Genericity with Infinitely Many Parameters[1]

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Abstract

Genericity analysis is widely used to show that desirable properties that fail in certain "knife-edge" economic situations nonetheless obtain in "typical" situations. For finite-dimensional spaces of parameters, the usual notion of genericity is full Lebesgue measure. For infinite dimensional spaces of parameters (for instance, the space of preferences on a *finite*-dimensional commodity space), no analogue of Lebesgue measure is available; the lack of such an analogue has prompted the use of less compelling topological notions of genericity. Christensen (1974) and Hunt, Sauer and Yorke (1992) have proposed a measure-theoretic notion of genericity, which Hunt, Sauer and Yorke call *prevalence*, which coincides with full Lebesgue measure in Euclidean space and which extends to infinite-dimensional vector spaces. This notion is not directly applicable in most economic settings because the relevant parameter sets are small subsets of vector spaces — especially cones or order intervals — not vector spaces themselves. We adapt the notion to economically relevant environments by defining two notions of prevalence relative to a convex set in a topological vector space. The first notion is very easy to understand and apply, and has all of the properties one would desire except that it is not closed under countable unions; the second notion contains the first and has all the good properties of the first notion except simplicity; it *is* closed under countable unions.

We provide four economic applications: 1) generic existence of equilibrium in financial models, 2) generic finiteness of the number of pure strategy Nash equilibria and Pareto inefficiency of "non-vertex" Nash equilibria for games with a continuum of actions and smooth payoffs, 3) generic regularity of exchange economies when some agents are constrained to have 0 endowment of some goods, 4) generic single-valuedness of the core of transferable utility games.

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1 Introduction

Comparative statics analysis — whether in game theory or in general equilibrium theory — is possible and meaningful only when equilibria exist, are locally unique, and depend nicely on underlying parameters. Unfortunately, there are many economically important circumstances in which these criteria are not met. Even nonpathological exchange economies with two goods and nice preferences, or simple two-person games, may possess a continuum of equilibria, or equilibria that do not depend continuously on underlying parameters. If we expand our horizons to include rational expectations economies, or incomplete markets economies, or economies with an infinite number of commodities, we are faced with situations in which equilibria do not even exist. In such circumstances, we may be content with a "second best" theory: that equilibria exist, that equilibria are locally unique, and that equilibria depend nicely on parameters — not for all parameter values but rather for a set of parameter values which is "large" or "generic" in some appropriate sense. Diverse examples in the literature include a) the work of Debreu (1970) on determinacy of Walrasian equilibrium for generic finite-dimensional exchange economies; b) the work of Debreu (1975), Grodal (1975), H. Dierker (1975) and Geller (1987) on the rate and form of core convergence for generic sequences of finite economies; c) the work of Duffie and Shafer (1985,1986) on the existence of equilibrium for generic incomplete markets economies; d) the work of Wilson (1971), Harsanyi (1973) and Dubey (1986) on generic finiteness and Pareto inefficiency of the set of Nash equilibria for games in normal form.

When the parameter space is finite dimensional, it is common to identify "generic" sets as those whose complements have Lebesgue measure 0. When the parameter space is infinite dimensional, however, there is no natural analogue of Lebesgue measure to which we might appeal.[1] Indeed, *no*

[1] Lest the reader think that infinite-dimensional parameter spaces seem like an arcane concern, it should be kept in mind that spaces of preferences are usually infinite-dimensional. Indeed, as noted by Chichilnisky (1976), the space of smooth strictly convex preferences on $\mathbf{R}^2_+$ is already infinite-dimensional.

infinite dimensional Banach space admits *any* translation invariant measure which assigns finite, strictly positive measure to each open set. Because of this, much of the literature has appealed to an alternative, topological notion of genericity. Recall that a subset of a complete metric space is *first category* if it is contained in the countable union of closed sets with empty interior; a subset is *second category* or *residual* if its complement is first category. Because the Baire Category theorem guarantees that first category sets have empty interior, first category sets are usually regarded as small and second category (residual) sets are usually considered large. However, it should be kept in mind that this topological notion of genericity is very different from the measure-theoretic notion of genericity. In particular, there are second category subsets of $\mathbf{R}^n$ which have measure 0 and full measure subsets of $\mathbf{R}^n$ which are first category. Because our formulation of probability theory is measure-theoretic, not topological, the measure-theoretic notion of genericity corresponds much more closely to the idea that a "typical" or "random" element lies in the generic set. As Mas-Colell (1985, page 318) writes, topological genericity "has to be thought of as much less sharp than the measure-theoretic concept available in the finite-dimensional case."

In this paper, we develop a measure-theoretic notion of typicality and exceptionality for infinite-dimensional spaces. To demonstrate the usefulness of this notion, we go on to apply this notion to four economic problems:

1) the generic existence of equilibrium in financial models with continuous trading

2) the generic finiteness of the number of pure strategy Nash equilibria, and Pareto inefficiency of "non-vertex" Nash equilibria in games with continuum action spaces and smooth payoff functions

3) the generic finiteness of equilibria in exchange economies with a finite number of commodities in which some consumers may have 0 endowment of some goods

4) the generic single-valuedness of the core of transferable utility games with a continuum of players

Our notions of typicality and exceptionality generalize notions due (independently) to Christensen (1974) and Hunt, Sauer and Yorke (1992). Motivation for all of these notions comes from an observation about Lebesgue measure in $\mathbf{R}^n$. Translation-invariance of Lebesgue measure implies that a subset $E \subset \mathbf{R}^n$ has Lebesgue measure 0 if and only if all its translates have Lebesgue measure 0. Less obviously, E has Lebesgue measure 0 if and only if there is *some* positive measure μ on $\mathbf{R}^n$ such that E and all its translates have μ-measure 0. Christensen and Hunt, Sauer and Yorke therefore define a Borel subset E of a complete metric topological vector space X to be *shy* if there is some positive Borel measure μ with the property that the set and *all* its translates have measure 0; a set is *prevalent* if its complement is shy.[2] Hunt, Sauer and Yorke show that shyness and prevalence have the essential properties that one should demand for notions of small and large, and go on to demonstrate a wide variety of genericity results.

Unfortunately, the parameter spaces of interest in economics are frequently not vector spaces, or open subsets of vector spaces, but rather small subsets of the ambient vector spaces, and these small subsets are typically shy in the ambient vector spaces. In our first application, for instance, the parameter space is the space of endowments of a continuous time economy (the fundamental model in modern finance). Because endowments must be nonnegative, the space of endowments is the cone of *nonnegative* random variables (with finite mean and variance) — a shy cone in the vector space of all random variables (with finite mean and variance). In our third application, the parameter space is the space of smooth concave utility functions satisfying a boundary condition — a shy cone in the space of all smooth utility functions. In our fourth application, the parameter space is the space of nonnegative population distributions (measures) that are absolutely continuous with respect to a given population distribution (measure) — a shy cone in the vector space of all measures absolutely continuous with respect to the given measure. Without a *relative* notion of shyness it would make no sense to ask whether "most" endowments lead to economies for which equilibrium

[2]Christensen works in the more general framework of topological groups, and uses different terminology.

exists, or whether "most" utility functions lead to economies for which equilibria are regular, or whether "most" population distributions lead to games for which the core is a singleton. To address these questions, one must either modify the economic model to bring it into a domain on which shyness is applicable or modify the notion of shyness to make it applicable.[3]

Our contribution is precisely the latter: we provide, in Section 2, notions of shyness and prevalence *relative* to a convex subset C of a topological vector space X. Unfortunately, this is not a straightforward matter, and the "correct" relativizations are subtle and not easy to grasp. (As various examples in Appendix B demonstrate, the difficulty is intrinsic in the problem.) For most purposes, however, we do not need to use the definitions of these notions, but only the following few basic properties.

Fact 1 Every subset of a set that is shy in C is shy in C.

Fact 2 If E is shy in C then $E + x$ is shy in $C + x$ for each $x \in X$.

Fact 3 The countable union of sets shy in C is shy in C.

Fact 4 No relatively open subset of C is shy in C.

Fact 5 If $X = \mathbf{R}^n$ and $C \subset \mathbf{R}^n$ is a convex subset with nonempty interior, then $E \subset C$ is shy in C if and only if the Lebesgue measure of E is 0.

Fact 6 If there is a finite dimensional subspace $L \subset X$ such that $L \cap (E - x)$ has zero Lebesgue measure in L for every $x \in X$ and $L \cap (C - y)$ has positive Lebesgue measure in L for some $y \in X$ (we say that E is *finitely shy* in C) then E is shy in C.

Facts 1-5 embody the basic properties one would desire of any measure-theoretic notion of "smallness;" Fact 6 provides a useful sufficient condition for shyness. As the reader will see, a natural method for establishing that a set is shy in C is to exhibit it as a countable union of sets that are finitely

[3] In his work on generic determinacy of Walrasian equilibrium, Araujo (1985) takes the former route. Because the machinery of Christensen and Hunt, Sauer and Yorke applies only to open subsets of a vector space (in this case, the space of C^2 functions), Araujo necessarily makes assumptions about preferences that are quite different from Debreu's.

shy in C (much as a natural method for establishing that a set is Borel is to exhibit it as a countable union of closed sets, or a countable intersection of such countable unions, and so forth). Indeed, our own heuristic is to think of shyness as obtained from finite shyness by closing under countable unions, but we warn the reader that this heuristic is not literally true; see Example 1 in Appendix B.

Our first application, to the general equilibrium foundations of continuous-time finance, is in Section 3. Just as the ordinary Capital Asset Pricing Model (CAPM) is the most fundamental model in discrete-time finance, so Breeden's continuous-time version (CCAPM) is the most fundamental model in continuous-time finance. Duffie and Zame (1989) found sufficient conditions on primitives (utilities and endowments) for existence of an equilibrium satisfying Breeden's requirements (especially nonvanishing consumptions): separable utilities, Inada conditions (infinite marginal utility at 0 consumption), aggregate endowment bounded away from 0. Unfortunately, the last of these conditions is nongeneric in the topological sense. Indeed, Araujo and Monteiro (1991) showed that, given separable utility functions satisfying Inada conditions, the set of endowments for which equilibrium fails to exist is residual; in this sense, "most" endowments lead to an economy for which equilibrium does not exist. This appears to be very unfortunate, since the assumptions used in Duffie and Zame (and related papers by Araujo and Monteiro (1989), Karatzas, Lehoczky, and Shreve (1990), Dana and Pontier (1992) and Dana (1993)) are the only ones known to guarantee existence of an equilibrium satisfying the conditions required in finance. Our measure-theoretic notions, however, yield exactly the opposite result about generic existence: the set of endowments which are bounded away from 0, and in particular the set of endowments for which equilibrium exists, is prevalent (indeed, finitely prevalent). To the extent that prevalence is the more satisfactory notion, this result reverses the very negative result of Araujo and Monteiro (1991).

Our second application, to games with a continuum of actions, is presented in Section 4. Recall that a game form consists of a finite set of agents and a corresponding action space for each agent; a game consists of a game form and an assignment of a payoff profile to each action profile. When action spaces are finite, these payoff assignments can be identified with vectors in $\mathbf{R}^n$. It is well known (Wilson (1971); Harsanyi (1973)) that, for a set of payoff assignments that is open and dense and has full measure, the resulting game has only a finite number of (mixed strategy) Nash equilibria. Here we consider a game form in which action spaces are compact polyhedra in Euclidean space[4] and we consider payoff functions that are smooth in actions. Dubey (1986) showed that, for an open and dense set of payoff functions, the resulting game has finitely many pure strategy Nash equilibria. We show that for a finitely prevalent set of such payoff functions, the resulting game has at most a finite number of pure strategy Nash equilibria.

Our third application, to regular economies, is presented in Section 5. Debreu (1970) defined an economy to be regular if the aggregate excess demand function is smooth and its derivative is of maximal rank at every Walrasian equilibrium (which is a zero of aggregate excess demand). A regular economy has only a finite number of Walrasian equilibria, and the equilibria depend (locally) smoothly on the underlying parameters of the economy (endowments in particular). Thus, for regular economies, comparative statics are meaningful. It is easy to construct nonpathological economies which are not regular, but Debreu (1970) showed that, if preferences are smooth and indifference surfaces have nonvanishing Gaussian curvature and satisfy a boundary condition, then the set of endowments which lead to a regular economy contains an open set of full Lebesgue measure in the space $\mathbf{R}_+^{KN}$ of all endowment profiles. (Here K is the number of commodities and N is the number of consumers.) However, Debreu's Theorem has nothing to say about the situation in which consumers' endowments of some goods are restricted *a priori* to be 0. This is a situation of great interest; as Mas-Colell (1985, page 336) writes, " ...it does not make sense in a reasonably rich model to allow the initial endowment vectors to be perturbed in any direction (implying that for a consumer to have a zero endowment of some commodity is a pathological situation!)" Here we show that, even when we allow for some consumers to have 0 endowments of some commodities, the

[4] More generally, we can allow for action spaces that are stratified by smooth manifolds.

regular economies form an open prevalent set. Indeed, if we fix endowments of all consumers and the utility functions of all but a single consumer, then the set of utility functions for that consumer for which the resulting economy is regular is open and prevalent.[5]

Our fourth application, to the cores of transferable utility games, is presented in Section 6. Following Ostroy (1984), Makowski and Ostroy (1992) and Gretsky, Ostroy and Zame (1992, 1999), we give a formulation of transferable utility games in terms of population distributions that abstracts a more familiar description of transferable utility economies in terms of population distributions. The core of such a game can be identified with the subdifferential of the total gains function at the population distribution, so the core is a singleton exactly when the gains function is Gâteaux differentiable. By establishing an infinite dimensional version of Rademacher's theorem (generic differentiability of Lipschitz functions), we show that the set of population distributions for which the gains function is differentiable, and hence the core is a singleton, is prevalent; within this set, the core is a continuous function of the population distribution.

We caution the reader that, in the infinite dimensional context, prevalence does not correspond to any obvious notion of beliefs. As Stinchcombe (2001) points out, every regular Borel probability measure on an infinite dimensional Banach space lives on a set which is the countable union of compact sets. Since compact sets in infinite dimensional Banach spaces are shy, and countable unions of shy sets are shy, this means that every regular Borel probability measure on an infinite dimensional Banach space lives on a shy set. But we think that, in this case, failure to correspond to any obvious notion of beliefs is a virtue. To see why, it is useful to think about the interpretation of genericity in various contexts.

- Consider first a model that involves some parameter $x \in \mathbf{R}$. Sup-

[5]We treat economies with a finite number of commodities. Determinacy with an infinite-dimensional commodity space is a more difficult problem; see Kehoe and Levine (1985), Kehoe, Levine, Mas-Colell and Zame (1989), Chichilnisky and Zhou (1996), and especially Shannon (1999) and Shannon and Zame (1999).

pose the model exhibits some particular nice behavior when x is not an integer. Because the set $\mathbf{Z}$ of integers has Lebesgue measure 0 in $\mathbf{R}$, it might seem natural to conclude that the model "typically" exhibits this nice behavior. But economics is full of instances in which the parameters are *necessarily* integer-valued. In such a circumstance viewing non-integer values of the parameter as typical and integer values as exceptional would make no sense at all. Whatever notion of typicality and exceptionality we use in a particular circumstance *must* be informed by economic modeling; we cannot expect one notion to suit all circumstances.

- Consider next a model in which the initial conditions include the supply of two commodities. If the model exhibits some particular nice behavior except for a set of initial conditions having two-dimensional Lebesgue measure 0, it might seem natural to conclude that this particular nice behavior is "typical." But such a conclusion could be very misleading if the commodities in question occur only in matched pairs — new pairs of shoes for instance. The suitability of 2-dimensional Lebesgue measure as a criterion for typicality depends on a *presumption* — that initial holdings could "just as well" be $(1, 3)$ as $(3, 1)$ or $(2, 2)$ or anywhere else in the positive orthant, that we don't know any prior restrictions on the parameters.

- Finally, consider a model that involves some parameter x whose values are in some infinite dimensional Banach space X. If the presumption is that x lies in some sigma-compact set, then of course we should use a notion of typicality that takes this into account. But sigma-compact subsets of infinite dimensional Banach spaces are very small. In particular, if $G \subset X$ is sigma-compact then there exists an infinite set $\{x_i\} \subset X$ having the property that G and all the translates $G + x_i$ are pairwise disjoint. A presumption that x lies in some sigma-compact set is therefore inconsistent with the presumption that the values of the parameter could "just as well" be *anywhere* in X. As we have noted, every regular Borel measure on X lives on some sigma-compact set, so a presumption that the parameter x is belongs to this set is again inconsistent with the presumption that the values of the parameter could "just as well" be *anywhere* in X.

The preceding considerations might have lead one to guess that there is *no* notion of typicality that is consistent with the presumption that the values of the parameter could "just as well" be anywhere in X. Being a residual set (in the sense of the Baire category theorem) provides such a notion of typicality — but it is a notion that is quite different from any measure-theoretic notion (even in the finite-dimensional context), and hence is not generally regarded as completely satisfactory. It seems all the more remarkable, therefore, that prevalence provides a notion of typicality that is one the one hand grounded in a measure-theoretic approach and on the other hand consistent with the presumption that the values of the parameter could "just as well" be anywhere in the space.

2 Shyness and Prevalence in a Convex Set

Hunt, Sauer and Yorke (1992) make the following definitions. (The setting in Christensen (1974) is more general and the terminology is different, but the notions are equivalent.)

Definition 2.1 Let X be a completely metrizable topological vector space. A Borel set $E \subset X$ is *shy* if there exists a regular[6] Borel probability measure μ on X with compact support such that $\mu(E + x) = 0$ for every $x \in X$. A (not necessarily Borel) subset $F \subset X$ is shy if it is contained in a shy Borel set. A subset $Y \subset X$ is *prevalent* if $X \setminus Y$ is shy.

[6] A Borel measure μ is regular if, for every Borel set B,

$$\mu(B) = \sup\{\mu(C) : C \subset B, C \text{ compact}\} = \inf\{\mu(D) : D \supset B, D \text{ open}\}.$$

As we have noted in the Introduction, these definitions are not satisfactory for our purposes because we are interested in parameter spaces which are themselves small subsets of the ambient topological vector space. What we want, therefore, are definitions of shyness and prevalence relative to a subset $C \subset X$. However, it is not at all obvious what these definitions should be. It seems obvious that we should require that the measure μ be supported on C, but that will not be enough to yield the properties we desire. To understand the difficulties it is useful to consider the Hunt, Sauer and Yorke (1992) proofs that the union of two shy sets is shy and that the union of countably many shy sets is shy:

1) Let $E_1, E_2 \subset X$ be shy Borel sets. To show that the union $E_1 \cup E_2$ is shy, choose compactly supported measures μ_1, μ_2 such that $\mu_1(E_1 + x) = 0 = \mu_2(E_2 + x)$ for every $x \in X$. The convolution $\mu = \mu_1 * \mu_2$ is the measure defined by:

$$\mu(A) = \int_X \mu_1(A - x) d\mu_2(x)$$

   It is easily checked that $\mu_1 * \mu_2$ is a compactly supported measure and that $(\mu_1 * \mu_2)(E + x) = 0$ for every $x \in X$. Hence $E_1 \cup E_2$ is shy.

2) Let $E_1, E_2, \ldots \subset X$ be shy Borel sets. To show that the union $\bigcup E_n$ is shy, choose for each n a compactly supported measure μ_n such that $\mu_n(E_n + x) = 0$ for every $x \in X$. The infinite convolution $\mu_1 * \mu_2 \ldots$ may not converge, but we can remedy this situation by modifying the measures μ_n. Choose, for each n, a point $y_n \in \text{supp } \mu_n$ and define μ'_n by $\mu'_n(A) = \mu_n(A - y_n)$, so $0 \in \text{supp } \mu'_n$. Write B_n for the closed ball of center 0, radius 2^{-n} and let ν_n be the restriction of μ'_n to B_n, normalized to have total mass 1. It is now easily checked that the infinite convolution $\nu = \nu_1 * \nu_2 * \ldots$ does converge and that $\nu(E + x) = 0$ for each $x \in X$.

If we attempt to relativize the definition of shyness simply by adding the requirement that the measure be supported on C, we encounter difficulties in each of these arguments. In the first argument, the difficulty is that the support of the convolution of two measures is the sum of the supports; hence if μ_1, μ_2 are supported on C then the convolution $\mu_1 * \mu_2$ is supported on $C + C$ and not on C. In the second argument, one difficulty is that the translates μ'_n might not be supported in C. A second difficulty is that, even if the translates μ'_n happen to be supported in C, the infinite convolution $\nu_1 * \nu_2 * \ldots$ will not be supported in C.

These considerations point the way to the following definition.

Definition 2.2 Let X be a topological vector space and let $C \subset X$ be a convex subset which is completely metrizable in the relative topology induced from X.[7] Let $c \in C$. We say that a set $E \subset C$ which is universally measurable in X[8] is *shy in* C *at* c if for each $\delta > 0$ and each neighborhood W of 0 in X, there is a regular Borel probability measure μ on X with compact support such that supp $\mu \subset [\delta(C - c) + c] \cap (W + c)$ and $\mu(E + x) = 0$ for every $x \in X$.[9] [Note that convexity of C entails that $\delta(C - c) + c = \delta C + (1 - \delta)c$ is a subset of C.] We say that E is *shy in* C if it is shy at each point $c \in C$. An arbitrary subset $F \subset C$ is *shy in* C if it is contained in a shy universally measurable set. A subset $Y \subset C$ is *prevalent in* C if its complement $C \setminus Y$

[7]Although the definition would make perfectly good sense if C were not complete or metrizable, we will see below that, without these restrictions, we would be unable to establish the desired properties. If X is competely metrizable then C will also be, provided that C is open, or closed, or the intersection of a closed set and a open set, or the intersection of a closed set and countable many open sets (see Christensen (1974), page 56). We prefer to make assumptions directly on C rather than on X because that is more general, and the generality is useful. For instance, if K is a compact metric space then the cone $M_+(K)$ of nonnegative measures on K is completely metrizable in the weak star topology, but the entire space $M(K)$ of all measures on K is not.

[8]Recall that a subset $C \subset X$ is *universally measurable* if it is measurable with respect to the completion of every regular Borel probability measure on X. The extension to universally measurable sets is useful in some applications (see Shannon (1998), for instance) because the continuous image of a Borel set (indeed the projection of a Borel set) need not be Borel, but is always universally measurable.

[9]Since μ is supported on C, and E is universally measurable, C and E are measurable with respect to the completetion of μ. To ease the notation, we shall not distinguish between μ and its completion.

is shy in C.[10]

When there is no danger of confusion, we omit the explicit reference to C, and simply say that a set is *shy* or *prevalent*.

The definition seems awkward because the point c seems to play a special role; as the following Fact shows, however, the special role is an illusion. We defer the proof of this and the other Facts to Appendix A.

Fact 0 If E is shy at some point of C then it is shy at every point of C and hence shy in C.

The following Facts record that shyness and prevalence in a convex set C satisfy all the properties we should expect of a measure-theoretic definition of smallness.

Fact 1 Every subset of a set that is shy in C is shy in C.

Fact 2 If E is shy in C then $E + x$ is shy in $C + x$ for every $x \in X$.

Fact 3 The countable union of sets that are shy in C is shy in C.

[10]When $C = X$, shyness for Borel subsets of C is equivalent to shyness for Borel sets in the sense of Hunt, Sauer and Yorke: If E is Borel and shy in the sense of Hunt, Sauer and Yorke, there is a compactly supported probability measure μ such that $\mu(E + x) = 0$ for each $x \in X$. Given $c \in X$, $\delta > 0$ and a neighborhood W of $0 \in X$, pick any $c' \in \text{supp}\ \mu$ and define the translate μ' by

$$\mu'(A) = \mu(A + c' - c)$$

Clearly, $\mu'(E+x) = 0$ for every x. Because W is a neighborhood of $0 \in X$ and $c' \in \text{supp}\ \mu$ it follows that $\mu(W + c') > 0$, so $\mu'(W + c) > 0$. Since μ is regular, so is μ', so we may choose a compact set $K \subset W + c$ such that $\mu'(K) > 0$; let $\mu'' = \mu'_{|K}$ be the restriction, normalized to have total mass 1. It is evident that

$$\text{supp}\ \mu'' \subset K \subset W + c = [\delta(X - c) + c] \cap (W + c)$$

and that $\mu''(E + x) = 0$ for every $x \in X$. Hence E satisfies our definition of shyness.

Fact 4 No relatively open subset of C is shy in C.

Fact 5 If $X = \mathbf{R}^n$ and C has nonempty interior in X, then $E \subset C$ is shy in C if and only if the Lebesgue measure of E is 0.

Demonstrating that a set is shy might be a difficult task, but there is a simple sufficient condition. If $V \subset X$ is a finite dimensional subspace, we write λ_V for Lebesgue measure on V.[11]

Definition 2.3 A universally measurable subset $E \subset C$ is *finitely shy in* C if there is a finite-dimensional subspace $V \subset X$ such that $\lambda_V(C + a) > 0$ for some $a \in X$[12] and $\lambda_V(E + x) = 0$ for every $x \in X$. An arbitrary subset $F \subset X$ is finitely shy in C if it is contained in a finitely shy universally measurable set.

Fact 6 Every set that is finitely shy in C is shy in C.

Taken together, Facts 3 and 6 provides a simple but powerful method for demonstrating that a set is shy: exhibit it as the countable union of finitely shy sets. As an easy illustration, we demonstrate the shyness of the everywhere differentiable concave functions in the cone of all concave functions. Let K be a compact convex subset of $\mathbf{R}^n$ with nonempty interior. Write $C(K)$ for the space of continuous real-valued functions on K, equipped with the sup norm topology, and $C_{\text{con}}(K) \subset C(K)$ for the subset of concave functions; note that $C_{\text{con}}(K)$ is a closed convex cone. Gruber (1977) and Howe (1982) showed (independently) that the subset $C^{\text{diff}}_{\text{con}}(K) \subset C_{\text{con}}(K)$ consisting of functions that are everywhere differentiable on the interior of K is a residual

[11] Since V is finite dimensional, there is a continuous linear isomorphism $T : V \to \mathbf{R}^k$ for some k. If λ_k is Lebesgue measure on $\mathbf{R}^k$, we may define $\lambda_V(A) = \lambda_k(T(A))$ for every Borel set $A \subset V$. Of course this definition of λ_V depends on the choice of linear isomorphism T, but different choices lead to measures which are mutually absolutely continuous and hence have the same sets of measure 0. Since we are interested only in sets of measure 0, we find it convenient to abuse notation in this way.

[12] We allow the possibility that $\lambda_V(C + a) = \infty$.

subset of $C_{\rm con}(K)$. (By contrast, the nowhere differentiable continuous functions form a residual subset of $C(K)$.) However, in our measure-theoretic sense, the everywhere differentiable concave functions form a small set.

Theorem 2.4 *Let K be a compact convex subset of $\mathbf{R}^n$ with nonempty interior. The set $C_{\rm con}^{\rm diff}(K)$ of concave functions differentiable everywhere on the interior of K is shy (in fact finitely shy) in the cone $C_{\rm con}(K) \subset C(K)$ of concave functions.*

Proof Choose a point x_0 in the interior of K and write $D \subset C_{\rm con}(K)$ for the subset of functions which are differentiable at x_0. We assert that D is a Borel set and is finitely shy in $C_{\rm con}(K)$.

To see that D is a Borel set (which is much the harder part), choose a countable dense subset $\{x_i : i \in \mathbf{N}\}$ of the interior of K with $x_i \neq x_0$ for $i \geq 1$ and a countable dense subset $Q \subset R^n$. A function $f : K \to \mathbf{R}$ is differentiable at x_0 if and only if there is a vector $\gamma \in \mathbf{R}^n$ such that

$$\lim_{x \to x_0} \frac{|f(x) - f(x_0) - \gamma \cdot (x - x_0)|}{\|x - x_0\|} = 0$$

Equivalently, f is differentiable at x_0 if and only if for each integer $k > 0$ there is an integer $\ell > 0$ and a vector $q \in Q$ such that

$$\frac{|f(x_i) - f(x_0) - q \cdot (x_i - x_0)|}{\|x_i - x_0\|} < \frac{1}{k}$$

whenever $|x_i - x_0| < \frac{1}{\ell}$. For each k, ℓ, q, i, set

$$D(k, \ell, q, i) = \left\{ f \in C_{\rm con}(K) : \frac{|f(x_i) - f(x_0) - q \cdot (x_i - x_0)|}{\|x_i - x_0\|} < \frac{1}{k} \right\}$$

Note that $D(k, \ell, q, i)$ is an open subset of $C_{\rm con}(K)$ and that

$$D = \bigcap_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} \bigcup_{q \in Q} \Big(\bigcap_{0 < |x_i - x_0| < \frac{1}{\ell}} D(k, \ell, q, i) \Big)$$

so that D is a Borel set.

To see that D is finitely shy in C, define a function $h \in C(K)$ by

$$h(x) = -|x - x_0|$$

Note that h is continuous and concave, but is not differentiable at x_0. Let $V \subset C(K)$ be the one-dimensional space spanned by h. Since th is concave for each $t \geq 0$, it is evident that $\lambda_V(C_{\text{con}}(K)) > 0$. On the other hand, we claim that $\lambda_V(D + g) = 0$ for every $g \in C(K)$. To see this, it suffices to show that $(V - g) \cap D$ is empty or a singleton. If this were not so, we could find distinct real numbers $t_1 \neq t_2$ and distinct functions $d_1, d_2 \in D$ so that $d_1 = t_1 h - g, d_2 = t_2 h - g$. The difference of two functions that are differentiable at a given point is again differentiable at that point, so $d_1 - d_2$ is differentiable at x_0. On the other hand, $d_1 - d_2 = (t_1 - t_2)h$, which, by construction, is not differentiable at x_0. This is a contradiction, so $(V - g) \cap D$ is empty or a singleton, and hence D is finitely shy in $C_{\text{con}}(K)$. Because $C^{\text{diff}}_{\text{con}}(K) \subset D$ and every subset of a finitely shy set is finitely shy, we conclude that $C^{\text{diff}}_{\text{con}}(K)$ is finitely shy, as desired. ■

Two more Facts will be useful. The first asserts that a prevalent subset of a prevalent set is prevalent.

Fact 7 If E is prevalent in F and F is prevalent in G then E is prevalent in G.

Finally we show that, just as the special role of the distinguished point c is an illusion, so is the special role of the ambient vector space X.

Fact 8 Let X, Y be topological vector spaces, let $C \subset X$ and $C \subset Y$ such that X and Y induce the same topology on C, and this topology is completely metrizable.[13] Let $E \subset C$. Then E is shy in C (viewed within X) if and only if E is shy in C (viewed within Y).

[13] In particular, we might have $C \subset X$ and X a subspace of Y.

3 Equilibrium in Financial Models

We recall the standard continuous-time financial model; for more detail see Duffie and Huang (1985), Duffie (1986) or Duffie and Zame (1989). Our economy evolves over a finite time interval $[0, T]$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space, so that $(\mathcal{F}_t)$ is an increasing family of sub-σ-algebras of $\mathcal{F}$, indexed by $t \in [0, T]$, having the property that $\mathcal{F}_T = \mathcal{F}$. We interpret $\mathcal{F}_t$ as representing information available up to time t. The consumption space L^2 consists of square integrable predictable stochastic processes; that is, functions $x : \Omega \times [0, T] \to \mathbf{R}$ for which the consumption rate $x(\cdot, t)$ depends only on information available up to time t, and for which

$$E\left[\int_0^T x(\cdot, t)^2 dt\right] < \infty$$

(All expectations are with respect to P.) Write L^2_+ for the cone of nonnegative functions in L^2.[14]

We consider a pure exchange economy with n agents. Agent i is described by a consumption set, which we take to be L^2_+, an endowment $e_i \in L^2_+$, and a utility function $U_i : L^2_+ \to \mathbf{R}$, which we assume to be of the form

$$U_i(x) = E\left[\int_0^T v_i\left(x(t, \cdot), t\right) dt\right]$$

where, for each $t \in T$, the function $v_i(\cdot, t) : \mathbf{R}_+ \to \mathbf{R}$ is smooth, strictly concave, increasing, and satisfies the uniform Inada condition:

$$\frac{\partial v_i}{\partial x}(x, t) \to \infty \quad \text{as } x \searrow 0 \text{ , uniformly for } t \in [0, T]$$

Duffie and Zame (1989) establish the following theorem.

Theorem 3.1 *If, in addition to the assumptions above, the aggregate endowment $\sum e_i$ is uniformly bounded away from 0, then there exists a Walrasian equilibrium, and every Walrasian equilibrium has the property that consumptions x_i are uniformly bounded away from 0.*

[14] As usual, we identify functions which are equal almost everywhere.

Inada conditions are assumed in order to guarantee that equilibrium consumptions are everywhere nonvanishing, a requirement for Breeden's CAPM. However, imposing Inada conditions means that utility functions will necessarily fail to admit supporting prices at some points; see Mas-Colell (1985) and Mas-Colell and Zame (1991). Assuming that the aggregate endowment is uniformly bounded away from 0 guarantees the existence of supporting prices at the "right" allocations, and hence the existence of equilibrium. However, this is a strong assumption. Indeed, the set of functions in L^2_+ which are bounded away from 0 is first category, and thus exceptional in the topological sense. Araujo and Monteiro (1989) have shown that, for given utility functions satisfying Inada conditions, the set of endowments $(e_1, \ldots, e_n)$ for which an equilibrium exists is first category in $(L^2_+)^n$.

We show, however, that our measure-theoretic notion of typicality leads to exactly the opposite conclusion.

Theorem 3.2 *The set of endowments*

$$\{(e_1, \ldots, e_n) \in (L^2_+)^n : \sum_{i=1}^{n} e_i \textit{ is uniformly bounded away from 0}\}$$

(and a fortiori *the set of endowments for which equilibrium exists) is finitely prevalent in* $(L^2_+)^n$.

This result follows immediately from the following abstract result.

Theorem 3.3 *For every Banach lattice*[15] X *and every* $e \in X_+$*, the set*

$$P_e = \{x \in L_+ : \exists \varepsilon > 0\,,\, x \geq \varepsilon e\}$$

is finitely prevalent in X_+.

Proof Note first that $P_e = \cup_{n \in \mathbf{N}} \left\{x : x \geq \frac{e}{n}\right\}$ is a countable union of closed sets, and hence is a Borel set. Let $E = X_+ \setminus P_e$. Let $V \subset X$ be the one-dimensional subspace spanned by e, and let $x \in X$. We claim $\lambda_V(E+x) = 0$;

[15]See Aliprantis and Burkinshaw (1985) for instance.

indeed, we claim that $V \cap (E+x)$ is empty or a singleton. To see this, suppose not. Then we can find real numbers $t_1 > t_2$ and elements $w_1, w_2 \in E$ such that $t_1 e = w_1 + x, t_2 e = w_2 + x$. Then

$$w_1 = w_2 + (t_1 - t_2)e \geq (t_1 - t_2)e$$

so that $w_1 \in P$, which is a contradiction. We conclude that $V \cap (E + x)$ is empty or a singleton, so that $\lambda_V(E+x) = 0$, as asserted. Since $\lambda_V(X_+) > 0$, it follows that P is finitely prevalent in X_+, as asserted. ■

If we take $X = (L^2)^n$ and $e = (\mathbf{1}, \ldots, \mathbf{1})$ (where $\mathbf{1}$ is the function which is identically 1), it follows immediately that

$$\{(e_1, \ldots, e_n) \in (L^2_+)^n : \exists \varepsilon > 0\,,\ (e_1, \ldots, e_n) \geq \varepsilon e\ \}$$

is finitely prevalent. Since the set of endowment profiles for which the social endowment is bounded away from 0 contains this set, Theorem 3.2 follows immediately.

4 Pure Strategy Nash Equilibria

Fix a set of $N > 1$ players and strategy spaces $A_1, \ldots, A_N$. For simplicity, assume that each $A_n \subset \mathbf{R}^{k_n}$ is a compact convex polyhedron (i.e., the intersection of a finite number of closed half-spaces), and (without loss) has nonempty interior. (As we discuss below, these assumptions are much stronger than necessary; all we need is that strategy spaces are nice unions of smooth manifolds of various dimensions.) Write $K = k_1 + \ldots + k_N$, so that $A = A_1 \times \ldots \times A_N \subset \mathbf{R}^K$ is the space of action profiles. Let $C^2(A)$ be the space of functions from A to $\mathbf{R}$ that have twice continuously differentiable extensions to some neighborhood of A, equipped with the C^2 norm (supremum of the absolute value of the function and its first and second derivatives); this is a Banach space.[16] Let $\mathcal{G} = C^2(A)^N$ denote the set of C^2 payoff functions $u = (u_1, \ldots, u_N) : A \to \mathbf{R}^N$. Each $u \in \mathcal{G}$ defines a game

$$\Gamma_u = (\{1, \ldots N\}, (A_n), u)$$

[16] See Malgrange (1966) and Dunford and Schwartz (1957, II.4.13, page 72).

Dubey (1986)[17] showed that there is an open, dense set $G \subset \mathcal{G}$ such that for all $u \in G$,

- Γ_u has finitely many pure strategy Nash equilibria
- if $(x_1, \ldots, x_n)$ is a Nash equilibrium of Γ_u and no x_i is a vertex of A_i then $(x_1, \ldots, x_n)$ is Pareto inefficient

Here we refine Dubey's result by showing that G may also be chosen to be finitely prevalent.

Theorem 4.1 *There is an open and finitely prevalent set $G \subset \mathcal{G}$ such that for all $u \in G$,*

- *Γ_u has a finite number of pure strategy Nash equilibria*[18]
- *if $(x_1, \ldots, x_n)$ is a Nash equilibrium of Γ_u and no x_i is a vertex of A_i then $(x_1, \ldots, x_n)$ is Pareto inefficent*

Proof By assumption, A_n is a polyhedron of dimension k_n, and hence has a (unique) partition into open faces, each of which is an open polyhedron (i.e., the intersection of a subspace and a finite number of open halfspaces). The unique open face of dimension k_n is the interior of A_n, and the faces of dimension 0 are the vertices of A_n. It is convenient to consider equilibrium

[17] Dubey's setting differs in minor ways from ours. He required that the strategy space of each player be a simplex, and he considered payoffs that are C^2 on a fixed open set $V \supset A$.

[18] Since we do not assume payoff functions are concave in own strategy, the number of pure strategy equilibria may in fact be zero. However, the set $\mathcal{G}_{\text{con}}$ of payoff functions with the property that, for each i, the second derivative of player i's payoff is negative definite in player i's strategy (everywhere on A) is a convex open set in $\mathcal{G}$. If $u \in \mathcal{G}_{\text{con}}$, the game Γ_u admits at least one pure strategy Nash equilibrium. $\mathcal{G}_{\text{con}}$ is completely metrizable (Christensen (1974), page 56). It follows immediately from Theorem 4.1 that there is a prevalent subset of $\mathcal{G}_{\text{con}}$ such that games with payoff functions in this set admit at least one but only a finite number of pure strategy Nash equilibria.

profiles in each array of open faces (one for each player) because, in an open face, we can analyze equilibrium through the first order conditions.

We therefore let $\mathcal{B}_n$ be the set of open faces of A_n and let $\mathcal{B} = \Pi \mathcal{B}_n$ be the set of profiles of open faces. If $\beta = (B_1, \ldots, B_N) \in \mathcal{B}$, let ℓ_n be the dimension of B_n. Write $B = B_1 \times \ldots \times B_N$ and $L = \ell_1 + \cdots + \ell_N$. We may assume without loss of generality that each B_n is an open subset of $\mathbf{R}^{\ell_n}$ (which we identify as the subspace of $\mathbf{R}^{k_n}$ consisting of vectors whose last $k_n - \ell_n$ coordinates vanish), and that $0 \in B_n$. For each n write $\overline{B}_n$ for the closure of B_n and $\overline{B} = \overline{B}_1 \times \ldots \times \overline{B}_N$ (the closure of B).

We first show that the set of pure strategy Nash equilibria is generically finite. For $\alpha = (\alpha^1, \ldots, \alpha^N) \in \mathbf{R}^L$, define a perturbation[19] $\pi^\alpha : \mathbf{R}^L \to \mathbf{R}^N$ by

$$\pi^\alpha_n(x^1, \ldots, x^N) = \sum_{i=1}^{\ell_n} \alpha^n_i x^n_i$$

Given $u \in \mathcal{G}$ and $\beta \in \mathcal{B}$, define $F_{u\beta} : \overline{B} \times \mathbf{R}^L \to \mathbf{R}^L$ by

$$F_{u\beta}(x, \alpha) = \left(\frac{\partial (u + \pi^\alpha)_1}{\partial x^1}, \ldots, \frac{\partial (u + \pi^\alpha)_N}{\partial x^N} \right) \Bigg|_{(x,\alpha)}$$

Note that for all $u \in \mathcal{G}$ and all $\beta \in \mathcal{G}$, F is C^1 on a neighborhood of $\overline{B} \times \mathbf{R}^L$.

Consider the $L \times L$ matrix

$$J_{u\beta}(x, \alpha) = \frac{\partial F}{\partial (x^1, \ldots, x^N)} \Bigg|_{(x,\alpha)}$$

Set $G(\beta) = \{u : F_{u\beta}(x, 0) = 0 \Rightarrow \det (J_{u\beta}(x, 0)) \neq 0\}$ and $G = \bigcap_{\beta \in \mathcal{B}} G(\beta)$. We claim that

- every game $u \in G$ has finitely many pure strategy Nash equilibria
- G is open

[19] Dubey (1986) uses similar perturbations. Our perturbations affect the dependence of player i's payoff on player i's strategy, while Dubey's perturbations affect the dependence of each player's payoff on the strategies of all the players.

- G is finitely prevalent, hence prevalent (Fact 6)

To see the first claim, note that if $u \in G(\beta)$, the inverse function theorem guarantees that the $x \in \overline{B}$ for which $F_{u\beta}(x,0) = 0$ are locally isolated; since $\overline{B}$ is compact, there are only finitely many such $x \in \overline{B}$. Hence, there are only finitely many pure strategy equilibria of Γ_u in B.[20] Any pure strategy Nash equilibrium of Γ_u either occurs at a vertex (i.e. a 0-dimensional face of A) — of which there are only a finite number — or in the interior of some face (i.e. in B for some $\beta \in \mathcal{B}$), so if $u \in G$ then u has only finitely many pure strategy Nash equilibria.

To establish the second claim, we show that $\mathcal{G} \setminus G(\beta)$ is closed for each $\beta \in \mathcal{B}$. To this end, suppose that $u_n \in \mathcal{G} \setminus G(\beta)$ and $u_n \to u \in \mathcal{G}$. Find $x_n \in \overline{B}$ such that $F_{u_n\beta}(x_n, 0) = 0$ and $\det\left(J_{u_n\beta}(x_n, 0)\right) = 0$. Since $\overline{B}$ is compact, choose a convergent subsequence $x_{n_j} \to 0$. Then $F_{u\beta}(x,0) = \lim_{j\to\infty} F_{u_{n_j}\beta}(x_{n_j}, 0) = 0$ and $\det J_{u\beta}(x,0) \lim_{j\to\infty} \det(J_{u_{n_j}\beta}(x_{n_j}, 0)) = 0$, so $u \in \mathcal{G} \setminus G(\beta)$, proving the second claim.

To establish the third claim, set $E(\beta) = \mathcal{G} \setminus G(\beta)$ for each β. Each π^α belongs to $\mathcal{G}$, so

$$H = \{\pi^\alpha : \alpha \in \mathbf{R}^L\} \subset \mathcal{G}$$

is an L-dimensional subspace of $\mathcal{G}$. Obviously $\lambda_H(\mathcal{G}) > 0$; we show that $\lambda_H(E(\beta) - u) = 0$ for every $u \in \mathcal{G}$. To this end, write

$$H_u = \{\alpha \in \mathbf{R}^L : u + \pi^\alpha \in E(\beta)\}$$

for each $u \in \mathcal{G}$. Since λ_H is supported on H, it will suffice to show that $\lambda_H(H_u) = 0$ for each $u \in \mathcal{G}$. To this end, consider the Jacobian matrix of $F_{u\beta}$. This is an $L \times 2L$ matrix of the form $(J_{u\beta}|I)$, where I is an $L \times L$ identity matrix, and so has has rank L for every (x, α). By the Thom Transversality theorem, there is a subset $W \subset \mathbf{R}^L$ of full Lebesgue measure such that if

[20] This argument does *not* establish that there are only a finite number of pure strategy Nash equilibria of Γ_v in $\overline{B}$, because boundary equilibria need not satisfy the first order conditions. This will not be a problem, however, because boundary equilibria of a face are interior equilibria of lower dimensional faces.

$\alpha \in W$ and $F_{u\beta}(x, \alpha) = 0$ then $\det(J_{u\beta}(x, \alpha)) \neq 0$, i.e. $u + \pi^\alpha \notin E(\beta)$. Since W has full Lebesgue measure, we conclude that $\lambda_H(H_u) = 0$. Hence $G(\beta)$ is finitely prevalent. Since we are concerned with finite prevalence in the ambient space, our notion coincides with that of Hunt, Sauer and York (1992); these authors show that the finite intersection of finitely prevalent sets is finitely prevalent, so we conclude that G is finitely prevalent, as claimed. This establishes generic finiteness of the set of Nash equilibria.

Now, we turn to the proof of generic Pareto inefficiency. The structure of the proof is similar to that of the proof of generic finiteness of equilibrium, but the perturbations are different. To highlight the common elements of the two proofs, we use the same notation for an object here as was used for the corresponding object in the proof of generic Pareto inefficiency. Given $u \in \mathcal{G}$ and $\beta \in \mathcal{B}$ with $\ell_n \geq 1$ $(1 \leq n \leq N)$, define u_β to be the restriction of u to $\overline{B}$, so $u_\beta : \overline{B} \to \mathbf{R}^N$. The Jacobian $Ju_\beta|_x$ is an $N \times L$ matrix. Let $M_{u\beta}(x)$ be the $N \times N$ submatrix consisting of the columns of $Ju_\beta|_x$ corresponding to $x_1^1, x_1^2, \ldots, x_1^n$, the first strategy direction for each player; let $f_{u\beta(x)} = \det M_{u\beta}(x)$. Let $F_{u\beta} : \overline{B} \to \mathbf{R}^{N+1}$ be the function whose first N components are the diagonal entries of $M_{u\beta}(x)$, and whose $N+1^{\text{st}}$ component is $f_{u\beta}(x)$.[21] Note two things:

- If $x \in B$ is a Nash equilibrium then the diagonal entries of $M_{u\beta}$ are all zero.
- If $x \in B$ is a Pareto optimum then the rows of $Ju_\beta|_x$ are linearly dependent. (If they were linearly independent then the matrix $Ju_\beta|_x$ would define an onto map from $\mathbf{R}^L$ to $\mathbf{R}^N$, so $\mathbf{1} = (1, \ldots, 1)$ would be in the range of this map; say $\mathbf{1} = \mathbf{Ju}_\beta|_\mathbf{x}(\tau)$. But then, for sufficiently small $\varepsilon > 0$, $x + \varepsilon\tau$ would be a strategy profile yielding a higher payoff than x to all players, contradicting Pareto optimality.) Hence, $Ju_\beta|_x$ has rank less than N, so $M_{u\beta}(x)$ has rank less than N, whence $f_{u\beta}(x) = 0$.

Thus, if $x \in B$ is a Pareto optimal Nash equilibrium, then $F_{u\beta}(x) = 0$.

[21] Since u has a C^2 extension to an open neighborhood of A, $J_{u\beta}$ extends uniquely to $\overline{B}$.

Set $G(\beta) = \{u \in \mathcal{G} : 0 \notin F_{u\beta}(\overline{B})\}$ and $H(\beta) = \mathcal{G} \setminus G(\beta)$. Since $\overline{B}$ is compact, $H(\beta)$ is closed, so $G(\beta)$ is open.

Given a matrix $\alpha \in \mathbf{R}^{L\times N}$, we can define an element of $\mathcal{G}$ by $v_\alpha(x) = \alpha x$.[22] Let $V_\beta = \{v_\alpha : \alpha \in \mathbf{R}^{L\times N}\}$ and let μ_β denote the extension to $\mathcal{G}$ of Lebesgue measure on V_β. Then $\mu_\beta(\mathcal{G}) = \infty \neq 0$. For any $w \in \mathcal{G}$,

$$\begin{aligned} \mu_\beta((H(\beta)+w)) &= \mu_\beta((H(\beta)+w)\cap V_\beta) \\ &= \mu_\beta(H(\beta)\cap(V_\beta - w)) \\ &= \mu_\beta(\{v \in V_\beta : 0 \in F_{(v-w)\beta}(\overline{B})\}) \end{aligned}$$

¿From the proof of generic finiteness of equilibrium, we know that for almost all $v_\alpha \in V_\beta$, there are finitely many solutions $x \in B$ to the equations

$$F^n_{(v_\alpha - w)\beta}(x) = 0 \quad 1 \le n \le N$$

Moreover, these solutions depend *only* on the diagonal terms of α. Fix the diagonal terms of such an α, and let the solutions be $x_1, \ldots, x_m$. Now vary the off-diagonal terms of α and note that $f_{(v_\alpha - w)\beta}(x_i)$ is (holding x_i and the diagonal terms of α fixed) a polynomial in the off-diagonal elements of α; it is not the identically zero polynomial, since there are terms in the determinant that do not involve any of the diagonal entries of α, so the set of zeroes of this polynomial is a null set with respect to Lebesgue measure on the off-diagonal elements of α (see Bochnak, Coste and Roy (1987)). Thus, for almost all choices of off-diagonal elements of α, $f_{v_\alpha - w}(x_i) \neq 0$, and hence $f_{v_\alpha - w}(x_i) \neq 0$ $(1 \le i \le m)$. Thus, $\mu_\beta(\{v \in V_\beta : 0 \in F_{(v-w)\beta}(\overline{B})\}) = 0$, so $\mu_\beta((H(\beta)+w)) = 0$, so $H(\beta)$ is closed and finitely shy, so $G(\beta)$ is open and finitely prevalent.

Set $G = \bigcap_\beta G(\beta)$. Since G is a finite intersection of open, finitely prevalent sets, G is open and finitely prevalent. If $u \in G$, and x is a pure strategy Nash equilibrium, then either x_i is a vertex of A_i for some i or x lies in the interior of some face β of A whose dimension is $L = \sum_{n=1}^N \ell_n \ge \sum_{n=1}^N 1 = N$, in which case x cannot be Pareto optimal.■

[22] αx denotes multiplication of the matrix α times the column vector x. This gives a map from $\overline{B}$ to $\mathbf{R}^N$, which is extended to all of A in the obvious way.

Two extensions of Theorem 4.1 are worth noting. The first follows from the observation that the polyhedral nature of strategy sets is not required. What is required is that each of the strategy sets A_n be *stratified by smooth manifolds*, in the sense that

$$A_n = \bigcup_{j=0}^{k_n} A_n^j$$

where each A_n^j is a (possibly empty) smooth j-dimensional submanifold of some open subset of $\mathbf{R}^{k_n}$, $A_n^j \cap A_n^k = \emptyset$ if $j \neq k$, and for each k, the union $\bigcup_{j=0}^{k} A_n^j$ is closed in A_n. (Because A_n^0 is is a 0-dimensional manifold and compact it must then be finite.) This will certainly be the case if A_n is a polyhedron, or a compact manifold, or a compact manifold with boundary, or the finite intersection of compact manifolds with boundaries that intersect transversally, or a semi-algebraic set, or a finitely sub-analytic set; see Blume and Zame (1993). Abusing terminology, we refer to elements of A_n^0 as vertices.

The second extension of Theorem 4.1 follows from the observation that the only requirements on the space of payoff functions are that they be completely metrizable and contain the affine functions. Thus we have the following more general result.

Theorem 4.2 *Assume that each strategy space A_n is stratified by smooth manifolds. Let $\mathcal{G}' \subset \mathcal{G}$ be a subspace of payoff functions on K which contains the affine functions and is completely metrizable in some topology for which the inclusion mapping $\mathcal{G}' \to \mathcal{G}$ is continuous. There is an open and finitely prevalent set $G \subset \mathcal{G}'$ such that for all $u \in G$,*

- *Γ_u has a finite number of pure strategy Nash equilibria*
- *if $(x_1, \ldots, x_n)$ is a Nash equilibrium of Γ_u and no x_i is a vertex of A_i then $(x_1, \ldots, x_n)$ is Pareto inefficent*

For example, $\mathcal{G}'$ might be the space of affine functions, or the space of polynomials of degree at most d, or the space of C^r functions, or the space of real analytic functions on a fixed neighborhood of K.[23] Note that in this setting, it is possible that $A_n^0 = \emptyset$ for each n (so that strategy spaces have no vertices) in which case generically *all* equilibria are Pareto inefficient.

For a game with finite action spaces, payoff functions on the product of the spaces of pure strategies extend affinely to affine functions on the product of the simplices of mixed strategies (which are finite polyhedra), so Theorem 4.2 contains the familiar Wilson (1971) and Harsanyi (1973) result that generic games with finite strategy spaces have a finite number of mixed strategy equilibria.

It would be extremely interesting to have extensions of Theorems 4.1 and 4.2 to mixed strategy equilibria.

5 Regular Exchange Economies

As we have noted in the Introduction, Debreu's (1970) theorem, which asserts that generic exchange economies are regular, has nothing to say about the situation in which some consumers' endowments of some goods are restricted *a priori* to be 0. Even the most casual empiricism, however, suggests that it is common for most consumers to be endowed with a rather small number of commodities which they trade for the much larger number of commodities they consume. It had been conjectured that the conclusions of Debreu's theorem would still obtain even if some agents were constrained to have zero endowment of certain goods, as long as the number of degrees of freedom is at least as large as the number of commodities. However, Minehart (1997) provided a surprising example to show that at least the most natural version of this conjecture is false.[24] On the other hand, Mas-Colell (1985; Proposition 8.7.3 and pages 341-2) showed that, parameterizing economies by endowments *and* preferences, and constraining some consumers' endowments of some commodities to be 0, the set of regular economies is an open

[23] See Malgrange (1966) for the appropriate topology.

[24] Whether a stronger condition on the number of degrees of freedom would suffice to restore generic regularity is unknown.

and dense set.

Here we obtain an analogue of Mas-Colell's result using our measure-theoretic notion of genericity. We show that, even when we constrain some consumers' endowments of some commodities to be 0, the regular economies form an open prevalent set. Indeed, even if we fix endowments of all consumers and utility functions of all but one consumer, the set of utility functions for that consumer for which the resulting economy is regular is open and prevalent.

Although our work is closely related to Araujo (1985), there are important differences. Araujo fixes a compact box B in the commodity space and considers utility functions in $\mathcal{U}_B$, the set of differentiably strictly monotone, differentiably strictly concave utility functions on this box. Because $\mathcal{U}_B$ is an open subset of $C^2(B)$, Araujo is able to appeal to prevalence in $C^2(B)$. However, Araujo's assumptions on utility functions are incompatible with Debreu's, and do not rule out equilibria with boundary consumptions; Araujo therefore defines an economy to be regular if all of its Walrasian equilibria with interior consumptions are regular. Araujo shows that, for fixed strictly positive individual endowments, the set of utility functions which lead to economies that are regular in this sense is a prevalent subset of $\mathcal{U}_B$. Since we use a notion of relative prevalence, we are able to work directly in Debreu's setting, and also to consider individual endowments that are 0 in some commodities.

Our formulation will be slightly unusual in two ways. First, we take utility functions — rather than preferences — as primitives. We need to take this step because, while there is a natural vector space structure on the space of utility functions, there is no obvious vector space structure on the space of preferences. In our context, however, the difference between utility functions and preferences is not large one: Mas-Colell (1985, Propositions 2.3.5 and 2.3.9) shows that every C^r preference relation on $\mathbf{R}^K_{++}$ with connected indifference surfaces admits a C^r utility representation u with no critical point and that every C^r utility function $u : \mathbf{R}^K_{++} \to \mathbf{R}$ with no critical point represents a C^r preference. The utility representation is not unique: if u is a

C^r utility function with no critical point, and $h : \mathbf{R} \to \mathbf{R}$ is any C^r function with $h'(t) > 0$, then $h \circ u$ is a C^r utility function with no critical point representing the same preference as u. On the other hand, if we normalize utility representations by requiring $u(t, \cdots, t) = t$ for all $t > 0$, then the space of C^r monotonic preferences is homeomorphic under the natural map to the space of normalized monotonic C^r utility functions with no critical points (in the topology of C^r convergence on compact subsets of $\mathbf{R}^K_{++}$); see Mas-Colell (1985, Proposition 2.4.3).

The second unusual element of our formulation is that we restrict attention to concave utility functions, rather than to quasiconcave utility functions. The reason is that we need the ambient space of utility functions to be convex and, while a convex combination of two concave functions is concave, a convex combination of two quasiconcave functions need not be quasiconcave. This is a restriction, but not a stringent one. Mas-Colell (1985, Proposition 2.6.4) shows that, given any C^2 monotone differentiably strictly convex preference relation $\succeq$ and any compact $X \subset \mathbf{R}^K_{++}$, there is a C^2 utility function $\hat{u}$ with no critical point representing $\succeq$ and such that the restriction of $\hat{u}$ to X is differentiably strictly concave. Given fixed endowments $e(1), \ldots, e(N)$ and strongly monotonic preferences $\succeq_2, \ldots, \succeq_N$, there is a compact set P in the open simplex of strictly positive prices such that no price outside of P can be an equilibrium (because there is some good such that the demand of agents $2, \ldots, N$ for that good exceeds the social endowment of that good). Then as long as $e(1) \neq 0$, there is a consumption vector $y \in \mathbf{R}^K_{++}$ such that $p \cdot y \leq p \cdot e(1)$ for all $p \in D$. Then whether or not $\succeq_1$ results in a regular economy depends only on the behavior of $\succeq_1$ on $\{x : p \cdot x \leq p \cdot e(1)$ for all $p \in D, x \succeq_1 y\}$, which, under our assumptions, is a compact subset of $\mathbf{R}^K_{++}$, so $\succeq_1$ has a utility representation which is concave on this set.

We consider exchange economies with K goods $k = 1, \ldots, K$ and N agents $n = 1, \ldots, N$. We define the relevant spaces of utilities below.

Definition 5.1 Let $\mathcal{D}$ be the vector space of functions $u : \mathbf{R}^K_+ \to \mathbf{R}$ which are continuous on $\mathbf{R}^K_+$, twice continuously differentiable on $\mathbf{R}^K_{++}$, and identically 0 on $\mathbf{R}^K_+ \setminus \mathbf{R}^K_{++}$. We equip $\mathcal{D}$ with the topology of uniform convergence on compact subsets of $\mathbf{R}^K_+$ and uniform C^2 convergence on compact subsets of $\mathbf{R}^K_{++}$; this is a complete separable metric topology. Let d be any (translation invariant) metric for this topology.

Let $\mathcal{U}$ be the set of functions $u \in \mathcal{D}$ which are differentiably monotone on $\mathbf{R}^K_{++}$ (i.e., $\nabla u|_x \geq 0$ for all $x \in \mathbf{R}^K_{++}$) and differentiably concave on $\mathbf{R}^K_{++}$ (i.e., the Hessian matrix of second partial derivatives of u is negative semi-definite at each point of $\mathbf{R}^K_{++}$). Note that $\mathcal{U}$ is a closed convex cone in $\mathcal{D}$, and hence is a complete separable metric space in the relative topology.

Let $\mathcal{U}^0$ be the set of functions $u \in \mathcal{U}$ which are differentiably strictly monotone on $\mathbf{R}^K_{++}$ (i.e., $\nabla u|_x \gg 0$ for all $x \in \mathbf{R}^K_{++}$) and differentiably strictly concave on $\mathbf{R}^K_{++}$ (i.e., the Hessian matrix of second partial derivatives of u is negative definite at every point of $\mathbf{R}^K_{++}$). Note that $\mathcal{U}^0$ is a countable intersection of open sets in $\mathcal{U}$, and hence is completely metrizable (Christensen (1974), page 56).

In our framework, a consumer is described as an endowment e_n and a utility function u_n and the data of an exchange economy is a set finite set $\{(e_n, u_n) : n = 1, \ldots, N\}$ of consumer. Our main result is:

Theorem 5.2 *Fix N endowment vectors $e_1, \ldots, e_N$ with $e_1 \neq 0$ and $N-1$ utility functions $u_2, \ldots u_N \in \mathcal{U}^0$. The set $\mathcal{U}^*$ of utility functions $u_1 \in \mathcal{U}^0$ for which the exchange economy $\{(e_n, u_n) : n = 1, \ldots, N\}$ is regular is open and prevalent in $\mathcal{U}^0$. Moreover, $\mathcal{U}^0$ is prevalent in $\mathcal{U}$, so $\mathcal{U}^*$ is also prevalent in $\mathcal{U}$.*

Proof We show first that $\mathcal{U}^0$ is finitely prevalent in $\mathcal{U}$. To this end, write $\mathcal{U}^1 = \mathcal{U} \setminus \mathcal{U}^0$; we show that $\mathcal{U}^1$ is finitely shy in $\mathcal{U}$.

We assert first of all that $\mathcal{U}^1$ is a Borel set. Choose a countable family $\{L_k : k \in \mathbf{N}\}$ such that $\cup_{n \in \mathbf{N}} L_k = \mathbf{R}^K_{++}$. Define

$$\mathcal{U}_{kn} = \left\{ u \in \mathcal{U} : \nabla u|_x \gg \left(\frac{1}{n}, \ldots, \frac{1}{n} \right) \text{ for all } x \in L_k \right\} \tag{1}$$

Observe that each $\mathcal{U}_{kn}$ is open, and that the set of differentiably strictly monotonic functions u is $\cap_k \cup_n \mathcal{U}_{kn}$, which is therefore a Borel set. Similarly, we see that the set of differentiably strictly concave utility functions is a Borel set. Hence $\mathcal{U}^0$, which is the intersection of the set of differentiably strictly monotonic utility functions with the set of differentiably strictly concave utility functions, is a Borel set, so its complement, $\mathcal{U}^1$, is also Borel.

Choose and fix $u_0 \in \mathcal{U}^0$, and let $V \subset \mathcal{D}$ be the one-dimensional space spanned by u_0. Clearly $\lambda_V(\mathcal{U}) > 0$. Fix $w \in \mathcal{D}$. We claim that $(\mathcal{U}^1 + w) \cap V$ contains at most one point. If not, we can find $u_1, u_2 \in \mathcal{U}^1$ such that

$$\begin{aligned} u_1 + w &= t_1 u_0 \\ u_2 + w &= t_2 u_0 \end{aligned}$$

with $t_1 > t_2$, so

$$u_1 - u_2 = (t_1 - t_2) u_0 \in \mathcal{U}^0 \tag{2}$$

We are in one of two cases:

1: $\nabla u_1|_x \not\gg 0$ for some $x \in \mathbf{R}^K_{++}$. Then since $\nabla u_2|_x \geq 0$, $\nabla(u_1 - u_2)|_x \not\gg 0$, a contradiction.

2: the Hessian of u_1 is not negative definite at some $x \in \mathbf{R}^K_{++}$; but since the Hessian of u_2 is negative semidefinite, the Hessian of $u_1 - u_2$ is not negative definite at x, again a contradiction.

These contradictions show that $(\mathcal{U}^1 + w) \cap V$ contains at most one point, whence $\lambda_V(\mathcal{U}^1 + w) = 0$ so $\mathcal{U}^1$ is finitely shy, as desired.

The remainder of the proof makes use of perturbations of utility functions. This is a familiar idea; see Allen (1981), Araujo (1985) and Mas-Colell (1985) for instance. However, linear perturbations (as these authors use) will not work for our purposes because we require that utility functions be 0 on the boundary. We therefore use Cobb-Douglas perturbations.

Let

$$A = \{(\alpha_0, \ldots, \alpha_K) \in \mathbf{R}^{K+1}_+ : \alpha_k > 0 \ (0 \leq k \leq K), \ \sum_{k=1}^{K} \alpha_k = 1\}$$

be the set of parameters of Cobb-Douglas utility functions on $\mathbf{R}_+^K$, and define $\varphi : A \to \mathcal{D}$ by

$$\varphi(\alpha_0, \ldots, \alpha_K)(x) = \alpha_0 x_1^{\alpha_1} \cdots x_K^{\alpha_K}$$

Then $\mathcal{C} = \varphi(A)$ is the set of Cobb-Douglas utility functions on $\mathbf{R}_+^K$. Because the convergence notion in $\mathcal{D}$ is uniform convergence on compact sets in $\mathbf{R}_+^K$ and uniform C^2 convergence on compact sets in $\mathbf{R}_{++}^K$, φ is continuous.

In order to show that $\mathcal{U}^*$ is prevalent in $\mathcal{U}^0$, fix a neighborhood W of $0 \in \mathcal{D}$ and a real number $\delta > 0$; we construct the required measure μ. For $\rho > 0$, write

$$A_\rho = \{(\alpha_0, \ldots, \alpha_K) \in A : \alpha_0 \in [\rho, 2\rho], \alpha_k \geq \rho \ (1 \leq k \leq K)\}$$

The image $\varphi(A_\rho)$ is a compact subset of $\mathcal{D}$; choose $\rho > 0$ so small that $\mathcal{C}_\rho = \varphi(A_\rho) \subset W$. Let λ_ρ be normalized Lebesgue measure on A_ρ and let μ be the direct image measure:

$$\mu(E) = \lambda_\rho(\varphi^{-1}(E) \cap A_\rho)$$

By construction, μ is a probability measure with compact support $\operatorname{supp} \mu = \mathcal{C}_\rho \subset W$. Of course $\operatorname{supp} \mu \subset \delta \mathcal{U} = \mathcal{U}$.

For $u \in \mathcal{U}^0$, let $\chi(u)$ denote the exchange economy in which agent 1 has utility function u, agent n has utility function u_n $(2 \leq n \leq N)$, and agent n has endowment $e(n)$ $(1 \leq n \leq N)$. In this economy, let $D_n(p, u)$ denote the demand of agent n at the prices p. (Strict concavity implies that demand is a singleton.) It will be convenient to normalize prices using the Euclidean length, i.e. $|p| = 1$. For $n > 1$, the characteristics of agent n do not depend on u, so $D_n(p, u)$ is independent of u for $n > 1$. Let $\mathbf{C} = \{u \in \mathcal{U}^0 : \ \chi(u) \text{ is a critical economy}\}$.

Fix a compact set $L \subset \mathbf{R}_{++}^K$, and let

$$\begin{aligned} \mathbf{C}_L \ = \ \{u \in \mathbf{C} : \ & \text{there is a critical equilibrium} \\ & \text{price } p \text{ such that } D_1(p, u) \in L\} \end{aligned}$$

We will show that $\mathbf{C}_L$ is shy for each compact set L; because $\mathbf{C}$ is a countable union of sets $\mathbf{C}_L$, shyness of $\mathbf{C}$ will be an easy consequence.

We need to show that $\mu(\mathbf{C}_L + w) = 0$ for every translation $w \in \mathcal{D}$; fix such a w. Observe that $\mu(\mathbf{C}_L + w) = 0$ if and only if $\mu(\{c \in \mathcal{C} : c - w \in \mathbf{C}_L\}) = 0$. Let $\mathcal{U}^0_L = \{u \in \mathcal{D} : u \text{ is differentiably strictly monotonic and differentiably strictly concave on } L\}$. Observe that $\mathcal{U}^0_L$ is open in $\mathcal{D}$. Therefore, $(\mathcal{U}^0_L + w) \cap \mathcal{C}$ is an open subset of $\mathcal{C}$ in the relative topology, and hence $\varphi^{-1}\left([\mathcal{U}^0_L - w] \cap \mathcal{C}\right)$ is an open subset of A. Observe also that $(\mathbf{C}_L) \subset \mathcal{U}^0 \subset \mathcal{U}^0_L$.

Notice that if $u = \varphi(\alpha) - w$ with $\alpha \in A$ and $u \in \mathcal{U}^0_L$, then for all $x \in L$,

$$\begin{aligned} \nabla u|_x &= \nabla\varphi(\alpha)|_x - \nabla w|_x \\ &= (\varphi(\alpha))(x)\left(\frac{\alpha_1}{x_1}, \ldots, \frac{\alpha_K}{x_K}\right) - \nabla w|_x \end{aligned}$$

so that small changes in α allow us to perturb the normalized gradient[25]

$$f(x, \alpha) = \frac{\nabla(\varphi(\alpha) - w)|_x}{|\nabla(\varphi(\alpha) - w)|_x|}$$

in any direction on the $(K-1)$-sphere, where K is the number of goods[26]; in other words, the Jacobian $J_\alpha(f(x, \alpha))$ has rank $K-1$ for all $x \in L$ and all α in the open set $\varphi^{-1}\left([\mathcal{U}^0_L + w] \cap \mathcal{C}\right)$. Since $D_1(p, \varphi(\alpha) - w)$ satisfies the equation $f(x, \alpha) = p$, and the Jacobian $J_x(f(x, \alpha))$ (with $x \in L$ restricted to satisfy $p \cdot x = e(1)$) has rank $K-1$ because $\varphi(\alpha) - w = u \in \mathcal{U}^0_L$, the Implicit Function Theorem guarantees that the Jacobian $J_\alpha(D_1(p, \varphi(\alpha) - w))$ has rank $K-1$ provided that $D_1(p, \varphi(\alpha) - w) \in L$. Since D_n is independent of $c = \varphi(\alpha)$ for $2 \leq n \leq N$, the Jacobian $J_\alpha(D(p, \varphi(\alpha) - w)) = J_\alpha(D_1(p, \varphi(\alpha) - w))$ has rank

[25] Since $\varphi(\alpha) - w = u \in \mathcal{U}^0_L$ and $x \in L$, $\nabla(\varphi(\alpha) - w)|_x \gg 0$, and in particular $\nabla(\varphi(\alpha) - w)|_x \neq 0$.

[26] Assume first that the normalized gradient of w, $(\nabla w|_x) / |\nabla w|_x|$, is not equal to the normalized gradient of $\varphi(\alpha)$, $(\nabla\varphi(\alpha)|_x) / |\nabla\varphi(\alpha)|_x|$. For directions on the geodesic joining $-(\nabla w|_x) / |\nabla w|_x|$ to $(\nabla\varphi(\alpha)|_x) / |\nabla\varphi(\alpha)|_x|$, adjusting the parameter α_0 suffices. For all directions in the tangent to the sphere of normalized gradients which are perpendicular to this geodesic, adjusting the parameters $\alpha_1, \ldots, \alpha_K$ suffices to move $\nabla\varphi(\alpha)|_x$ in the desired direction; adjusting α_0 to counteract the effect of the changes in $\alpha_1, \ldots, \alpha_K$ on the scale factor $(\varphi(\alpha))(x)$ which appears in $\nabla\varphi(\alpha)|_x$, will keep the weightings on $-\nabla w|_x$ and $\nabla\varphi(\alpha)|_x$ which appear in the *normalized* gradient $f(x, \alpha)$ unchanged. If the normalized gradients of w and $\varphi(\alpha)$ are equal, simply adjust the parameters $\alpha_1, \ldots, \alpha_K$ to move $\nabla u|_x$ in the desired direction.

$K-1$ for all $\alpha \in \varphi^{-1}\left([\mathcal{U}_L^0+w]\cap C\right)$ and all p satisfying $D_1(p,\varphi(\alpha)-w) \in L$. Hence, the full Jacobian $J_{p\alpha}(D(p,\varphi(\alpha)-w))$ has rank $K-1$ whenever $\alpha \in ((\mathcal{U}_L^0+w)\cap C)$ and $D_1(p,\varphi(\alpha)-w)\in L$. Note that $\varphi^{-1}\left([\mathcal{U}_L^0+w]\cap C\right)$ is an open set in A, and A is diffeomorphic to an open subset of $\mathbf{R}^K$. Hence we can apply the Thom Transversality Theorem to conclude that the set of $\alpha \in \varphi^{-1}\left([\mathcal{U}_L^0+w]\cap C\right)$ for which the Jacobian $J_p(D(p,\varphi(\alpha)-w))$ has rank $K-1$ whenever $D(p,\varphi(\alpha)-w)=0$ and $D_1(p,\varphi(\alpha)-w)\in L$ is of full Lebesgue measure. In other words, for almost all choices of α, the economy $\chi(\varphi(\alpha)-w)$ has no critical price p with $D_1(p,\varphi(\alpha)-w)\in L$. Thus, $\mu(\{c\in\mathcal{C} : c-w\in\mathbf{C}_L\})=0$, so $\mu(\mathbf{C}_L+w)=0$, so $\mathbf{C}_L$ is shy.

Now write $L_m=\left[\frac{1}{m},m\right]^K$. If $u\in\mathcal{U}^0$ then $D_1(p,u)\in\mathbf{R}_{++}^K$ for all strictly positive p, so

$$\mathbf{C}=\bigcup_{m\in\mathbf{N}}\mathbf{C}_{L_m}$$

Hence $\mathbf{C}$ is shy in $\mathcal{U}^0$, and the set of utility functions u which induce regular economies is prevalent in $\mathcal{U}^0$.

Next, we show that $\mathbf{C}$ is closed in $\mathcal{U}^0$, so that $\mathcal{U}^*$ is open in $\mathcal{U}^0$. Note that since $u_2,\ldots,u_n\in\mathcal{U}^0$, $\left|\sum_{n=2}^N D_n(p,u)\right|\to\infty$ as p tends to the boundary of the price simplex. Thus, there is a compact set K of strictly positive prices such that $p\notin K$ implies $\left|\sum_{n=2}^N D_n(p,u)\right| > \left|\sum_{n=1}^N e(n)\right|$, so $p\notin K$ implies that p is not an equilibrium price for $\chi(u)$ for any $u\in\mathcal{U}^0$. Now suppose $u_m\to u$, and let p_m be an equilibrium price vector for $\chi(u_m)$ which is not regular. Since $p_m\in K$, we can (by taking a subsequence) assume that $p_m\to p$ for some $p\in K$. Since $p_m\to p >> 0$, there is a compact set containing $\cup_{m\in\mathbf{N}}\{x\in\mathbf{R}_+^K : p_m\cdot x\le p_m\cdot e(1)\}$, so u_m converges uniformly to u on this set, and hence $D_1(p_m,u_m)\to D_1(p,u)$. Consequently, p is an equilibrium price for $\chi(u)$. Since $u\in\mathcal{U}^0$, $D_1(p,u)\in\mathbf{R}_{++}^K$, so the second partial derivatives of $u_m|_{D_1(p_m,u_M)}$ converge to those of $u|_{D_1(p,u)}$; hence the Jacobian of $\sum_{n=1}^N D_n(p_m,u_m)$ converges to the Jacobian of $\sum_{n=1}^N D_n(p,u)$, so p is a nonregular equilibrium price for $\chi(u)$. Thus, $\mathbf{C}$ is closed in $\mathcal{U}^0$.

Finally, since $\mathcal{U}^*$ is prevalent in $\mathcal{U}^0$ and $\mathcal{U}^0$ is prevalent and Borel in $\mathcal{U}$, $\mathcal{U}^*$ is prevalent in $\mathcal{U}$ by Fact 7. ■

6 Differentiability of Lipschitz Functions and the Core of a Game

In this Section we establish the generic differentiability of Lipschitz functions in infinite dimensional Banach spaces and use this result to establish the generic single-valuedness of the core for a class of transferable utility games with a continuum of players.

To motivate our description of the class of games of interest, consider a familiar description of a transferable utility economy with a continuum of consumers. We take as given a measurable space $(A, \mathcal{A})$ of consumer characteristics — endowments and utility functions. A positive measure μ on $(A, \mathcal{A})$ defines a population distribution. Given any such population distribution μ, the total gains available to the population μ are

$$g(\mu) = \sup \left\{ \int_A u_a(x_a) d\mu(a) : \int_A x_a d\mu(a) = \int_A e_a d\mu(a) \right\}$$

(the assumption of transferable utility making this a meaningful quantity). It is easily checked that g is super-additive and homogeneous of degree 1 — hence concave. Moreover, familiar assumptions (see Aumann and Shapley (1974), Hart (1977), and Gretsky, Ostroy and Zame (1992)) guarantee that g is Lipschitz; i.e., there is a constant K such that if μ_1, μ_2 are population distributions then

$$|g(\mu_1) - g(\mu_2)| \leq K \|\mu_1 - \mu_2\|$$

We can describe a transferable utility game with a continuum of players in a similar way. We take as given a measurable space $(A, \mathcal{A})$ of players, or player characteristics; for technical reasons, we assume $\mathcal{A}$ is generated as a σ-algebra by some countable subset. As before, we interpret a positive measure μ on $(A, \mathcal{A})$ as a population distribution. We take as given a reference population distribution λ, and consider all population distributions that are absolutely continuous with respect to λ. In view of the Radon-Nikodym theorem, we may identify the set of such population distributions with measures with the positive cone $L^1(\lambda)_+ \subset L^1(\lambda)$; we find it convenient to view the elements of $L^1(\lambda)_+$ as measures rather than as functions. Because $(A, \mathcal{A})$ is countably generated, $L^1(\lambda)$ is a separable Banach space. We take as given a total gains function $g : L^1_+(\lambda) \to \mathbf{R}$. By analogy with tranferable utility economies, we assume that g is super-additive and homogenous of degree 1 (hence concave) and Lipschitz. Each population distribution $\mu \in L^1_+(\lambda)$ induces a game with player set A and characteristic function v_μ defined by:

$$v_\mu(B) = g(\mu|_B) \text{ for } B \in \mathcal{A}$$

The core of v_μ may be identified with the subdifferential of g at the population distribution μ. (See Ostroy (1984), Makowski and Ostroy (1992) and Gretsky, Ostroy and Zame (1992).) In particular, the core of v_μ is a singleton if and only if g is Gâteaux differentiable at f.[27]

The main result of this section is that for a prevalent set of population distributions the core is a singleton.

Theorem 6.1 *The set of population distributions $\mu \in L^1_+(\lambda)$ for which the core of v_μ is a singleton is a prevalent subset of $L^1_+(\lambda)$.*

Because the core of v_μ is a singleton exactly when g is Gâteaux differentiable at μ, Theorem 6.1 is equivalent to the assertion that g is Gâteaux differentiable at every point of a prevalent set, which is an immediate consequence of the following infinite dimensional version of Rademacher's theorem (almost everywhere differentiability of Lipschitz functions on $\mathbf{R}^n$).[28]

[27]If C is a convex subset of a Banach space X, the function $u : C \to \mathbf{R}$ is *Gâteaux differentiable* at $x \in C$ if there is a continuous linear functional $\varphi \in X^*$ such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} |u(x + \varepsilon y) - u(x) - \varphi(\varepsilon y)| = 0$$

for each $y \in X$ having the property that $x + ty \in C$ for $t > 0$ sufficiently small. If u is Gâteaux differentiable at c, the linear functional φ is uniquely determined on the smallest closed subspace of X containing $C - x$, but indeterminate elsewhere.

[28]Verona (1989) establishes the weaker result that concave Lipschitz functions are Gâteaux differentiable at every point of a residual set.

Theorem 6.2 *Let X be a separable Banach space, let $C \subset X$ be a closed convex subset, and let $u : C \to \mathbf{R}$ be a Lipschitz function. The set*

$$\{x \in C : u \text{ is Gâteaux differentiable at } x\}$$

is a prevalent subset of C.

Proof The proof is in three steps: 1) construct a sufficiently large countable set of finite dimensional subspaces of X; 2) for each finite dimensional subspace V in this collection, use Rademacher's theorem to show that the set of points in C at which u is Gâteaux differentiable in directions in V is a prevalent set; 3) use the Lipschitz nature of u to conclude that u is Gâteaux differentiable at every point of the intersection of this countable family of prevalent sets, which is again a prevalent set.

Step 1 Choose and fix a countable dense subset $\{x_n\}$ of C. For each pair (i, j) of indices with $i \neq j$, set $z_{ij} = x_i - x_j$, let V_{ij} be the one-dimensional subspace spanned by z_{ij}, and let

$$Z_{ij} = \{x \in C : x + tz_{ij} \in C \text{ for some } t > 0\}$$

Let $\mathcal{I}$ be the set of all finite subsets of the set of all index pairs $\{(i, j) : i \neq j\}$, and let $\mathcal{J}$ be the subset of $\mathcal{I}$ consisting of symmetric sets (so $(i, j) \in \mathcal{J} \Rightarrow (j, i) \in \mathcal{J}$). For each $J \in \mathcal{J}$, let $V_J \subset X$ be the linear subspace of X spanned by $\{x_{ij} : (i, j) \in J\}$.

Step 2 For each $J \in \mathcal{J}$, we analyze the set of points at which u is Gâteaux differentiable in directions in V_J. Because we restrict attention to points at which small translations in the directions in V_J remain in C, we write

$$Z_J = \bigcap_{(i,j) \in J} Z_{ij}$$

and define

$$Y_J = \{y \in Z_J : u|_{[(V_J + y) \cap C]} \text{ is Gâteaux differentiable at } y\}$$

Informally speaking, $y \in Y_J$ just when u is Gâteaux differentiable at y in the directions in the subspace V_J. We show that each Y_J is finitely prevalent in C.

It is convenient to show first that each Z_{ij} is finitely prevalent in C. To see this, note first that $x_j, x_i \in C$, so $0, (x_i - x_j) \in C - x_j$; of course $0, (x_i - x_j) = z_{ij} \in V_{ij}$. Because C is convex and V_{ij} is a 1-dimensional subspace, it follows that $(C - x_j) \cap V_{ij}$ contains a relatively open subset of V_{ij}; in particular, $\lambda_{V_{ij}}(C - x_j) > 0$. On the other hand, we claim that $\lambda_{V_{ij}}\Big((C \setminus Z_{ij}) + y\Big) = 0$ for each $y \in X$; indeed, we assert that the intersection $V_{ij} \cap \Big((C \setminus Z_{ij}) + y\Big)$ is either empty or a singleton. If not, choose two distinct points q_1, q_2 in the intersection, and write

$$q_1 = c_1 + y = t_1 z_{ij} \ , \ q_2 = c_2 + y = t_2 z_{ij}$$

Without loss of generality, we may suppose $t_1 > t_2$, whence

$$c_1 = c_2 + (t_1 - t_2) z_{ij}$$

But this implies that $c_2 \in Z_{ij}$, so $q_2 \in Z_{ij} + y$, which is a contradiction. We conclude that $V_{ij} \cap \Big((C \setminus Z_{ij}) + y\Big)$ is either empty or a singleton, and hence that $\lambda_{V_{ij}}\Big((C \setminus Z_{ij}) + y\Big) = 0$, for each $y \in X$, so that Z_{ij} is finitely prevalent, as asserted.

To see Y_J is finitely prevalent in C, note first that, because each Z_{ij} is finitely prevalent in C, so is each Z_J; in particular, $Z_J \neq \emptyset$. Now write $E_J = Z_J \setminus Y_J$; we show that E_J is finitely shy. To this end, write

$$F_J = \{w \in (V_J + y) \cap Z_J : u|_{[(V_J + y) \cap Z_J]} \text{ is not Gâteaux differentiable at } w\}$$

Convexity of C and symmetry of J imply that $(V_J + y) \cap Z_J$ is a relatively open subset (possibly empty) of $V_J + y$. Rademacher's theorem (see Federer (1969, 3.1.6), for instance) guarantees that a locally Lipschitz function defined on an open set in a finite dimensional space is Gâteaux differentiable almost everywhere, so $\lambda_{V_J}(F_J) = 0$. Hence $\lambda_{V_J}(E_J - y) = \lambda_{V_J}\Big((V_J + y) \cap E_J\Big) = 0$. Since this is true for each $y \in X$, we conclude that E_J is finitely shy in C and hence that Y_J is finitely prevalent in C, as asserted.

Step 3 Write

$$Y_0 = \bigcap_{J \in \mathcal{J}} Y_J$$

Because it is the countable intersection of sets that are finitely prevalent in C, Y_0 is prevalent in C (Facts 3 and 6). To complete the proof it suffices to show that u is Gâteaux differentiable at each point of Y_0.

To this end, fix $y \in Y_0$. For each $J \in \mathcal{J}$, write φ_J for the Gâteaux derivative of $u|_{[(V_J+y)\cap Z_J]}$ at y. By assumption, u is locally Lipschitz, so there is a neighborhood Q of y in C such that u is Lipschitz on Q with constant K. It follows that the norm of φ_J (which is a linear functional on the finite dimensional subspace $V_J \subset X$) is at most K. Write

$$\Phi_J = \{\psi \in X^* : \psi|_{V_J} = \varphi_J, \|\psi\| \leq K\}$$

The Hahn-Banach theorem guarantees that φ_J has an extension to all of X having the same norm, so $\Phi_J \neq \emptyset$. Alaoglu's theorem guarantees that each Φ_J is a weak star compact subset of the dual space X^*. Because the Gâteaux derivative is unique when it exists, it follows that $\Phi_J \supset \Phi_{J'}$ if $J \subset J'$. Hence $\bigcap_{J\in\mathcal{J}} \Phi_J \neq \emptyset$; let φ be any point in the intersection.

We show that φ is a Gâteaux derivative of u at y.[29] To this end, let $z \in X$ and suppose that $y + t_0 z \in C$ for some $t_0 > 0$. Replaying z by $t_0 z$, we may assume $y + z \in C$; by convexity of C, $y + tz \in C$ for all $t \in [0, 1]$. Fix $\delta > 0$.

Because $\{x_n\}$ is a dense subset of C, we can choose x_i, x_j such that $\|x_j - y\| < \frac{\delta}{4K+2}$ and $\|x_i - (y+z)\| < \frac{\delta}{4K+2}$, whence $\|z_{ij} - z\| < \frac{\delta}{2K+1}$. Because $Y_0 \subset Z_J$ for every J, it follows that $y + \varepsilon z_{ij} \in C$ for sufficiently small $\varepsilon > 0$. Using the fact that u is Lipschitz with constant K and that φ is linear and has norm bounded by K, we can estimate:

$$\begin{aligned}
&|u(y+\varepsilon z) - u(y) - \varphi(\varepsilon z)| \\
&= \quad |u(y+\varepsilon z) - u(y+\varepsilon z_{ij}) + u(y+\varepsilon z_{ij}) \\
&\qquad - u(y) - \varphi(\varepsilon z_{ij}) + \varphi(\varepsilon z_{ij}) - \varphi(\varepsilon z_{ij})|
\end{aligned}$$

[29] As we have noted, the Gâteaux derivative at y is uniquely determined only on the closed subspace spanned by $C - y$.

$$
\begin{aligned}
&\le |u(y+\varepsilon z) - u(y+\varepsilon z_{ij})| \\
&\quad + |u(y+\varepsilon z_{ij}) - u(y) - \varphi(\varepsilon z_{ij})| \\
&\quad + |\varphi(\varepsilon z) - \varphi(\varepsilon z_{ij})| \\
&\le K\varepsilon\|z - z_{ij}\| \\
&\quad + |u(y+\varepsilon z_{ij}) - u(y) - \varphi(\varepsilon z_{ij})| \\
&\quad + K\varepsilon\|z - z_{ij}\| \\
&\le \frac{2K\varepsilon\delta}{2K+1} + |u(y+\varepsilon z_{ij}) - u(y) - \varphi(\varepsilon z_{ij})|
\end{aligned}
$$

If $(i,j) \in J$, our construction guarantees that the restriction of φ to V_J is the Gâteaux derivative of $u|_{[(V_J+y)\cap Z_J]}$, so the second term will be smaller than $\frac{\varepsilon\delta}{2K+1}$, provided that ε is small enough. We conclude that

$$
\frac{|u(y+\varepsilon z) - u(y) - \varphi(\varepsilon z)|}{\varepsilon} < \frac{2K\delta}{2K+1} + \frac{\delta}{2K+1} = \delta
$$

provided that ε is small enough, which is the desired estimate. Hence φ is a Gâteaux derivative of u at y, as asserted. ■

Appendix A: Proofs of Facts 0 - 8

Here we collect the proofs of Facts 0-8.

Proof of Fact 0 Suppose that E is shy at $c_0 \in C$; let $c \in C$ be any other point. Assume without loss that E is a universally measurable set. Given $\delta > 0$ and a neighborhood W of 0 in X, use continuity of addition to choose a neighborhood W_0 of 0 such that $W_0 + W_0 \subset W$ and use continuity of scalar multiplication to choose a $\delta_0, 0 < \delta_0 < \delta$ such that $\delta_0(c_0 - c) \in W_0$. T he definition of shyness at c_0 provides a compactly supported Borel probability measure μ_0 such that

$$
\text{supp } \mu_0 \subset \Big(\delta_0(C - c_0) + c_0\Big) \cap (W_0 + c_0) \tag{3}
$$

and $\mu_0(E + x) = 0$ for every $x \in X$. Set $c_1 = (1-\delta_0)c + \delta_0 c_0$, and define a probability measure μ by:

$$
\mu(A) = \mu_0(A + c_0 - c_1)
$$

for each universally measurable set $A \subset X$. It is easily checked that

$$\text{supp } \mu = \text{supp } \mu_0 - c_0 + c_1 \tag{4}$$

Because translation is a homeomorphism, (4) implies that supp μ is compact. Combining (3), (4), the convexity of C and the fact that $c_1 = (1-\delta_0)c + \delta_0 c_0$ yields

$$\begin{aligned} \text{supp } \mu &= \text{supp } \mu_0 - c_0 + c_1 \\ &\subset \delta_0(C - c_0) + c_0 - c_0 + c_1 \\ &= \delta_0(C - c_0) + (1 - \delta_0)c + \delta_0 c_0 \\ &= \delta_0(C - c) + c \\ &\subset \delta(C - c) + c \end{aligned}$$

Moreover, because $\delta_0(c_0 - c) \in W_0$ and $W_0 + W_0 \subset W$ it follows that

$$\begin{aligned} \text{supp } \mu &= \text{supp } \mu_0 - c_0 + c_1 \\ &\subset (W_0 + c_0) - c_0 + c_1 \\ &= W_0 + (1 - \delta_0)c + \delta_0 c_0 \\ &= W_0 + \delta_0(c_0 - c) + c \\ &\subset W + c \end{aligned}$$

Combining these last two calculations we see that:

$$\text{supp } \mu \subset \Big(\delta(C - c) + c\Big) \cap (W + c)$$

Because μ is a translation of μ_0, $\mu(E + x) = 0$ for every $x \in X$. Since δ, W were arbitrary, we conclude that E is shy in C at c, as asserted. ■

Proof of Fact 1 Obvious. ■

Proof of Fact 2 Obvious. ■

Proof of Fact 3 The argument is adapted from Hunt, Sauer and Yorke (1992). Let $\{E_n\}$ be a countable family of sets, each shy in C; we show that the union $E = \bigcup E_n$ is shy in C. There is no loss in assuming that $0 \in C$ and that each E_n is a universally measurable set, and it suffices (in view of Fact 0) to establish that E is shy in C at 0. Fix $\delta > 0$ and an open neighborhood W of $0 \in X$; we construct a measure that satisfies the support condition and has the property that every translate of E has measure 0. The measure we construct is an infinite convolution; guaranteeing that this infinite convolution is well-defined (that is, that the sequence defining it converges) requires some care.

By assumption, C is completely metrizable. Choose and fix a complete metric d^* on C. (If $C = X$, we could choose d^* to be translation invariant. In general, however, C is a proper subset of X so it may not be possible to choose d^* to be translation invariant. This makes the construction below rather fussy.) Define the metric d on δC by

$$\mathrm{d}(x, y) = \mathrm{d}^*(\delta^{-1}x, \delta^{-1}y)$$

Because multiplication by δ is a homeomorphism from C to δC and d^* is a complete metric on C, it follows that d is a complete metric on δC.

Write $W_0 = W$. We use induction to construct a sequence (W_n) of open neighborhoods of $0 \in X$ and a sequence (μ_n) of regular Borel probability measures on X such that for every $n \geq 1$:

(i) $K_n = \mathrm{supp}\ \mu_n$ is compact

(ii) $K_n \subset W_n \cap 2^{-n}\delta C$

(iii) $\mu_n(E_n + x) = 0$ for every $x \in X$

(iv) $\overline{W_n + W_n} \subset W_{n-1}$

(v) if $x \in K_1 + \ldots + K_n$ and $w \in W_{n+1} \cap (\delta C)$ then $\mathrm{d}(x, x + w) < 2^{-n}$

(vi) $\mu_n(E_n + x) = 0$ for all $x \in X$

To construct W_1, μ_1, choose open neighborhoods Q_1, Q_2 of $0 \in X$ such that $\overline{Q}_1 \subset W_0$ and $Q_2 + Q_2 \subset Q_1$; set $W_1 = Q_1 \cap Q_2$. Now use shyness of

E_1 in C to choose a compactly supported probability measure μ_1 such that $\mu_1(E_1 + x) = 0$ for every $x \in X$ and

$$K_1 = \text{supp } \mu_1 \subset W_1 \cap 2^{-1}\delta C$$

Now suppose that $\mu_1, \ldots, \mu_n$ and $W_1, \ldots, W_n$ have been chosen satisfying (i)–(vi). Use compactness of $K_1 + \cdots + K_n$ and continuity of the distance function d to choose open neighborhoods Q_1, Q_2, Q_3 of $0 \in X$ such that $\overline{Q_1} \subset W_n$, $Q_2 + Q_2 \subset Q_1$, and if $x \in K_1 + \ldots + K_n$ and $q \in Q_3 \cap (\delta C)$ then $x + q \in \delta C$ and $\mathrm{d}(x, x+q) < 2^{-n}$. Set $W_{n+1} = Q_1 \cap Q_2 \cap Q_3$. Because E_{n+1} is shy in C at 0 there is a compactly supported probability measure μ_{n+1} such that $\mu_{n+1}(E_{n+1} + x) = 0$ for all $x \in X$ and

$$K_{n+1} = \text{supp } \mu_{n+1} \subset W_n \cap 2^{-(n+1)}\delta C$$

It is easily checked that $\mu_1, \ldots, \mu_{n+1}$ and $W_1, \ldots, W_{n+1}$ satisfy (i)-(vi), so the inductive construction is complete.

In order to define the infinite convolution $\mu = \mu_1 * \mu_2 * \cdots$ write

$$\hat{K} = \prod_{n=1}^{\infty} K_n$$

In the product topology, $\hat{K}$ is a compact metric space (because each K_n is). Moreover, the Borel σ-algebra on $\hat{K}$ coincides with the product σ-algebra.[30]

[30] By definition, the product σ-algebra is generated by sets of the form $\prod_{n=1}^{\infty} B_n$, where each B_n is a Borel subset of K_n and $B_n = K_n$ for all but a finite number of indices n. Each of the generating sets is in the σ-algebra generated by sets of the form $\prod_{n=1}^{\infty} T_n$, where each T_n is an open subset of K_n and $T_n = K_n$ for all but a finite number of the indices n; since these sets are open in the product topology, the product σ-algebra is contained in the Borel σ-algebra generated by the product topology. To see that the Borel σ-algebra generated by the product topology is contained in the product σ-algebra, note that since each K_n is compact metric, there is a countable collection $\mathcal{T}_n$ (with $K_n \in \mathcal{T}_n$) which forms a base for the topology on K_n. A base for the product topology is given by sets of the form $\prod_{n=1}^{\infty} T_n$, where $T_n \in \mathcal{T}_n$ for all n and $T_n = K_n$ for all but finitely many values of n. Notice that this base is countable, and consists entirely of sets that are product measurable, so every product open set is product measurable, and hence every Borel set is product measurable.

Let

$$\hat{\mu} = \prod_{n=1}^{\infty} \mu_n$$

be the product measure on $\hat{K}$. Thus

$$\hat{\mu}\left(\prod_{n=1}^{\infty} B_n\right) = \prod_{n=1}^{\infty} \mu_n(B_n)$$

Define $T : \hat{K} \to \delta C$ by

$$T(x_n) = \sum_{n=1}^{\infty} x_n$$

We assert that T is well-defined (i.e., for each (x_n), the infinite sum converges to an element of δC), that T is continuous, and that $T(\hat{K}) \subset \delta C \cap W$.

To see that the $T(x_n)$ is well-defined, consider the partial sums $s_k = \sum_{n=1}^{k} x_n$. Fix n, m. The triangle inequality implies that

$$d(s_n, s_m) \leq \sum_{k=0}^{m-n-1} d(s_{n+k}, s_{n+k+1})$$

By definition, $s_{n+k+1} - s_{n+k} = x_{n+k+1}$, $s_{n+k} \in K_1 + \ldots + K_{n+k}$ and $x_{n+k+1} \in K_{n+k+1} \subset W_{n+k+1}$ so (v) guarantees that

$$\mathrm{d}(s_{n+k}, s_{n+k+1}) \leq 2^{-(n+k)}$$

Hence

$$d(s_n, s_{n+m}) \leq 2^{-(n-1)}$$

In particular, the sequence of partial sums is Cauchy. Because δC is complete, the sequence of partial sums converges to an element of δC, so T is well-defined.

To see that T is continuous, fix $x \in \hat{K}$ and $\varepsilon > 0$. Choose an index N so that $2^{-(N-1)} < \varepsilon/4$. Define $T^N : \hat{K} \to \delta C$ by

$$T^N(x) = \sum_{n=1}^{N} x_n$$

(the N-th partial sum). Continuity of addition guarantees that there is a neighborhood V of x in $\hat{K}$ such that

$$\mathrm{d}\left(T^N(x), T^N(y)\right) < \frac{\varepsilon}{2}$$

whenever $y \in V$. Combining this with the triangle inequality and the calculations of the preceding paragraph, we find that if $y \in V$ then

$$\begin{aligned}
\mathrm{d}\left(T(x), T(y)\right) &\le \mathrm{d}\left(T(x), T^N(x)\right) + \mathrm{d}\left(T^N(x), T^N(y)\right) + \mathrm{d}\left(T^N(y), T(y)\right) \\
&\le 2^{-(N-1)} + \mathrm{d}\left(T^N(x), T^N(y)\right) + 2^{-(N-1)} \\
&\le \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \\
&= \varepsilon
\end{aligned}$$

Hence T is continuous.

Write $K = T(\hat{K})$; continuity of T implies that K is a compact subset of X. By construction, $K_n \subset 2^{-n}\delta C$ for each n. Because C is convex and $0 \in C$ it follows that

$$K_1 + \cdots + K_N \subset \frac{2^{-1} - 2^{-N-1}}{2^{-1}} \delta C = \left(1 - 2^{-N}\right) \delta C \subset \delta C$$

Since elements of K are limits of Cauchy sequences in δC, they therefore belong to δC; thus $K \subset \delta C$. Combining (ii) with repeated applications of (iv) yields that

$$K_1 + \cdots + K_N \subset W_1 + \cdots + W_N \subset W_1 + W_1$$

so $K \subset \overline{W_1 + W_1} \subset W_0 = W$.

We define the infinite convolution measure

$$\mu = \mathop{*}_{n=1}^{\infty} \mu_n$$

by

$$\mu(B) = \hat{\mu}\left(\left\{(x_n) \in \hat{K} : \sum_{n=1}^{\infty} x_n \in B\right\}\right) \tag{5}$$

for each Borel set $B \subset X$. Notice that μ is a probability measure and that $\mu(K) = 1$, so that supp $\mu \subset K \subset \delta C \cap W$.

Now fix $y \in X$. For each m, let $\hat{K}_m = \prod_{n \neq m} K_n$ and $\hat{\mu}_m = \prod_{n \neq m} \mu_n$. E_m is universally measurable, hence so are all of its translates, so their characteristic functions are measurable in the completion of the product measure $(\hat{\mu}_m, \mu_m)$ on $\hat{K}_m \times K_m$, so Fubini's Theorem applies. Thus,

$$\begin{aligned}
\mu(E_m + y) &= \int_{\hat{K}} \mathbf{1}_{E_m + y} \left(\sum_{n=1}^{\infty} x_n \right) d\hat{\mu} \\
&= \int_{\hat{K}_m} \int_{K_m} \mathbf{1}_{E_m + y} \left(\sum_{n=1}^{\infty} x_n \right) d\mu_m d\hat{\mu}_m \\
&= \int_{\hat{K}_m} \int_{K_m} \mathbf{1}_{\left(E_m + y - \sum_{n \neq m} x_n \right)} (x_m) d\mu_m d\hat{\mu}_m \\
&= \int_{\hat{K}_m} 0 d\hat{\mu}_m \\
&= 0
\end{aligned}$$

Because μ is countably additive and $E = \cup E_m$

$$\mu(E + y) \leq \sum_{m=1}^{\infty} \mu(E_m + y) = 0$$

Hence E is shy in C at 0; in view of Fact 0, E is shy in C, as desired. ■

Proof of Fact 4 Suppose that $Q \subset C$ is relatively open and shy in C. In view of Fact 2, there is no loss in assuming that $0 \in Q$. Choose an open set $U \subset X$ such that $U \cap C = Q$.

We show first that, for each positive integer n, $nQ \cap C$ is shy in C at 0. Given $\delta > 0$ and a neighborhood W of 0 in X, choose a compactly supported probability measure μ such that

$$\text{supp } \mu \subset \frac{\delta}{n} C \cap \frac{1}{n} W$$

and $\mu(Q + x) = 0$ for every $x \in X$. Define a measure ν by

$$\nu(A) = \mu\left(\frac{1}{n} A\right)$$

It is immediate that ν is a probability measure, that

$$\text{supp}\, \nu = n\, \text{supp}\, \mu \subset \delta C \cap W$$

and that ν has compact support. If $y \in X$, then

$$\begin{aligned} \nu\left((nQ \cap C) + y\right) &= \mu\left(\frac{1}{n}(nQ \cap C) + \frac{1}{n}y\right) \\ &= \mu\left(\left(Q \cap \frac{C}{n}\right) + \frac{y}{n}\right) \\ &\leq \mu\left(Q + \frac{1}{n}y\right) \\ &= 0 \end{aligned}$$

Hence $nQ \cap C$ is shy in C at 0 and hence (by Fact 0) shy in C.

We now obtain a contradiction. Because $0 \in C$ and C is convex, $C/n \subset C$, so

$$nQ \cap C = n\left(Q \cap \frac{C}{n}\right) = n\left(U \cap \frac{C}{n}\right) = nU \cap C$$

Because U is a neighborhood of 0 in X, $\bigcup_{n=1}^{\infty} nU = X$, so

$$\begin{aligned} C &= \left(\bigcup_{n=1}^{\infty} nU\right) \cap C \\ &= \bigcup_{n=1}^{\infty} (nU \cap C) \\ &= \bigcup_{n=1}^{\infty} (nQ \cap C) \end{aligned}$$

Hence C is the union of countably many subsets that are shy in C. Since this contradicts Fact 3, we conclude that Q is not shy in C, as desired. ■

Proof of Fact 5 Suppose E is shy in C. Choose a universally measurable set $Z \supset E$ and a probability measure ν with $\nu(C) = 1$ such that $\nu(Z+x) = 0$ for all $x \in \mathbf{R}^k$. Then

$$\lambda(Z) = \int_{\mathbf{R}^k} \lambda(Z) d\nu(z)$$

$$
\begin{aligned}
&= \int_{\mathbf{R}^k} \lambda(Z-z)d\nu(z) \\
&= \int_{\mathbf{R}^k}\int_{\mathbf{R}^k} \mathbf{1}_Z(x+z)d\lambda(x)d\nu(z) \\
&= \int_{\mathbf{R}^k}\int_{\mathbf{R}^k} \mathbf{1}_Z(x+z)d\nu(z)d\lambda(x) \\
&= \int_{\mathbf{R}^k} \nu(Z-x)d\lambda(x) \\
&= \int_{\mathbf{R}^k} 0 d\lambda(x) = 0
\end{aligned}
$$

so Z is a set of Lebesgue measure zero, whence E is a set of Lebesgue measure zero. As in the proof of Fact 3, the use of Fubini's Theorem is justified because Z is universally measurable, hence $\mathbf{1}_Z(x+z)$ is measurable in the completion of the product measure.

If the Lebesgue measure of E is 0 then E is finitely shy in C; Fact 6 below tells us that E is shy in C, as desired. ■

Proof of Fact 6 Let $E \subset C$ be finitely shy, so there is a finite dimensional subspace V and a point $a \in X$ such that $\lambda_V(C+a) > 0$ and $\lambda_V(E+x) = 0$ for every $x \in X$. In view of Fact 2, there is no loss in assuming that $a = 0$, so $\lambda_V(C) > 0$. Because C is convex and V is finite dimensional, $V \cap C$ contains a convex relatively open subset Q of V; there is again no loss in assuming $0 \in Q$. Given $\delta > 0$ and a neighborhood W of 0 in X, the set $\delta Q \cap (W \cap V)$ is a neighborhood of 0 in V, so it contains a compact set K with 0 in its relative interior. Define

$$
\mu = \frac{\lambda_{V|K}}{\lambda_V(K)}
$$

By construction, μ is a regular compactly supported probability measure and supp $\mu \subset \delta Q \cap W \subset \delta C \cap W$. Because μ is absolutely continuous with respect to λ_V, it follows that $\mu(E+x) = 0$ for every $x \in X$. Because δ, W were arbitrary, it follows that E is shy in C at 0, and hence (Fact 0) that E is shy in C, as desired. ■

Before turning to the proofs of Facts 7 and 8, we need to establish the following technical result. This may well be known, but we have not been able to find a reference.

Proposition 6.3 *If X is a Hausdorff topological space and $C \subset X$ is completely metrizable in the relative topology, then C is a G_δ subset of its closure $\overline{C} \subset X$. In particular, C is a Borel subset of X.*

Proof Choose and fix a complete metric d on C. For $A \subset C$, write diam $(A) = \sup\{d(x,y) : x, y \in A\}$ for the diameter of A with respect to the metric d. For $x \in X$ define the *oscillation* of C at X to be

$$\omega(x) = \inf \operatorname{diam}(U \cap C)$$

where the infimum is taken over all open neighborhoods U of x (with respect to X). Note that $\omega(x) = 0$ if $x \in C$ and that $\omega(x) = -\infty$ if $x \notin \overline{C}$ (for then there is a neighborhood U of x which misses C, whence $U \cap C = \emptyset$ and $\operatorname{diam}(U \cap C) = \operatorname{diam}(\emptyset) = -\infty$).

We assert that if $x \in \overline{C}$ and $\omega(x) = 0$ then $x \in C$. To see this, note that every open neighborhood U of x meets C, so for each U we can choose a point $x_U \in U \cap C$. The open neighborhoods of x are directed by set inclusion so $\{x_U\}$ is a net in C. The assumption that $\omega(x) = 0$ implies that for every $\varepsilon > 0$ there is an open neighborhood V of x with $\operatorname{diam}(V \cap C) < \varepsilon$, so if $U, U' \subset V$ then $\mathrm{d}(x_U, x_{U'}) < \varepsilon$. That is, $\{x_U\}$ is a Cauchy net in C. Because C is complete in the metric d, this net converges to a point $x' \in C$. On the other hand, the construction of $\{x_U\}$ guarantees that it converges to $x \in X$, and X is Hausdorff, so limits of nets are unique. Hence $x = x' \in C$ as asserted.

Let $W_n = \{x \in X : \omega(x) < 1/n\}$. If $x \in W_n$, then there is a neighborhood U of x such that diam $(U \cap C) < 1/n$. For any other $y \in U$, $\omega(y) \leq$ diam $(U \cap C) < 1/n$, so $y \in W_n$. Therefore, W_n is an open subset of X, so

$$C = \{x \in \overline{C} : \omega(x) = 0\} = \overline{C} \cap \bigcap_{n=1}^{\infty} \{x \in X : \omega(x) < 1/n\}$$

is a G_δ subset of $\overline{C}$, as asserted. ■

Proof of Fact 7 Suppose that E is prevalent in F and F is prevalent in G. $F \setminus E$ is shy in F, so there is a universally measurable set $H \subset F$ that is shy in F and $(F \setminus E) \subset H$. Fix $f \in F$, $\delta > 0$, and a neighborhood W of 0 in X. There is a regular Borel probability measure μ with compact support $\subset ([\delta(F - f) + f] \cap (W + f))$ such that $\mu(H + x) = 0$ for all $x \in X$. $F \subset G$ so $f \in G$ and $([\delta(F - f) + f] \cap (W + f)) \subset ([\delta(G - f) + f] \cap (W + f))$, so H is shy at f in G; by Fact 0, H is shy in G. $G \setminus F$ is shy in G, so there is a universally measurable set $I \subset G$ that is shy in G and $(G \setminus F) \subset I$. Then $G \setminus E = (F \setminus E) \cup (G \setminus F) \subset H \cup I$; since this is the union of two shy sets in G, it is shy in G, so $G \setminus E$ is shy in G. Therefore, E is prevalent in G. ■

Proof of Fact 8 Because X and Y induce the same topology on C and this topology is completely metrizable, Proposition 6.3 guarantees that C is a Borel subset of X and of Y. If E is shy in C with respect to X, there is a universally measurable (with respect to X) set $F \subset C$ such that $E \subset F$ and F is shy in C with respect to X.

We show first that F is universally measurable with respect to Y. Let ν be a regular Borel measure on Y; we want to show that F is measurable with respect to the completion of ν. To this end, let ν_C be the restriction of ν to C; since C is a Borel subset of Y, ν_C is a regular Borel measure on C. Define ν_X to be the trivial extension of ν_C to X; ν_X is a regular Borel measure on X. Since F is universally measurable with respect to X, F is measurable with respect to the completion of ν_X, so there exist Borel sets A, B in X with $A \subset F \subset B$ and $\nu_X(B \setminus A) = 0$. A and $B \cap C$ are Borel sets in C, hence Borel sets in Y, $A \subset F \subset B \cap C$, and $\nu((B \cap C) \setminus A) = \nu_X(B \setminus A) = 0$. Hhence F is measurable with respect to the completion of nu. Since ν is arbitary, F is a universally measurable subset of Y.

Fix $c \in C$, $\delta > 0$, and a neighborhood V of 0 in Y. Then $(c + V) \cap C$ is a neighborhood of c in C, so there exists a neighborhood W of 0 in X such that $(c + W) \cap C \subset (c + V) \cap C$. Since F is shy in C with respect to X, there is a regular Borel probability measure μ on X with compact support $\subset [\delta(C - c) + c] \cap (W + c)$ and $\mu(F + x) = 0$ for every $x \in X$. Define a Borel probability measure on Y by $\mu_Y(B) = \mu(B \cap C)$. μ_Y is compactly supported, with support $\subset ([\delta(C-c)+c] \cap (W+c)) \subset ([\delta(C-c)+c] \cap (V+c))$. If $x \in X \cap Y$, $\mu_Y(F+x) = \mu(F+x) = 0$. If $y \in Y \setminus X$, $(F+y) \cap X = \emptyset$, so $\mu_Y(F+y) = 0$. Thus, F is shy in C with respect to Y, whence E is shy in C with respect to Y. The converse follows by interchanging the roles of X and Y. ■

Appendix B: Examples

We present here four examples. The first example delineates the difference between shyness and finite shyness by providing a shy set that is not contained in the union of countably many finitely shy sets. The second and third examples provide motivation for our definitions by showing that if the support conditions were weakened then the desired properties would not obtain. The fourth example shows that prevalence does not satisfy the obvious analog of Fubini's theorem.

Before beginning, it is useful to collect some notation and a few facts that will be used repeatedly. Let λ be Lebesgue measure on the unit interval $[0,1]$. Recall that $L^1 = L^1(\lambda)$ is the Banach space of (equivalence classes) of Lebesgue integrable functions on $[0,1]$. For $f \in L^1, t \in [0,1]$ we write f_t for the rotation of f by t:

$$f_t(s) = f(s-t)$$

(We carry out subtraction modulo 1.) Translation-invariance of Lebesgue measure implies that $f_t \in L^1$ for each t. We need 2 lemmas.

Lemma A *The rotation map* $t \mapsto f_t : [0,1] \to L^1$ *is continuous.*
Proof See Rudin (1990, page 3). ■

Lemma B *Let* $U \subset [0,1]$ *be a dense open set with* $\lambda(U) < 1$*, and let* $f = 1 - \chi_U$ *be the characteristic function of the complement of* U*. If* $A \subset [0,1]$ *is a set of positive Lebesgue measure, then* $\inf_{a \in A} f_a = 0$ *(the infimum computed in* L^1*).*

Proof We show first that if $B \subset [0,1]$ is any dense set and $A \subset [0,1]$ is any set of positive Lebesgue measure, then $A + B$ has Lebesgue measure 1. To see this, choose a countable subset $B_0 \subset B$ which is dense in $[0,1]$. Because B_0 is countable, it is certainly measurable, so $A + B_0$ is the countable union of translates of A and is thus measurable. Suppose $\lambda(A + B_0) < 1$, so that $D = [0,1] \setminus (A + B_0)$ has positive measure. Choose points of density of $a \in A, d \in D$ respectively.[31] Find $\delta > 0$ such that

$$\begin{aligned} \lambda(D \cap [d-\delta, d+\delta]) &> \frac{3\delta}{2} \\ \lambda\left(A \cap [a-\delta, a+\delta]\right) &> \frac{3\delta}{2} \end{aligned}$$

Since B_0 is dense, we can find $b \in B_0 \cap \left(d - a - \frac{\delta}{2}, d - a + \frac{\delta}{2}\right)$. Then

$$\lambda\Big((A+b) \cap [d-\delta, d+\delta]\Big) > \frac{3\delta}{2} - \frac{\delta}{2} = \delta$$

so

$$\lambda\Big((A+B_0) \cap D\Big) > \delta + \frac{3\delta}{2} - 2\delta = \frac{\delta}{2} > 0$$

In particular, $(A+B_0) \cap D \neq \emptyset$. This is a contradiction, so we conclude that $\lambda(A + B_0) = 1$. Because $B_0 \subset B$ and every superset of a set of Lebesgue measure 1 is Lebesgue measurable, we conclude that $\lambda(A + B) = 1$.

To show that $\inf_{a \in A} f_a = 0$, take $B = U$ and conclude from the above argument that $\lambda(A + U) = 1$. Because f_a is zero on $a + U$, it follows that $\inf_{a \in A} f_a(c) = 0$ for almost every $c \in [0,1]$. It would be tempting to conclude that $\inf_{a \in A} f = 0$, but it would be wrong[32] because this infimum is to be computed in L^1_+, not pointwise. Instead, suppose $\inf_{a \in A} f_a \neq 0$. Then there

[31] Recall that x is a point of density of $E \subset [0,1]$ if

$$\lim_{h \to 0+} \frac{1}{2h} \lambda\left((x-h, x+h) \cap E\right) = 1$$

The Density Theorem (see Aliprantis and Burkinshaw (1990, Theorem 29.5)) asserts that almost every point of E is a point of density of E.

[32] We are grateful to Richard M. Nixon for pointing this out.

exists a set E of positive measure such that $\inf_{a\in A} f_a$ is positive almost everywhere on E. Because $\lambda(A+U)=1$, $\lambda(E\cap(A+U))=\lambda(E)>0$, so we can find $e\in(E\cap(A+U))$ and $\delta_1>0$ such that $\lambda(E\cap(e-\delta,e+\delta))>0$ whenever $\delta<\delta_1$. Since $e\in A+U$, we can find $a\in A$ such that $e\in a+U$. Since U is open, we can find $\delta_2>0$ such that $(e-\delta_2,e+\delta_2)\subset a+U$. Setting $\delta=\min\{\delta_1,\delta_2\}$, we see that f_a is zero almost everywhere on $(e-\delta,e+\delta)$, so $\inf_{a\in A} f_a$ is zero almost everywhere on $(e-\delta,e+\delta)$, which contradicts the fact that $\lambda(E\cap(e-\delta_1,e+\delta_1))>0$. We conclude that $\inf_{a\in A} f_a=0$. ■

With the preliminaries out of the way, we turn to our examples. We begin with an example of a set that is shy but not finitely shy and not contained in the union of a countable family of finitely shy sets.

Example 1 We assert that the positive cone $L^1_+([0,1])\subset L^1([0,1])$ is shy, but is not contained in a countable union of finitely shy sets.

Because the Hunt, Sauer and Yorke notion of shyness coincides with our notion of shyness in X, to verify that L^1_+ is shy in L^1, we need only construct a compactly supported probability measure μ such that $\mu(L^1_++x)=0$ for each $x\in L^1$. To accomplish this, define $f\in L^1$ by

$$f(s)=-[\min\{s,1-s\}]^{-1/2}$$

and let $F=\{f_t : t\in[0,1]\}$ be the set of rotations of f. Because rotation is a continuous map (Lemma A), F is compact. Let μ be the image in F of Lebesgue measure on $[0,1]$: for $A\subset T$, $\mu(A)=\lambda(\{t\in[0,1] : f_t\in A\})$.

Suppose that $\mu(L^1_++x)=\gamma>0$ for some $x\in L^1$. Let $x=x^+-x^-$ be the decomposition of x into its positive and negative parts. Let $A=\{a\in[0,1] : f_a\in L^1_++x\}$; thus, $a\in A$ if and only if $f_a\geq x$ if and only if $-x^-\leq f_a$. Fix an integer N and partition the interval $[0,1]$ into N equal subintervals $I_n=\left[\frac{n}{N},\frac{n+1}{N}\right)$ $(n=1,\ldots,N)$. Because $\lambda(A)=\gamma$, at least $N\gamma$ of these subintervals must contain a point of A. If I_n is one of these subintervals and $a\in A\cap I_n$, then

$$\int_{I_n}(-x^-)d\lambda \;\leq\; \int_{I_n} f_a d\lambda$$

$$
\begin{aligned}
&= -\int_{I_n} \frac{1}{\sqrt{|s-a|}} d\lambda(s) \\
&\leq -2\frac{1}{\sqrt{N}}
\end{aligned}
$$

Hence

$$\int_{[0,1]} -x^- d\lambda \leq N\gamma[-2\frac{1}{\sqrt{N}}] = -2\gamma\sqrt{N}$$

Because N is arbitrary, we conclude that

$$\int_{[0,1]} -x^- d\lambda = -\infty$$

so $x \notin L^1$, which is a contradiction. Hence $\mu(L^1_+ + x) = 0$ for every $x \in L^1$, so L^1_+ is shy, as asserted.

It remains to show that L^1_+ is not contained in the countable union of finitely shy sets. Suppose it were. Without loss of generality, we may assume these finitely shy sets are universally measurable; since L^1_+ is closed, their intersections with L^1_+ are finitely shy universally measurable sets, whose union equals L^1_+. Thus, we suppose (in order to derive a contradiction) that $L^1_+ = \bigcup_{n=1}^\infty A_n$ where each A_n is a finitely shy universally measurable set. In order to derive a contradiction, we show first that each A_n is finitely shy in L^1_+. To accomplish this, use the definition of finite shyness to choose a finite dimensional subspace $V \subset L^1$ for which $\lambda_V(L^1) > 0$ but $\lambda_V(A_n + x) = 0$ for each $x \in X$. To show that A is finitely shy in L^1_+ we need only show that $\lambda_V(L^1_+ + y) > 0$ for some $y \in L^1$. To this end, choose a basis $f_1, \ldots, f_m$ of V. For each i, decompose $f_i = f_i^+ - f_i^+$ as the difference of its positive and negative parts, and set $y = -(f_1^- + \ldots + f_m^-)$. Because $f_1, \ldots, f_m$ form a basis for V, the set

$$Z = \left\{ \sum_{i=1}^m \alpha_i f_i : 0 \leq \alpha_i \leq 1 \right\}$$

contains a relatively open subset of V. However,

$$
\begin{aligned}
\alpha_1 f_1 + \cdots + \alpha_n f_n &= \left(\alpha_1 f_1^+ + \cdots + \alpha_n f_n^+\right) - \left(\alpha_1 f_1^- + \cdots + \alpha_n f_n^-\right) \\
&\geq -\left(f_1^- + \cdots + f_n^-\right) \\
&= y
\end{aligned}
$$

Hence $Z \subset L^1_+ + y$, whence $\lambda_V(L^1_+ + y) \geq \lambda_V(Z) > 0$. We conclude that A_n is finitely shy in L^1_+ for each n.

In view of Fact 6, A_n is shy in L^1_+ for each n, so L^1_+ is the countable union of sets, each of which is shy in L^1_+, and hence (Fact 3) L^1_+ is shy in itself, a contradiction. We conclude that L^1_+ is not contained in a countable union of finitely shy sets. $\diamondsuit$

The next two examples show why the definition of shyness makes such stringent requirements on the supports of the measures. Example 2 shows that if the requirement that the support of the measure be contained in arbitrarily small contractions of C toward a point $c \in C$ were eliminated from the definition, then it would be possible to find a set C and two relatively shy subsets whose union is all of C.

Example 2 Choose an open dense set $U \subset [0,1]$ with $\lambda(U) < 1$. Let χ_U be the characteristic function of U and let $f = 1 - \chi_U$ be the characteristic function of the complement of U. Set

$$K = \{f_t : t \in [0,1]\} \subset L^1_+$$

Because rotation is continuous (Lemma A), K is the continuous image of a compact set, and hence is compact. Let $C = \overline{\text{conv}} K$ be the closed convex hull of K. Because L^1 is a Banach space, C is compact[33] and (of course) convex. Write $Y = C \setminus K$.

We construct probability measures $\lambda_\alpha, \lambda_\beta$ supported on C for which

$$\lambda_\alpha(g + Y) = 0 \quad \text{for every } g \in L^1 \tag{6}$$
$$\lambda_\beta(g + K) = 0 \quad \text{for every } g \in L^1 \tag{7}$$

On the other hand, there is certainly no probability measure supported on C for which C has measure 0.

[33]See Dunford and Schwartz (1957), Theorem V.2.6, page 416.

To construct the desired measures $\lambda_\alpha, \lambda_\beta$, define mappings $\alpha, \beta : [0,1] \to C$ by

$$\begin{aligned} \alpha(t) &= f_t \\ \beta(t) &= \frac{2}{3} f_t + \frac{1}{3} f_{1-t} \end{aligned}$$

and let $\lambda_\alpha, \lambda_\beta$ be the direct image measures:

$$\lambda_\alpha(A) = \lambda(\alpha^{-1}(A))$$
$$\lambda_\beta(A) = \lambda(\beta^{-1}(A))$$

We first establish Equation (6). Let $g \in L^1$; we are to show that $\lambda_\alpha(g + Y) = 0$. If $g = 0$ there is nothing to prove (because λ_α is supported on K, which is disjoint from $Y = C \setminus K$). Suppose therefore that $g \neq 0$ and that $\lambda_\alpha(g + Y) > 0$. Write

$$A = \{t \in [0,1] : f_t - g \in Y\}$$

By construction, $\lambda(A) = \lambda_\alpha(g + Y) > 0$. Decompose g as the sum of its positive and negative parts: $g = g^+ - g^-$. Because $Y \subset L^1_+$ it follows that

$$f_t - g = f_t - g^+ + g^- \geq 0$$

for each $t \in A$. Because $f_t \geq 0$ and $g^+ \wedge g^- = 0$, it follows that $f_t - g^+ \geq 0$, and hence that $f_t \geq g^+$ for each $t \in A$. Because $\lambda(A) > 0$, it follows from Lemma B that $\inf_{t \in A} f_t = 0$, whence $g^+ = 0$. Thus, for each $t \in A$ we have $f_t - g = f_t + g^-$. Because f, g^- are positive, it follows that $\|f_t - g\| = \|f_t\| + \|g^-\|$. On the other hand, the norm of each element of K is $1 - \lambda(U) > 0$, so $f_t + g^-$ can only belong to K if $g^- = 0$. We conclude that $g = 0$, which is a contradiction. This establishes Equation (6).

The argument that establishes Equation (7) is almost the same, except for one point: The image of β certainly is not disjoint from K; in particular, $\beta(0), \beta(1/2), \beta(1)$ all belong to K. We assert, however, that $\beta(t) \in K \Rightarrow t$ is rational; in particular, $\{t \in [0,1] : \beta(t) \in K\}$ has measure 0. To see this, suppose $\beta(t) = \frac{2}{3} f_t + \frac{1}{3} f_{1-t} \in K$. Since K consists entirely of characteristic functions, it follows that $\beta(t)$ must also be a characteristic function. Hence $f_t(s) = 0 \Leftrightarrow f_{1-t}(s) = 0$, from which it follows that $U + t = U + (1 - t)$, or equivalently that $U = U + (1 - 2t)$. Because U is an open set, it can be written in a unique way as a countable union of disjoint open intervals $U = \bigcup_{n \in \mathbf{N}} U_n$. Because $U = U + (1-2t)$, each of the translation $U_1 + (1-2t)$ must coincide with one of the intervals $U_1, U_2, \ldots$.[34] In particular, for every $n, m \geq 0$ the translates $U_1 + n(1 - 2t)$ and $U_1 + m(1 - 2t)$ must be identical or disjoint. Since any two such translates are subsets of $[0, 1]$ and have the same positive measure, they cannot all be disjoint. That is, there are positive integers $n \neq m$ for which $U_1 + n(1-2t) = U_1 + m(1-2t)$, which implies that $(n - m)(1 - 2t)$ is congruent to zero modulo 1; in particular, $1 - 2t$ must be rational, whence t must also be rational. We conclude that $\beta(t) \in K \Rightarrow t$ is rational; in particular, $\{t \in [0, 1] : \beta(t) \in K\}$ has measure 0. Equation (7) now follows by essentially the same argument as we used for Equation (6).

We note that the measures λ_α and λ_β have supports which are curves but are not contained in translates of small neighborhoods of $0 \in L^1$, but this additional requirement is easily satisfied: For $\varepsilon > 0$, let $\lambda^\varepsilon_\alpha, \lambda^\varepsilon_\beta$ be the direct images of the restriction of Lebesgue measure to the interval $[0, \varepsilon]$ (suitably normalized). $\diamondsuit$

Example 3 shows that if the assumption that the support of the measure be contained in an arbitrarily small neighborhood of $0 \in X$ were eliminated from the definition, then it would be possible to find a set C and a countable family of relatively shy subsets whose union is all of C.

Example 3 For each n, let $p_n = \chi_{[0,1/n]}$ and let

$$C_n = \{y \in L^1_+ : p_n \cdot y \leq 1\}$$

It is clear that each C_n is a closed subset of L^1_+, that $C_n \subset C_{n+1}$, and that $\bigcup_{n=1}^{\infty} C_n = L^1_+$.

We construct compactly supported probability measures μ_n for which $\mu_n(C_n + x) = 0$ for every $x \in L^1$, and for which the supports supp μ_n all lie

[34] As always, we compute translations modulo 1.

in a bounded subset of L^1_+. To accomplish this, fix an index n, and choose a dense open set $U_n \subset [0,1]$ such that

$$\begin{aligned} \rho_n &= \lambda(U_n \cap [0, \tfrac{1}{n}]) < \tfrac{1}{n} \\ \sigma_n &= \lambda(U_n \cap [\tfrac{1}{n}, 1]) < \tfrac{n-1}{n} \end{aligned}$$

Define $f_n \in L^1_+$ by

$$f_n(s) = \begin{cases} \frac{3n}{1-n\rho_n}(1-\chi_{U_n}(s)) & \text{if} \quad s \le \frac{1}{n} \\ \\ \frac{3n}{n-1-n\sigma_n}(1-\chi_{U_n}(s)) & \text{if} \quad s > \frac{1}{n} \end{cases}$$

Note that $\|f_n\| = 6$ and that $p_n \cdot f_n = 3$. Because rotation is continuous (Lemma A), we can find γ_n with $0 < \gamma_n < 1/n$ such that $\|(f_n)_t\| = 6$ and $p_n \cdot (f_n)_t > 2$ for each $t \in [0, \gamma_n]$. Define $\alpha_n : [0, \gamma_n] \to L^1_+$ by $\alpha_n(t) = (f_n)_t$ and let λ_n be the direct image of $(1/\gamma_n)\lambda$:

$$\lambda_n(A) = \frac{1}{\gamma_n}\lambda(\{t \in [0, \gamma_n] : (f_n)_t \in A\})$$

Note that λ_n is a probability measure and that supp $\lambda_n = \alpha_n([0, \gamma_n])$ is a compact subset of L^1_+. Arguing exactly as in Example 2, we see that $\inf_{a \in A}(f_n)_a = 0$ for every set $A \subset [0, \gamma_n]$ of positive measure, and hence that $\lambda_n(C_n + x) = 0$ for every $x \in L^1$.

Obviously, however, there is no probability measure μ supported on $L^1_+ = \bigcup_{n=1}^{\infty} B_n$ for which $\mu(L^1_+) = 0$.

A final comment about this example: We can choose a point $c \in L^1_+$ such that for each n and each $\delta > 0$, there is a compactly supported probability measure μ_n with supp $\mu_n \subset \delta(L^1_+ - c) + c$ and $\mu_n(L^1_+ + x) = 0$ for every $x \in L^1$. And, for every n, we can choose a point $z_n \in L^1_+$ such that for each neighborhood W of 0, there is a compactly supported probability measure ν_n with supp $\nu_n \subset W + z_n$ and $\nu_n(L^1_+ + x) = 0$ for every $x \in L^1$. But we *cannot* choose the z_n's to coincide with c (nor with each other).$\diamondsuit$

Because shyness is an analogue of Lebesgue measure 0, it is natural to conjecture that an analogue of Fubini's theorem should be true. As our last example demonstrates, however, the obvious analogue of Fubini's theorem for shyness is false. First recall Fubini's theorem. Let $(A, \mathcal{A}, \alpha), (B, \mathcal{B}, \beta)$ be σ-finite measure spaces and let $E \subset A \times B$ belong to the product σ-algebra $\mathcal{A} \times \mathcal{B}$. For each $b \in B$, define the section

$$E_b = \{a \in A : (a, b) \in E\}$$

Fubini's theorem says that

$$(\alpha \times \beta)(E) = \int_B \alpha(E_b) d\beta$$

Hence if $\alpha(E_b) = 0$ for almost every $b \in B$, it follows that $(\alpha \times \beta)(E) = 0$. A natural analogous conjecture for shyness would be:

> **Conjecture** If X, Y are Banach spaces, $E \subset X \times Y$ is a Borel set, and the sections
>
> $$E_y = \{x \in X : (x, y) \in E\}$$
>
> are shy in X for each $y \in Y$, then E is shy in $X \times Y$.

As the following example shows, however, this conjecture is false.[35]

Example 4 Let $X = \mathbf{R}$ and let Y be the (separable) Banach space of real-valued Lipschitz functions on $[0, 1]$.[36] Let

$$E = \{(x, f) \in (0, 1) \times Y : f \text{ is not differentiable at } x\}$$

We assert that E is a Borel set. To see this, we proceed as in the proof of Theorem 2.4. Choose a countable dense subset $\{x_i\} \subset [0, 1]$ and a countable dense subset $Q \subset \mathbf{R}$. For positive integers k, ℓ, $q \in Q$ and index i write

$$E(k, \ell, q, i) = \left\{(x, f) \in (0, 1) \times Y : \frac{|f(x_i) - f(x) - q \cdot (x_i - x)|}{|x_i - x|} < \frac{1}{k}\right\}$$

[35] Hunt, Sauer and Yorke (1992, Fact 4, page 224) show that E is shy under the much stronger assumption that there is a *single* measure ν such that $\nu(E_y + x) = 0$ for all x.

[36] A norm on Y is given by

$$\|f\| = \sup\{|f(x)| : x \in [0, 1]\} + \inf\{c > 0 : |f(x) - f(y)| \leq c|x - y| \ , \forall x, y \in [0, 1]\}$$

Each of the sets $E(k, \ell, q, i)$ is open in $(0, 1) \times Y \subset \mathbf{R} \times Y$, and

$$E = \bigcup_{k=1}^{\infty} \bigcap_{\ell=1}^{\infty} \bigcap_{q \in Q} \bigcap_{|x_i - x| < \frac{1}{\ell}} E(k, \ell, q, i)$$

so E is a Borel set, as asserted.

Rademacher's theorem (see Federer (1969, 3.1.6), for instance) tells us that a Lipschitz function is differentiable almost everywhere. Hence, for each $y \in Y$, the section

$$E_y = \{x \in \mathbf{R} : y \text{ is not differentiable at } x\}$$

has Lebesgue measure 0, and in particular is shy in $X = \mathbf{R}$.

The complement of E is

$$F = \{(x, y) \in X \times Y : y \text{ is differentiable at } x\}$$

Arguing just as in Theorem 2.4, we see that, for each $x \in X$ the section F_x is shy in Y.

On the other hand, E and F cannot both be shy subsets of $X \times Y$ since their union is all of $X \times Y$.[37] Thus the conjecture cannot hold for both E, F.$\diamondsuit$

[37] In fact, we believe — but are not certain — that *neither* E nor F is shy.

References

[1] Aliprantis, Charalambos D. and Owen Burkinshaw (1985), *Positive Operators*. New York: Academic Press.

[2] Aliprantis, Charalambos D. and Owen Burkinshaw (1990), *Principles of Real Analysis*. Boston: Harcourt, Brace, Jovanovich.

[3] Allen, Beth (1981b), "Utility Perturbations and the Equilibrium Price Set," *Journal of Mathematical Economics*, 8(1981), 277-307.

[4] Araujo, Aloisio (1985), "Regular Economies and Sets of Measure Zero in Banach Spaces," *Journal of Mathematical Economics* 14, 61-66.

[5] Araujo, Aloisio and P. K. Monteiro (1991), "Generic Nonexistence of Equilibria in Finance Models," *Journal of Mathematical Economics* 20, 489-499.

[6] Aumann, Robert J. and Lloyd S. Shapley (1974), *Values of Nonatomic Games*. Princeton: Princeton University Press.

[7] Billingsley, Patrick (1968), *Convergence of Probability Measures*. New York: John Wiley & Sons.

[8] Blume, Lawrence E. and William R. Zame (1993), "The Algebraic Geometry of Competitive Equilibrium," in W. Neuefeind (editor), *Essays in General Equilibrium and International Trade: In Memoriam Trout Rader*, Springer-Verlag, New York.

[9] Bochnak, J., M. Coste and M.-F. Roy (1987), *Géométrie Algébrique Réelle*. Berlin: Springer-Verlag.

[10] Chichilnisky, Graciela (1976), "Manifolds of Preferences and Equilibria," Ph.D. dissertation, University of California, Berkeley.

[11] Chichilnisky, Graciela and Yuquing Zhou (1996), "Smooth Infinite Economies," Discussion Paper Series No. 9596-17, Department of Economics, Columbia University.

[12] Christensen, Jens Peter Reus (1974), *Topology and Borel Structure*, Volume 10 in *North Holland Mathematics Studies*. Amsterdam: North-Holland Publishing Company.

[13] Dana, Rose-Anne (1993), "Existence, Uniqueness and Determinacy of Arrow-Debreu Equilibria in Finance Models," *Journal of Mathematical Economics* 22, 563-579.

[14] Dana, Rose-Anne and M. Pontier (1992), "On the Existence of an Arrow-Radner Equilibrium in the Case of Complete Markets, A Remark," *Mathematics of Operations Research* 17, 148-163.

[15] Debreu, Gérard (1970), "Economies with a Finite Set of Equilibria," *Econometrica* 38, 387-392.

[16] Debreu, Gérard (1975), "The Rate of Convergence of the Core of an Economy," *Journal of Mathematical Economics*, 2:1-7.

[17] Dierker, Hildegard (1975), "Equilibria and Core of Large Economies," *Journal of Mathematical Economics*, 2:155-169.

[18] Dubey, Pradeep (1986), "Inefficiency of Nash Equilibria," *Mathematics of Operations Research* 11, 1-8.

[19] Duffie, Darrell and Chi-Fu Huang (1989), "Implementing Arrow-Debreu Equilibria by Continuous Trading of Few Long-lived Securities," *Econometrica* 53, 1337-1356.

[20] Duffie, Darrell and Wayne Shafer (1985), "Equilibrium in Incomplete Markets I: A Basic Model of Generic Existence," *Journal of Mathematical Economics* 14, 285-300.

[21] Duffie, Darrell and Wayne Shafer (1986), "Equilibrium in Incomplete Markets II: Generic Existence in Stochastic Economics," *Journal of Mathematical Economics* 15, 199-216.

[22] Duffie, Darrell and William R. Zame (1989), "The Consumption-Based Capital Asset Pricing Model," *Econometrica* 57, 1279-1297.

[23] Dunford, Nelson R. and Jacob T. Schwartz (1957), *Linear Operators Part I: General Theory.* New York: Interscience Publishers Inc., John Wiley and Sons.

[24] Federer, Herbert (1969), *Geometric Measure Theory.* New York: Springer-Verlag.

[25] Gretsky, Neil, Joseph Ostroy and William R. Zame (1992), "The Nonatomic Assignment Model," *Economic Theory* 2, 103-127.

[26] Gretsky, Neil, Joseph Ostroy and William R. Zame (1996), "Perfect Competition in the Continuous Assignment Model," UCLA Working Paper, forthcoming in *Journal of Economic Theory.*

[27] Grodal, Birgit (1975), "The Rate of Convergence of the Core for a Purely Competitive Sequence of Economies," *Journal of Mathematical Economics*, 2, 171-186.

[28] Gruber, Peter M. (1977), "Die meisten konvexen Körper sind glatt, aber nicht zu glatt," *Mathematische Annalen* 229, 259-266.

[29] Harsanyi, John (1973), "Oddness of the Number of Equilibrium Points: A New Proof," *International Journal of Game Theory* 2, 235-250.

[30] Hart, Sergiu (1977), "Asymptotic Value of Games with a Continuum of Players," *Journal of Mathematical Economics* 4, 57-80.

[31] Howe, Roger (1982), "Most Convex Functions are Smooth," *Journal of Mathematical Economics* 9, 37-39.

[32] Hunt, Brian R., Tim Sauer, and James A. Yorke (1992), "Prevalence: A Translation-Invariant 'Almost Every' on Infinite-Dimensional Spaces," *Bulletin (New Series) of the American Mathematical Society* 27, 217-238.

[33] Karatzas, Ioannis, J. P. Lehoczky, and S. E. Shreve (1990), "Existence and Uniqueness of Multi-Agent Equilibrium in a Stochastic Dynamic Consumption/Investment Model," *Operations Research* 15, 80-128.

[34] Kehoe, Timothy J. and David K. Levine (1985), "Comparative Statics and Perfect Foresight in Infinite Horizon Economies," *Econometrica* 53, 433-452.

[35] Kehoe, Timothy J., David K. Levine, Andreu Mas-Colell and William R. Zame (1989), "Determinacy of Equilibria in Large Square Economies," *Journal of Mathematical Economics* 18, 231-262.

[36] Makowski, Louis and Joseph Ostroy (1992), "Existence of Perfectly Competitive Equilibrium á la Wicksteed," in *Economic Analysis of Markets and Games*, D. Gale, O. Hart and E. Maskin, eds., MIT Press.

[37] Malgrange, Bernard (1966), *Ideals of Differentiable Functions*. London: Oxford University Press.

[38] Mas-Colell, Andreu (1985), *The Theory of General Economic Equilibrium: A Differentiable Approach*. Cambridge: Cambridge University Press.

[39] Minehart, Deborah Fairley (1997), "Generic Finiteness of the Set of Equilibria in a Finite Exchange Economy," *Mathematical Social Sciences*, 34(1), 75-80

[40] Ostroy, Joseph (1984), "A Reformulation of the Marginal Productivity Theory of Distribution," *Econometrica* 52, 599-630.

[41] Rudin, Walter (1990), *Fourier Analysis on Groups*. New York: John Wiley and Sons.

[42] Shannon, Chris (1998), "A Prevalent Transversality Theorem for Lipschitz Functions," UC Berkeley Working Paper.

[43] Shannon, Chris (1999), "Determinacy of Competitive Equilibria in Economies with Many Commodities," *Economic Theory* 14(1), 29-87.

[44] Shannon, Chris and William R. Zame (1999), "Quadratic Concavity and Determinacy of Equilibrium," UC Berkeley Working Paper.

[45] Stinchcombe, Maxwell B. (2001), "The Gap Between Probability and Prevalence: Loneliness in Vector Spaces," *Proceedings of the American Mathematical Society* 129, 451-457.

[46] Verona, Maria (1989), "On the Differentiability of Convex Functions," *Proceedings of the Centre of Mathematical Analysis* 20, 195-203.

[47] Wilson, Robert (1971), "Computing the Equilibria of n-person Games," *SIAM Journal of Applied Mathematics* 21, 80-87.