shows that, at equilibrium, consumer 1 buys, and consumer 2 sells, approximately .21 units of A_5 and that consumptions are (approximately):

$$\begin{aligned} x^1 &\approx (22.3, .79; (32.9, 0), (25, 0)) \\ x^2 &\approx (1.7, 0; (37.1, 1), (45, 1)) \end{aligned}$$

Equilibrium utilities are (approximately):

$$u^1 \approx 56 \ , \ u^2 \approx 57$$

Since $\varepsilon < \frac{1}{30}$, equilibrium utilities here are lower than equilibrium utilities in Example 5; thus, the higher collateral requirement leads to safer loans but to a *Pareto inferior* equilibrium. $\diamondsuit$

Given that the market chooses the asset structure, we are compelled to ask whether the market chooses the asset structure *efficiently*. Put differently, could a social planner effect a Pareto improvement by sequestering assets? Unfortunately we don't know the answer to this question. Indeed, because of the possibility of multiple equilibia, it is not absolutely clear how the question should be formulated; as the following example shows, an economy may admit both Pareto optimal and Pareto suboptimal equilibria.

Example 7 We begin with a 2 consumer Edgeworth box economy with two goods $0F, 0H$. Endowments are: Endowments are:

$$\begin{aligned} e_0^1 &= (9, 1) \\ e_0^2 &= (1, 9) \end{aligned}$$

We choose utility functions v^1, v^2 so that the economy is regular (see Mas-Colell (1985)) and has multiple equilibria, including ones in which allocations are:

$$\begin{aligned} \bar{x}^1 &= (5,5) \quad , \quad \bar{x}^2 = (5,5) \\ \hat{x}^1 &= (2,8) \quad , \quad \hat{x}^2 = (8,2) \end{aligned}$$

We now consider a world with 2 states of nature, two consumers and two goods F (food) and H (housing) at each date and state. Food is perfectly perishable, housing is perfectly durable. Endowments are:

$$\begin{aligned} e^1 &= (9,1,(5,1),(1,1)) \\ e^2 &= (1,9,(1,1),(5,1)) \end{aligned}$$

Utility functions are:

$$\begin{aligned} u^1 &= v^1(x_{0F}, x_{0H}) + \beta\left(\sqrt{x_{1F}+x_{1H}} + \sqrt{x_{2F}+x_{2H}}\right) \\ u^2 &= v^2(x_{0F}, x_{0H}) + \beta\left(\sqrt{x_{1F}+x_{1H}} + \sqrt{x_{2F}+x_{2H}}\right) \end{aligned}$$

where the date 0 utility functions v^1, v^2 are as in the Edgeworth box economy, and β is a small parameter, to be chosen.

Finally, we assume that all conceivable assets, with all conceivable collateral requirements are available for trade. Regularity entails that, for sufficiently small values of β, there are collateral equilibria arbitrarily close to the equilibria of the Edgeworth box economy.

Note that the collateral equilibrium close to the $(\bar{x}^1, \bar{x}^2)$ equilibrium of the Edgeworth box economy is Pareto optimal — indeed it has the same consumptions. However, the the collateral equilibrium close to the $(\hat{x}^1, \hat{x}^2)$ equilibrium of the Edgeworth box economy is Pareto suboptimal. To see this, not that because date 1 utility functions are identical and strictly concave and there is no aggregate risk, Pareto optimality entails that agents share date 1 risk perfectly. At the collateral equilibrium close to the $(\hat{x}^1, \hat{x}^2)$ equilibrium, date 1 risk can be shared perfectly because both agents hold enough housing to secure large loans. However, at the collateral equilibrium close to the $(\hat{x}^1, \hat{x}^2)$ equilibrium, consumer 1 holds too little collateral to secure a loan big enough to enable him to smooth date 1 consumption.

Note that the collateral equilibrium allocation for equilibrium B is Pareto dominated by a feasible allocation, but *not* by a collateral equilibrium allocation. $\diamondsuit$

In one circumstance, however, we can be sure that the market chooses the collateral structure efficiently.

Theorem 3 *Every set of collateral equilibrium plans is Pareto optimal among all sets of plans which (1) are socially feasible; (2) given whatever date 0 decisions are assigned, respect each consumer's budget set at every state s at date 1 at the* given *equilibrium prices; (3) call for deliveries on assets that are the minimum of the promise and the value of collateral. In particular, sequestering assets cannot lead to a Pareto improvement unless date 1 prices change; if date 1 prices do not change, the market chooses the asset structure efficiently.*

Proof Let $p, q, (\pi^i)$ be an equilibrium, and suppose that $(\tilde{\pi}^i)$ is a set of plans meeting the given conditions that Pareto dominates the equilibrium set of plans (π^i). By assumption, all the alternative plans are feasible, meet the budget constraints at each state at date 1, and call for deliveries that are the minumum of promises and the value of collateral, Optimality of the equilibrium plans π^i at prices p, q means, therefore, that *all* the alternative plans $\tilde{\pi}^i$ fail the budget constraints at date 0. Because the alternative set of plans is socially feasible, summing over consumers yields a contradiction. ■

7 Volatility and Market Crashes

We have seen that the collateral requirement can have a profound effect on prices and consumptions. In this section, we show that the collateral requirement can have a profound effect on price *volatility* as well. Of coure, the idea that collateral requirements can affect price volatility is not new. It has been widely argued, for instance, that low margin requirements were in part responsible for the stock market crash of 1929 (and for various crashes in real estate markets). The argument usually made is of the following sort: Optimistic speculators buy stock on margin; bad news about stock fundamentals causes the price of stock to fall, triggering margin calls from brokers; speculators who had purchased stock on margin are unable to meet these margin calls, causing their stock to be offered for sale — driving the price still lower, leading to another round of margin calls, and so forth.

Of course, this is crucially a partial equilibrium story. In a general equilibrium story, all commodities are offered for sale at every moment, so the crucial step in which the borrowers are forced to offer the collateral for sale would have no bite. More importantly, it seems to us that usual story does not address the central question: Given that there is bad news about stock fundamentals, why is the stock price lower in an environment in which margin requirements are low that it would be in an environment in which margin requirements are high?

We argue here that collateral requirements magnify shifts in the wealth distribution, and that it is these shifts in the wealth distribution that provide the missing link between collateral requirements and market crashes. Prices are determined by the marginal utility of the marginal buyer. When the most optimistic agents are wealthy, it is they who are most likely to be the marginal purchasers, so the stock price will be higher. Conversely, when the most optimistic agents are poor, the marginal purchaser of the stock is more likely to be someone less optimistic, so the stock price will be lower. The ability to buy stock on margin leverages purchasing power, in effect making the most optimistic agents wealthier, and driving up the stock price initially. Following bad news about stock fundamentals, however, margin purchasers of stock will be *poorer* than they would otherwise have been, precisely *because* their margin purchases of stock leave them with larger debt and a larger fraction of their wealth in stock. Because the more optimistic agents are poorer, the more pessimistic agents are more likely to be the marginal buyers of stock. It is this shift in ownership, from the optimists to the pessimists, with its attendant shift in prices, that is the hallmark of a margin crash.[5]

An example will make the point quite sharply. In the interests of simplicity and parallelism with the usual story of optimistic speculators, we offer an example which is driven by differences of opinion about the distribution of stock dividends, but the example could be recast in such a way that it would be driven by differences in risk aversion.

We consider a world with three dates $0, 1, 2$, three states of nature H, M, L (high, medium, low) at the terminal date, a single consumption good, and a stock which yields dividends only at the terminal date: 10 in the high state, 8 in the middle state, 0 in the low state. There are two agents $1, 2$ (pessimist and optimist, respectively) who are risk neutral and indifferent to the timing of consumption, but have different priors:

$$\begin{array}{rclrclrcl} \pi^1(H) & = & .5 & \pi^1(M) & = & .125 & \pi^1(L) & = & .375 \\ \pi^2(H) & = & .9 & \pi^2(M) & = & .074 & \pi^1(L) & = & ..025 \end{array}$$

Between date 0 and date 1, there is news about the terminal state. If the news is good, the terminal state will be H for certain; if the news is bad the terminal state will be either M or L.

To recast this story into our simple 2 date framework, we fold date 2 into date 1 by taking expectations. This leads to the following formal description.

Example 8 Consider a market with 2 dates $0, 1$, 2 states g, b (good, bad), 2 goods F, S (food, stock) available at each date and state, 2 agents $1, 2$ (pessimists, optimists). Food is perfectly perishable, stock is perfectly durable. Utility functions are:

$$\begin{array}{rclcl} u^1(x_{0F}, x_{0S}, (x_{gF}, x_{gS}), (x_{bF}, x_{bS})) & = & x_{0F} & + & .5(x_{gF} + 10x_{gS}) \\ & & & + & .5(x_{bF} + 2x_{bS}) \\ u^2(x_{0F}, x_{0S}, (x_{gF}, x_{gS}), (x_{bF}, x_{bS})) & = & x_{0F} & + & .9(x_{gF} + 10x_{gS}) \\ & & & + & .1(x_{bF} + 6x_{bS}) \end{array}$$

[5]See Kubler and Schmedders (2001) for other work on the influence of collateral on asset prices, and Aiyagarai and Gertler (1995), Kiyotaki and Moore (1997) and Geanakoplos (2001) for other work on the influence of collateral on volatility.

(Optimists assess the good state as more likely and have higher marginal utility for stock in the bad state — because they have more optimistic expectations about the future even when the bad state occurs.) Endowments are:

$$\begin{aligned} e^1 &= (40, 4; 40, 0; 40, 0) \\ e^2 &= (24, 0; 7, 0; 7, 0) \end{aligned}$$

(Pessimists are wealthy and own the stock.)

Suppose first that there are no assets, so that the stock cannot be purchased on margin. The allocation of stock and food in the good state is indeterminate; aside from that, the equilibrium is unique, and easy to calculate:

$$\begin{aligned} p &= (1, 8; 1, 10; 1, 6) \\ x^1 &= (64, 1; 40, 1; 47, 0) \\ x^2 &= (0, 3; 3, 3; 1, 4) \end{aligned}$$

At date 0, the optimists trade all their food for 3 shares of stock; they would be happy to buy all the stock, but are too poor to do so. At the date 0 price of 8, pessimists are exactly indifferent between the stock and food, since their marginal utility for stock is 10 in the good state, and the stock will sell for 6 in the bad state, an event to which they assign probability .5. In the bad state, optimists value the stock more highly than do pessimists, and so purchase the remaining 1 share at the equilibrium price of 6, leaving them enough income to purchase 1 unit of food. In the good state, both agents value the stock equally, so the allocation of stock and food in the good state is indeterminate. Because the allocation of food and stock in the good state affects neither equilibrium prices nor equilibrium utilities, we henceforth ignore it.

Now suppose we allow the stock to be purchased on margin. There is a single asset A_β which promises delivery of β times the price of food in each state, collateralized by a share of stock. We treat the promise β as an exogenous parameter, and consider the range $0 \leq \beta \leq \frac{5}{2}$.[6] Since stock yields no utility at date 0, it doesn't matter who holds the collateral. When

[6]In practice, margin requirements, which are set by the Federal Reserve, are usually expressed in terms of a cash down payment as a fraction of the sale price. As a fraction of the sale price, the margin requirement on the asset A_β is $\frac{p_{0S} - q_\beta}{p_{0S}}$.

$\beta = 0$ we are back in the environment in which stock cannot be purchased on margin.

The following simple observations facilitate the analysis:

- Write M for the collateral agent 2 buys at date 0. The optimist will always find it advantageous to borrow to the limit of his capacity, and hence will buy stock entirely on margin. Because one share of stock is required to secure one unit of the asset, $M = \psi^2$, the number of shares of stock sold by agent 2.
- The price of the stock in the good state will always be $p_{gS} = 10$.
- The price of the stock in the bad state will be at least $p_{bS} \geq 2$.
- Because the pessimists (the lenders) are risk neutral and indifferent to the timing of consumption, the price q_β of A_β will be exactly the pessimists' expectation of its delivery. Because deliveries will be the minimum of promises and the value of collateral, and the value of collateral will be 10 in the good state, it follows that:

$$q_\beta = \begin{cases} \beta & \text{if} \quad \beta \geq p_{bS} \\ \\ \frac{1}{2}(\beta + p_{bS}) & \text{if} \quad \beta < p_{bS} \end{cases}$$

- At date 0, the implicit marginal utility of stock to optimists is at least their expectation of the stock price at date 1, which is at least 9.2.
- At date 0, the implicit marginal utility of stock to pessimists is their expectation of the stock price at date 1, which is at most 8. Hence the pessimists will not buy any stock at date 0 if $p_{0S} > 8$.
- The pessimists will not buy any stock in the bad state if $p_{bS} > 2$.

We focus on p_{0S} and p_{bS}, the price of stock at date 0 and the price of stock in the bad state, as functions of the asset promise β. There are phase transitions at $\beta = \frac{8}{7}$ and $\beta = 2$ and a singularity at $\beta = \frac{7}{4}$; we analyze the parameter intervals delimited by these points separately.

$[0, \frac{8}{7}]$: At date 0, optimists sell all their food and borrow to buy stock on margin, but they cannot afford to buy all the stock; the pessimists buy the remaining stock, whence the equilibrium price is $p_{0S} = 8$. Recalling that $M = \psi^2$ is the total stock purchase of the optimist, the date 0 income/expenditure identity for the optimist is $24 + \beta M = 8M$, whence $M = \frac{24}{8-\beta}$. In the bad state, optimists spend βM to repay their loans, obtain M shares of stock that had been held as collateral, spend $6(4 - M)$ to purchase the remaining $4 - M$ shares of stock at the equilibrium price of 6, and use the remainder of their income $7 - \beta M - 6(4 - M)$ to purchase food. (For $\beta = \frac{8}{7}$, no income remains to buy food.) Stock prices are:

$$p_{0S} = 8 \ , \ p_{bS} = 6$$

Note that prices are *constant* on this interval.

$(\frac{8}{7}, \frac{7}{4})$: Optimists value the stock in the bad state more highly than do pessimists, their endowment in the bad state is no longer large enough to pay of all the remaining stock at a price of 6, so the price p_{bS} *falls* — just enough so that it remains possible for the optimists to buy all the remaining stock. The optimists cannot borrow enough to buy all the stock at date 0, so the pessimists hold some stock, and the price p_{0S} is the pessimist's expectation of the date 1 price. Hence p_{0S} also falls, to:

$$p_{0S} = \frac{1}{2}(10 + p_{bS})$$

Equating the income and expenditure of the optimists at date 0 gives

$$24 + \beta M = p_{0S} M$$

Equating the income and expenditure of the optimists at the bad state gives

$$p_{bS}(4 - M) = 7 - \beta M$$

Solving the last 3 equations yields the prices:

$$\begin{aligned} p_{0S} &= 5 + \frac{15 + 8\beta + \sqrt{(15 + 8\beta)^2 - 16(62\beta - 70)}}{16} \\ p_{bS} &= \frac{15 + 8\beta + \sqrt{(15 + 8\beta)^2 - 16(62\beta - 70)}}{8} \end{aligned}$$

Note that prices are *falling* on this interval, and that

$$\lim_{\beta \uparrow \frac{7}{4}} p_{0S} = 7.75 \ , \ \lim_{\beta \uparrow \frac{7}{4}} p_{bS} = 5.5$$

$(\frac{7}{4}, 2)$: Now the optimists can purchase *all* 4 shares of stock on margin at date 0. The budget equation for optimists at date 0 is:

$$4p_{0S} = 24 + 4q_\beta$$

The budget equation for optimists in the bad state is:

$$p_{bS}x^2_{bS} + 4\beta = 7 + 4p_{bS}$$

whence

$$0 < 4\beta - 7 = (4 - x^2_{bS})p_{bS} \leq 1$$

Hence $x^1_{bS} < 4$ and $x^2_{bS} > 0$; pessimists buy some of the stock in the bad state. Hence prices are:

$$p_{0S} = 6 + \beta \ , \ p_{bS} = 2$$

$[2, \frac{5}{2}]$: Now optimists default completely in the bad state, delivering the stock instead, and then buying back 3.5 shares at the equilibrium price of 2. Since the price of the asset A_β is $q_\beta = \frac{1}{2}(\beta + 2)$, the optimists' date 0 budget constraint is:

$$4p_{0S} - 24 + 4q_\beta = 24 + 4(\frac{1}{2})(\beta + 2)$$

Hence stock prices are:

$$p_{0S} = 7 + \frac{\beta}{2} \ , \ p_{bS} = 2$$

Figure 2 displays p_{0S} (the price of the stock at date 0) and p_{bS} (the price of the stock in the bad state at date 1) as functions of β. There is a crash as the margin parameter β crosses the threshold $\beta = \frac{7}{4}$. Just below this threshold, the optimists buy all the stock in the bad state. Just above this threshold, the optimists have acquired so much debt that they cannot afford to buy

Figure 2: A Margin Crash

all the stock in the bad state. Hence, below the threshold, the price p_{bS} is determined by the marginal utility of the optimists; above the threshold, the price p_{bS} is determined by the marginal utility of the pessimists. It is the shift in the distribution of income — and subsequently in ownership — that leads to the dramatic fall in prices and that we identify as the hallmark of a margin crash. $\diamondsuit$

A final point of this example is worth noting: Following a margin crash, there may be unusually good investment opportunities. More formally, suppose we add to the preceding example a third agent, infinitesimal in scale (so that his presence has no effect on prices), with preferences identical to that of the optimist, but *wealthy* in the bad state. Following the crash, this agent's marginal utility for stock will be 8, the price of stock will be 2 — and he will have sufficient wealth to make use of the opportunity.

Appendix

To prove Theorem 1, fix an economy $\mathcal{E} = \Big((e^i, u^i), (F_s^U, F_s^W), (A_j, C_j)\Big)$. In constructing an equilibrium for $\mathcal{E}$, we must confront the possibility that asset prices are 0; because of this, the argument is a bit delicate.[7] We first construct, for each $\rho > 0$, an auxiliary economy $\mathcal{E}^\rho$ in which asset deliveries are bounded below by ρ; in these auxiliary economies, equilibrium asset prices will be bounded away from 0. We then construct an equilibrium for $\mathcal{E}$ by taking limits as $\rho \to 0$.

We can choose the price normalization in each spot completely arbitrarily. For each $s = 0, 1, \ldots, S$, choose and fix an arbitary $\beta_s > 0$; write

$$\begin{aligned} \Delta_s &= \{(p_{s\ell}) \in \mathbf{R}^L_{++} : \sum_{\ell=1}^{L} p_{s\ell} = \beta_s\} \\ \Delta &= \Delta_0 \times \ldots \times \Delta_S \end{aligned}$$

(It is natural to think of β_s as the *price level*.) Write $\mathbf{1}_0 = (1, \ldots, 1) \in R^L_+$ and define

$$Q = \{q \in \mathbf{R}^J_+ : 0 \le q_j \le 2\beta_0 \mathbf{1}_0 \cdot C_j\}$$

We construct equilibria (for the auxiliary economies and then for our original economy) with commodity prices in Δ and asset prices in Q.

For each $\rho > 0$, let $\mathcal{E}^\rho$ be the economy which differs from $\mathcal{E}$ only in that asset promises are defined by:

$$A_j^\rho = A_j + \rho$$

We first construct truncated budget sets and demand and excess demand correspondences in this auxiliary economy. By assumption, collateral requirements for each asset are non-zero. Choose a constant M so large that, for each j,

$$MC_j \not\leq \bar{e}_0$$

[7] In fact, the equilibrium price of any asset whose yield is 0 in every state will *necessarily* be 0, and asset yields may well be identically 0 at some prices. For instance, an option to purchase an ounce of gold at $800 will be worthless if the price of gold never exceeds $799.

(Thus, to sell M units of the asset A_j^ρ would require more collateral than is actually available to the entire economy.) For each $(p,q) \in \Delta \times Q$ and each consumer i, define the truncated budget set

$$B_0^i(p,q) = \{\pi \in B^i(p,q) : 0 \le \varphi_j^i \le NM \ , \ 0 \le \psi_j^i \le NM \text{ for each } j\}$$

and the individual truncated demand correspondence

$$D^i(p,q) = \{\pi = (x,y,\varphi,\psi) \in B_0^i(p,q) : \pi \text{ is utility optimal in } B_0^i(p,q)\}$$

(Note that truncated demand exists at every price (p,q), because we bound asset purchases and sales. Absent such a bound, demands would certainly be undefined at some prices. For instance, if $q_j = 2\beta_0 \mathbf{1}_0 \cdot C_j$, agents could sell A_j^ρ for enough to finance the purchase of its collateral requirement, so there would be an unlimited arbitrage. Bounding asset sales bounds the arbitrage.) Write

$$D(p,q) = \sum_{i=1}^N D^i(p,q)$$

for the aggregate demand correspondence.

We now define excess demand $\zeta(\pi)$ for each plan π. Because goods are durable, we must take care that excess demand reflects date 1 commodities that come into being as the result of date 0 activities. Given a plan $\pi = (x,y,\varphi,\psi)$, the commodity excess demand at date 0 is the sum of consumption, storage, and collateral, minus endowments:

$$\zeta_0(\pi) = x_0 + y + \sum_j \psi_j C_j - \bar{e}_0$$

The commodity excess demand in state s is consumption minus endowment minus goods that come into being as the result of date 0 activities:

$$\zeta_s(\pi) = x_s - \bar{e}_s - F_s^W(y) - F_s^W\left(\sum_j \psi_j C_j^W\right) - F_s^U(x_0) - F_s^U\left(\sum_j \psi_j [C_j^B + C_j^L]\right)$$

Finally, the asset excess demand is

$$\zeta_a(\pi) = \varphi - \psi$$

Write

$$\zeta(\pi) = (\zeta_0(\pi), \ldots, \zeta_S(\pi); \zeta_a(\pi)) \in \mathbf{R}^{L(S+1)} \times \mathbf{R}^J$$

and define the aggregate excess demand correspondence

$$Z : \Delta \times Q \to \mathbf{R}^{L(S+1)} \times \mathbf{R}^J$$

by

$$Z(p, q) = \zeta(D(p, q))$$

It is easily checked that $Z(p, q)$ is non-empty, compact, and convex for each p, q and that the correspondence Z is upper hemi-continuous. Because consumptions asset sales are bounded, Z is also bounded below.

We assert that Z satisfies the usual boundary condition:

$$||Z(p, q)|| \to \infty \text{ as } (p, q) \to \text{bdy}\Delta \times Q \tag{2}$$

(It doesn't matter which norm we use; to be definite, use the supremum norm.) To establish the boundary condition for Z, it suffices to establish the corresponding boundary condition for D:

$$||D(p, q)|| \to \infty \text{ as } (p, q) \to \text{bdy}\Delta \times Q \tag{3}$$

To see this, suppose not, so that there is a sequence $(p^n, q^n) \in \Delta \times Q$ and aggregate demands $\mu^n \in D(p^n, q^n)$ such that

$$(p^n, q^n) \to (p^*, q^*) \in \text{bdy}\Delta \times Q$$

but $||\mu^n||$ is bounded. Because individual demands are non-negative, boundedness of aggregate demand implies that individual demands are also bounded. Hence (passing to a subsequence if necessary), we may assume that for each i we can find $\pi^i_n \in D^i(p^n, q^n)$ such that $\pi^i_n \to \pi^i$ as $n \to \infty$. By assumption, $p^*_{tk} = 0$ for some commodity tk. We distinguish 4 cases and reach a contradiction (to optimality of π^i_n) in each.

Case 1 $t = 0$ and commodity $0k$ is perishable. By assumption, the date 0 aggregate endowment is strictly positive, $\overline{e}_0 >> 0$, so there is some consumer i such that $p^*_0 \cdot e^i_0 > 0$. Consider the plan $\pi^i + (\delta_{0k}, 0, 0, 0)$ which coincides with π^i except that it calls for greater consumption of commodity $0k$. Because

$0k$ is perishable, strict monotonicity in perishable commodities guarantees that $u^i(\pi^i + (\delta_{0k}, 0, 0, 0)) > u^i(\pi^i)$. Continuity of utilities implies that we can choose $r > 0$ sufficiently small that $u^i((1-r)\pi^i + (\delta_{0k}, 0, 0, 0)) > u^i(\pi^i)$. Because $p_{0k} = 0$ and $p_0^* \cdot e_0^i > 0$, the plan $(1-r)\pi^i + (\delta_{0k}, 0, 0, 0)$ belongs to the truncated budget set $B_0^i(p^n, q^n)$ for n sufficiently large. Moreover, for n sufficiently large, continuity of utilities implies that $u^i((1-r)\pi^i + (\delta_{0k}, 0, 0, 0)) > u^i(\pi_n^i)$. This contradicts optimality of π_n^i.

Case 2 $t = 0$ and commodity $0k$ is non-perishable. As in Case 1, there is some consumer i for whom $p_0^* \cdot e_0^i > 0$. Write

$$\mu = (\delta_{0k} + U(\delta_{0k}) + W(\delta_{0k}), \delta_{0k}, 0, 0)$$

By assumption, $F_s^U(\delta_{0k}) + F_s^W(\delta_{0k}) > 0$ for some s. Now we can reason exactly as in Case 1: Strict monotonicity of utilities in date 1 commodities entails that $u^i(\pi^i + \mu) > u^i(\pi^i)$. We can choose r sufficiently small that $u^i((1-r)\pi^i + \mu) > u^i(\pi^i)$. For n sufficiently large, $u^i((1-r)\pi_n^i + \mu) > u^i(\pi_n^i)$ and $(1-r)\pi_n^i + \mu \in B_0^i(p^n, q^n)$. This contradicts optimality of π_n^i.

Since we have obtained a contradiction in Cases 1 and 2, we conclude that $p_0^* >> 0$.

Case 3 $t \geq 1$ and $p_t \cdot \overline{e}_t > 0$. Fix any consumer i for whom $p_t^* \cdot e_t^i > 0$. Again we can reason as in Case 1: Stict monotonicity of utilities implies that $u^i(\pi^i + (\delta_{tk}, 0, 0, 0)) > u^i(\pi^i)$. We can choose $r > 0$ sufficiently small that $u^i((1-r)\pi^i + (\delta_{tk}, 0, 0, 0)) > u^i(\pi^i)$. For n sufficiently large $u^i((1-r)\pi_n^i + (\delta_{tk}, 0, 0, 0)) > u^i(\pi_n^i)$ and $(1-r)\pi_n^i + (\delta_{tk}, 0, 0, 0) \in B_0^i(p^n, q^n)$. This contradicts optimality of π_n^i.

Case 4 $t \geq 1$ and $p_t \cdot \overline{e}_t = 0$. Set

$$\begin{aligned} \nu &= (\delta_{tk}, 0, 0, 0) \\ \gamma &= (\delta_{0\ell}, \delta_{0\ell}, 0, 0) \end{aligned}$$

Strict monotonicity in date 1 commodities implies that we can choose $r > 0$ sufficiently small that $u^i((1-r)\pi^i + \nu) > u^i(\pi^i)$. Having chosen r, we can then choose $r' > 0$ sufficiently small that $(1-r)\pi^i + r'\gamma + \nu \in B_)^i(p^*, q^*)$. Having chosen r, r', if n is sufficiently large then $u^i((1-r)\pi_n^i + r'\gamma + \nu) > u^i(\pi_n^i)$ and $(1-r)\pi_n^i + r'\gamma + \nu \in B_0^i(p^n, q^n)$. This contradicts optimality of π_n^i.

Having reached a contradiction in all 4 cases, we conclude that D satisfies the boundary condition (3), and hence that Z satisfies the boundary condition (2).

Now fix $\varepsilon > 0$, and set

$$\Delta^\varepsilon = \{p \in \Delta : p_{s\ell} \geq \varepsilon \text{ for each } s, \ell\}$$

Because Z is an upper hemi-continuous correspondence, it is bounded on $\Delta^\varepsilon \times Q$; set

$$K^\varepsilon = \{z \in \mathbf{R}^{L(S+1)} \times \mathbf{R}^J : ||z|| \leq \sup_{(p,q)\in\Delta^\varepsilon \times Q} ||Z(p,q)||\}$$

Define the correspondence

$$F^\varepsilon : \Delta^\varepsilon \times Q \times K^\varepsilon \to \Delta^\varepsilon \times Q \times K^\varepsilon$$
$$F(p,q,z) = \text{argmax}\,\{(p^*,q^*)\cdot z : (p^*,q^*) \in \Delta^\varepsilon \times Q\} \times Z(p,q)$$

For prices $(p,q) \in \Delta \times Q$ and a vector of excess demands $z \in \mathbf{R}^{L(S+1)} \times \mathbf{R}^J$, $(p,q)\cdot z$ is the value of excess demands. We caution the reader that, in this setting, Walras' law need not hold for arbitrary prices: given $(p,q) \in \Delta \times Q$, $z \in Z(p,q)$, the value of excess demand $(p,q)\cdot z$ need not be 0. We shall see, however, that Walras' law *does* hold at the prices we identify as candidate equilibria.

Our construction guarantees that F^ε is an upper-hemicontinuous correspondence, with non-empty, compact convex values. Kakutani's theorem guarantees that F^ε has a fixed point. We assert that for some $\varepsilon_0 > 0$ sufficiently small, the correspondences $F^\varepsilon, 0 < \varepsilon < \varepsilon_0$ have a *common fixed point*. To see this, write $\Gamma^\varepsilon \subset \Delta^\varepsilon \times Q \times K^\varepsilon$ for the set of all fixed points of F^ε; Γ^ε is a non-empty compact set. We show that for some $\varepsilon_0 > 0$ sufficiently small, the sets Γ^ε are nested and decrease as ε decreases; that is, $\Gamma^{\varepsilon_1} \subset \Gamma^{\varepsilon_2}$ whenever $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$.

To see this, note first that asset yields are bounded, because yields never exceed the value of collateral. Hence individual expenditures at budget feasible plans (and in particular at plans in the truncated demand set) are bounded, independent of prices, because income from endowments is bounded, asset prices and sales are bounded, and asset purchases and yields are bounded. Choose an upper bound $I > 0$ on individual expenditures at budget feasible plans. Because commodity demands are non-negative, individual excess demands are bounded below; choose a lower bound $-R < 0$ on individual exess demands.

Because excess demand is the sum of individual demands less the sum of endowments, it follows that if $z \in Z(p, q)$ then

$$\begin{aligned} (p, q) \cdot z &\leq NI \\ z_{s\ell} &\geq -R \text{ for each commodity } s\ell \end{aligned}$$

The boundary condition tells us that the norm of aggregate excess demand blows up as prices tend to the boundary of $\Delta \times Q$; because asset demands are bounded, it is the norm of commodity excess demand that blows up. Hence we can find $\varepsilon_0 > 0$ so that if $(p, q) \in \Delta \times Q$ and $p_{tk} < \varepsilon_0$ for some commodity tk and $z \in Z(p, q)$ then there is some commodity $t'k'$ such that

$$z_{t'k'} > \frac{1}{\beta_{t'} - (L-1)\varepsilon} \left[NI + \varepsilon R(L-1) + \max_s \beta_s R \right] \tag{4}$$

We assert that if $0 < \varepsilon < \varepsilon_0$ then $\Gamma^\varepsilon \subset \Delta^{\varepsilon_0} \times Q \times K^{\varepsilon_0}$. To see this, suppose that $(p, q, z) \in \Gamma^\varepsilon$ and $p \notin \Delta^{\varepsilon_0}$. Define $\hat{p} \in \Delta$ by

$$\hat{p}_{s\ell} = \begin{cases} \varepsilon & \text{if } s = t, \ell \neq k \\ \beta_{t'} - (L-1)\varepsilon & \text{if } s = t, \ell = k \\ \beta_s / L & \text{otherwise} \end{cases}$$

Direct calculation using equation (4)shows that

$$(\hat{p}, 0) \cdot z > NI$$

which is a contradiction. We conclude that $p \in \Delta^{\varepsilon_0}$ and hence that $(p, q, z) \in \Gamma_0^\varepsilon$ as desired.

The definition of F^ε implies immediately that if $0 < \varepsilon_1 < \varepsilon_2$ and $\Gamma^{\varepsilon_1} \subset \Delta^{\varepsilon_2} \times Q \times K^{\varepsilon_2}$ then $\Gamma^{\varepsilon_1} \subset \Gamma^{\varepsilon_2}$. We conclude therefore that for $0 < \varepsilon < \varepsilon_0$ the sets Γ^ε are nested and decrease as ε decreases.

A nested family of non-empty compact sets has a non-empty intersection so

$$\Gamma = \bigcap_{\varepsilon < \varepsilon_0} \Gamma^\varepsilon \neq \emptyset$$

Let $(p, q, z) \in \Gamma$; we assert that $z = 0$ and that p, q constitute equilibrium prices for the economy $\mathcal{E}^\rho$.

We first show that excess asset demand $z_a = 0$. If the excess demand for asset j is positive, the requirement that (p, q) maximize the value of excess demand implies that q_j is as big as possible: $q_j = 2\beta_0 \mathbf{1}_0 \cdot C_j$. But then agents can sell A_j^ρ for enough to finance the purchase of the collateral requirement, so there would be an unlimited arbitrage, whence the excess demand for A_j^ρ must be negative, a contradiction. We conclude that asset excess demand must be non-positive. If the excess demand for asset j is negative, the requirement that (p, q) maximize the value of excess demand implies that q_j is as small as possible: $q_j = 0$. But if the price of A_j^ρ is 0 then every agent will wish to buy it because its yield is the minimum of ρ and the value of collateral, which is strictly positive. Hence the excess demand for A_j^ρ must be positive, a contradiction.[8] We conclude that $z_a = 0$.

We show next that Walras' law holds for the prices p, q and the excess demand z: $(p, q) \cdot z = 0$. To see this, choose individual demands $\pi^i \in D^i(p, q)$ with the property that the corresponding aggregate excess demand is z:

$$\zeta(\sum_i \pi^i) = z$$

For each agent i, the plan π^i lies in the budget set at prices (p, q), so the date 0 expenditure required to carry out the plan π^i is no greater than the value of date 0 endowment. Because utility is strictly monotone in date 0 perishable commodities and in all commodities in state s, optimization implies that all individuals spend all their income at date 0, so we conclude that the date 0 expenditure required to carry out the plan π^i is precisely equal to the value of date 0 endowment. Put differently, the value of date 0 excess demand is 0 for each individual. Summing over all individuals, we conclude that the value of date 0 aggregate excess demand is 0: $p_0 \cdot z_0 + q \cdot z_a = 0$.

Now consider any state $s \geq 1$ at date 1. For individual i, we can argue exactly as above to conclude that the value of individual excess demand is equal to the net of deliveries on purchases and sales of assets. Thus, the value of aggregate excess demand in state s is the net of deliveries on aggregate

[8] Note that we could not obtain this conclusion in the original economy, because, at the prices (p, q) the asset A_j might promise 0 in every state.

purchases and sales of assets. However, $z_a = 0$ so aggregate purchases and sales of assets are equal. We conclude that the value of aggregate excess demand in state s is 0.

Summing over all spots we conclude that $(p, q) \cdot z = 0$, as asserted.

We show next that $z = 0$. If not, Walras' law entails that excess demand for some commodity is positive; say $z_{tk} > 0$. Define commodity prices $\hat{p}$ by:

$$\hat{p}_{s\ell} = \begin{cases} p_{s\ell} & \text{if} \quad s \neq t \\ \varepsilon & \text{if} \quad s = t, \ell \neq k \\ 1 - (L-1)\varepsilon & \text{if} \quad s = t, \ell = k \end{cases}$$

Because $(p, q) \cdot z = 0$ and $z_{tk} > 0$, $(\hat{p}, q) \cdot z$ will be strictly positive if ε is small enough. However, this would contradict our assumption that $(p, q, z) \in \Gamma$ and hence is a fixed point of F^{ε} for *every* sufficiently small ε. We conclude tha $z = 0$. It is clear that the prices p, q and plans (π^i) identified above constitute an equilibrium for the economy $\mathcal{E}^{\rho}$.

It remains to construct an equilibrium for the original economy $\mathcal{E}$. To this end, let $p(\rho), q(\rho), (\pi^i(\rho))$ be equilibrium prices and plans for $\mathcal{E}^{\rho}$ and let $\rho \to 0$. By construction, prices and plans lie in bounded sets, so we may choose a sequence $(\rho_n) \to 0$ for which the corresponding prices and plans converge; let the limits be $p, q, (\pi^i)$. Commodity prices p do not lie on the boundary of Δ (for otherwise the excess demands at prices $p(\rho_n), q(\rho_n)$ would be unbounded, rather than 0). It follows that $\pi^i(\rho)$ is utility optimal in consumer i's budget set at prices (p, q). Because the collection of plans (π^i) is the limit of collections of socially feasible plans, it follows that they are socially feasible and hence that the artificial bounds on asset purchases and sales do not bind at the prices p, q. Hence $p, q, (\pi^i)$ constitute an equilibrium for $\mathcal{E}$. This completes the proof. ■

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