6. Time

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A. The Fisherian two-period model

In the previous chapters we have considered a static or timeless model of individual choice and market equilibrium. We now extend the model to take account of time. Many aspects of time are intrinsically linked to uncertainty. For example, how much we save for the long-term depends very much on how likely we are to live far into the future. Despite this there are important issues associated with choice over time that can be analyzed without explicitly modeling uncertainty. In this chapter we will develop insights into the effect of an interest rate change on the distribution of consumption over time and hence on saving. We will also examine the determinants of interest rates. For a firm, we will analyze how rapidly it should expand production capacity when demand rises. Finally, we will look at optimal consumption plans for simple aggregate economies.

Initially we will examine the simplest case of an individual who consumes in only two periods. Later we consider general \( T \) period models. One issue on which we shall focus, is the effect of the length of the horizon on early decisions. For an important class of models we shall see that a distant time horizon has almost no affect on initial decisions. In such cases, it matters little whether we use a long finite horizon model or, as is often convenient, assume that the horizon is infinite.

Consider a consumer with a high first period income of \( I_1 \) and a low second period (retirement) income of \( I_2 \). Let \( x_t \equiv (x_{t1},...,x_{tn}) \) be his consumption of the \( n \) commodities in period \( t \). The vector of "spot" prices\(^1\) in period \( t \) is \( p_t \) and the consumer can borrow or lend at the nominal interest rate \( r \). Then if the consumer places \( S_1 \) in his savings account in the current (first) period, his final consumption must satisfy the following two constraints.

\[
p_t^1 \cdot x_t \leq I_1 - S_1 \quad \text{first period budget constraint}
\]
\[
p_t^2 \cdot x_t \leq I_2 + (1 + r)S_1 \quad \text{second period budget constraint}
\]

Dividing the second constraint by \( 1 + r \) and then adding the two constraints, we have the consumer’s “life-time” budget constraint

\[
p_t^1 \cdot x_t + \frac{p_t^2 \cdot x_t}{1+r} \leq I_1 + \frac{I_2}{1+r} \quad \text{life-time budget constraint}
\]

\[
PV_{\text{consumption}} \leq PV_{\text{income}}
\]

\(^1\) That is, \( p_t^j \) is the price of a unit of commodity \( j \) purchased "on the spot" in period \( t \).
Thus the consumer’s budget set is enlarged by a change in his lifetime income stream if and only if the change increases the discounted present value of lifetime earnings.

Ignoring any bequest motive, let \( U(x_1, x_2) \) be the consumer’s utility function. Then the consumer chooses his lifetime consumption plan to solve

\[
\max_{x_1, x_2} U(x_1, x_2) \mid p_1^i \cdot x_1 + \frac{p_2^i \cdot x_2}{1 + r} \leq I_1 + \frac{I_2}{1 + r}.
\]

(A.1)

Define \( x \equiv (x_1, x_2) \), to be the entire lifetime consumption vector, \( p \equiv (p_1^i, \frac{p_2^i}{1 + r}) \) to be the vector of current and discounted future spot prices, and \( W = I_1 + \frac{I_2}{1 + r} \) to be the present value of his income stream. Then the maximization problem can be rewritten as

\[
\max_{x} U(x) \mid p \cdot x \leq W.
\]

Thus, from a mathematical perspective, the problem is identical to the standard consumer problem analyzed in Chapter 3. However, from an economics perspective, we want to find a way to focus on the inter-temporal aspects of the problem. To do so it is useful to define \( c_t \) to be total consumption expenditures in period \( t \). We then divide the optimization problem into two stages.

**Stage 1**: Fix consumption expenditure \( c_t \) in period \( t, t=1,2 \) and maximize utility given these expenditure levels. Let \( x_t (c_1, c_2, p_t^i), \ t=1,2 \) be the solution. That is, these consumption vectors solve

\[
U^*(c_1, c_2; p_1^i, p_2^i) = \max_{x} U(x) \mid p_t^i \cdot x_t \leq c_t, \ t=1,2
\]

The solution of this first stage problem, \( U^*(c_1, c_2; p^i) \), is an indirect utility function.

It is a straightforward exercise to confirm that if \( U(x) \) is quasi-concave, \( U^*(c_1, c_2, p^i) \) is a quasi-concave function of \((c_1, c_2)\).

**Stage 2**: Solve for the optimal consumption expenditures in each period:

\[
\max_{c} U^*(c_1, c_2, p^i) \mid c_1 + \frac{c_2}{1 + r} = I_1 + \frac{I_2}{1 + r}.
\]

**The saving decision**

We need only look at the second stage problem to extract useful insights into the consumer’s saving decision. Figure A.1 depicts a case in which first period income is sufficiently high relative to second period income that optimal first period saving is
positive. We now consider the effect on saving of an increase in the interest rate from $r$ to $r'$.

![Diagram showing optimal saving and budget lines](image)

Fig. A.1: Optimal Saving

Given that consumption expenditures are broad aggregates, it is natural to assume that income effects are positive. A higher interest rate leaves a saver better off. Therefore the income effect of a higher interest rate is to increase consumption in both periods. Note that the slope of the budget line is $1+r$, thus the substitution effect of a higher interest rate is to move around the indifference curve, increasing $c_2$ and decreasing $c_1$. Equivalently, when the interest rate rises, the price of future spending $\frac{1}{1+r}$ falls so that the substitution effect is to increase second period consumption.

The total effect on second period consumption is thus unambiguous. A higher interest rate leads to higher spending in the second period. On the other hand, the substitution and income effects have opposing effects on first period spending and hence on saving. If saving is small, the income effect will be small and hence the substitution effect will likely dominate so that saving increases. However, for big savers, theory is silent. Given the absence of a clear-cut theoretical result, it is not surprising that econometric estimates of interest rate effects on saving are typically small.

For borrowers, however, unambiguous results do emerge. Figure A.2 depicts an individual with a current income which is low relative to future income. This might be an individual just starting work with a low current income or someone currently out of work. The substitution effect is as before. However, for a borrower, an increase in the interest rate makes him worse off. Therefore the income effect on first period consumption is negative, reinforcing the substitution effect. We may therefore conclude that borrowing declines as the interest rate rises.
Summarizing, the above results suggest that an increase in the interest rate has a small and ambiguous interest rate effect on saving and a negative effect on borrowing. Aggregating over savers and borrowers, it is tempting to conclude that the overall effect on aggregate net savings will be positive.

![Diagram showing optimal borrowing](image)

While this is consistent with the results of many econometric studies, there is one glaring defect in the above model. This is the assumption that the interest rate is the same whether or not an individual is a saver or a borrower. In reality there is a large gap between lending rates offered by savings institutions and rates for consumer borrowing. There are good reasons why there should be a gap between the two rates. Savings accounts are often fully insured by the government so a saver faces no default risk. For an institution to be willing to lend, however, it needs to be first convinced that the borrower is both able and willing to repay. Monitoring is costly and information is imperfect so some loans will default. The borrowing rate must therefore include a premium reflecting both monitoring costs and default risk.

Fig. A.3 reproduces Fig. A.2 except that now the interest cost of borrowing $\rho$ is higher than the interest earned on lending. Starting from the no-saving point N, an individual can increase his period 1 consumption by 1 unit if he is willing to forgo $1 + \rho$ units next period. An individual who decreases his consumption by 1 unit in period 1 has a period 2 increase of $1+r$. The boundary of the budget set therefore has a kink at the point N, where saving is zero. For the case depicted, the resulting optimum is the “no saving-no borrowing” point.

---

2 Compare the interest rate for your savings account with the interest rate on your credit card.
As long as the kink is quite large, it follows that a significant fraction of consumers end up at the kink. Interest rate effects for such consumers are zero. Aggregating over all consumers, it follows that the interest rate effect on aggregate net saving is likely to be small.

Exercise A.1:
A consumer has a quasi-concave utility function \(U(x_1, \ldots, x_T)\) where \(x_t\) is the consumption vector in period \(t\). Define
\[
U^*(c_1, \ldots, c_T) = \max_{x_t} \{U(x) \mid p_t \cdot x_t \leq c_t, \ t = 1, \ldots, T\}.
\]

(a) Show that \(U^*(c_1, \ldots, c_T)\) is quasi-concave
(b) If \(U(\cdot)\) is concave, show that \(U^*(c_1, \ldots, c_T)\) is concave.

Exercise A.2:
A consumer has utility function \(U(x) = x_1^\alpha x_2^{1-\alpha} + \delta x_1^{\beta} x_2^{1-\beta}\), \(0 < \alpha, \beta < 1\).

(a) Solve for the indirect utility function if he must spend \(c_t\) in period \(t\).
(b) Hence show that this consumer is always happiest if he consumes only in one period.
(c) Modify the utility function so ensure that it is strictly quasi-concave. Given your modification, would the consumer ever buy only one commodity in any period? Might he still consume only in one period?
**Exercise A.3:**
Consumer $h$ has the same income is each of two periods. He can lend at the rate $r$ and borrow at the rate $\rho = r + \nu$. There is a single consumption good in each period. His utility function is $U^h(x_1, x_2) = x_1^{\alpha_h} x_2^{1-\alpha_h}$.
(a) For what parameter values will this consumer borrow and when will he save?
(b) Suppose that there are $H$ consumers in the economy, each with a different preference parameter $\alpha_h$ but with the same income in each period. How does aggregate net saving vary with the interest rate?

---

**Equilibrium**

Suppose we abstract from differences among consumers and also differences in borrowing and lending rates and ask what factors determine the market rate of interest. To further simplify we will now interpret $c_1$ and $c_2$ as consumption of the single commodity in the economy. The aggregate production set of this economy is depicted in Figure A.4. If $\omega_1$, the initial endowment of “corn” is consumed, there are no ears of corn to be transformed into corn next period. If $z$ units of first period corn are withheld from the market and invested in next period’s crop, output in the second period is $y_2 = F(z)$.

The production set of the economy is therefore

$$Y = \{(y_1, y_2) \mid y_2 \leq F(z), \quad y_1 = \omega_1 - z\}$$

---

Equilibrium Diagram:

- $c_1$ and $c_2$ axes
- Production set $Y$ shaded
- Utility function $U(c_1, c_2) = \bar{U}$
- Slope of indifference curve $-(1 + r)$
- Initial endowment $\omega_1$ and consumption $z^*$
- Optimal consumption $(c_1^*, c_2^*)$

Fig. A.4: Fisherian Model with Production
The steepness of the boundary is the rate at which output in period 1 can be transformed into output in period 2.

\[-\frac{dy_2}{dy_1} = -\frac{dy_2}{dz} = F'(z) \quad \text{marginal rate of transformation}\]

Let \( p_t^s, t=1,2 \) be the spot price of corn in period \( t \). This is the price an individual pays if purchasing “on the spot.” Since we have only a single commodity in each period we may set the spot price of corn to be 1 in each period. Then the market interest rate, \( r \), is the real return in units of corn. Also the “futures price” of corn (the cost today for a unit to be delivered next period must be \( p_2 = \frac{1}{1+r} \). (Otherwise there are profitable arbitrage opportunities available.)

Consider a representative firm choosing its production plan. The present value of the firm is

\[ PV = \frac{1}{1+r} F(z) - z = \frac{y_2}{1+r} - (\omega_1 - y_1) = \frac{y_2}{1+r} + y_1 - \omega_1. \]

The present value maximizing production plan \( (y_1^*, y_2^*) \) is depicted in Figure A.4. Given this plan, the present value of the economy’s output is

\[ \frac{y_2^*}{1+r} + y_1^*. \]

Equivalently, the firm chooses a production plan to maximize

\[ p_1 y_1 + p_2 y_2 = y_1 + \frac{1}{1+r} y_2, \text{ where } (p_1, p_2) = (1, \frac{1}{1+r}) \]

Thus, if we express everything in terms of spot and futures prices, the firm simply maximizes profit. Expressed in this way there is nothing to distinguish the firm from a two product firm in the static model of the previous chapters.

Let \( \Pi(p) \) be the maximized present value (or profit) given the price vector \( p \). Our representative consumer has an endowment of period 1 corn \( \omega_1 \). He also receives a dividend equal to the maximized profit of the firm. He then chooses a consumption bundle \( (c_1^*, c_2^*) \) to solve
Max\{U(c)\mid p_1c_1 + p_2c_2 = p_1\bar{y}_1 + \Pi\}

As depicted in Figure A.4, there is excess demand for period 1 corn and excess supply of period 2 corn. Thus the relative price of future goods must fall. Equilibrium is depicted in Fig. A.5.

![Equilibrium Diagram](image)

Fig. A.5: Equilibrium

Other things equal, it is clear from Figure A.5, that the steeper the boundary of the production set (the greater the marginal rate of transformation) the greater will be the equilibrium rate of interest. On the demand side, the steeper the indifference curves, that is, the more one is willing to give up tomorrow for another unit of corn today, the greater will be the interest rate.

**Exercise A.4: Robinson Crusoe Economy**

Robinson Crusoe has two period utility function \( U = \sqrt{x_1x_2} \) where \( x_t \) is corn consumption in period \( t \). The farm has a current crop of 108 units of corn. If he plants \( z \) units of corn, his next period output \( y_2 = 12\sqrt{z} \). His first period consumption is then \( x_1 = 108 - z \).

(a) Solve for Robinson Crusoe's optimal first and second period consumption of corn

(b) What are the equilibrium spot and futures prices \( (p_1, p_2) \) in this economy?

(c) What is the equilibrium interest rate?

(d) How would your answers change if, instead, there are many people in this economy, all holding shares of the single firm and all with the same preferences as Robinson Crusoe?
B. Spot prices, futures prices and rational expectations

In the previous section we examined equilibrium for the simplest case in which there is a single commodity each period. In this section we begin with a general 2 period model and examine issues associated with trading through time. We then present an example in order to illustrate these issues.

In our two period economy, firm $f$ chooses a production plan $y^f = (y^f_1, y^f_2) \in Y^f$, where the subscript indicates the time period. As in the previous chapter, if $y^f_{jt}$ is positive, this is the output of commodity $j$ in period $t$ and if $y^f_{jt}$ is negative, this is the input of commodity $j$ in period $t$. In this general formulation, it is helpful to focus on the period 1 prices for delivery "on the spot" $p_1$ and in the future $p_2$.

Let $\Pi^f(p)$, $f = 1, \ldots, F$ be maximized profits of the $F$ firms at these prices. That is,

$$\Pi^f(p) = \max_{y^f} \{ p \cdot y^f \mid y^f \in Y^f \}$$  \hspace{1cm} (B.1)

Then if consumer $h$ has ownership shares of $\theta^h = (\theta_1^h, \ldots, \theta_F^h)$, his budget constraint is

$$p \cdot x^h = p_1 \cdot x_{21}^h + p_2 \cdot x_{22}^h \leq \sum_{f=1}^F \theta^f \Pi^f(p)$$

The optimal choice of consumer $h$ at these prices then solves

$$\max_{x^h} \{ U^h(x^h) \mid p \cdot x^h \leq \sum_{f=1}^F \theta^f \Pi^f(p) \}$$ \hspace{1cm} (B.2)

In equilibrium, supply equals demand, that is

$$\sum_{h=1}^H x_{jt}^h(p) \leq \sum_{f=1}^F y_{jt}^f(p), \text{ } t = 1, 2, \text{ with equality unless } p_{jt} = 0.$$ \hspace{1cm} (B.3)

Viewed as a mathematical construct, this description of equilibrium is identical to that of the previous chapter. Thus all our conclusions for the static model continue to hold. In particular, the equilibrium allocation is Pareto Efficient.

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3 We assume that consumers have no endowment other than their shareholdings. This may appear restrictive. However, we can always reinterpret a "firm" as a single production plan owned 100% by one consumer.
That is, if all firms and consumers can trade in all spot and futures markets in period 1, there is no allocation which is Pareto-preferred to the equilibrium allocation.

**Trading in future spot markets**

One uncomfortable feature of this analysis is that all trading takes place in period 1. But what about the further trading opportunities in period 2? Initially we shall suppose that no agent expects markets to reopen in period 2 so that the equilibrium trades are made in period 1. Let \((\bar{y}_1^f, \bar{y}_2^f), f = 1, ..., F\) and \((\bar{x}_1^h, \bar{x}_2^h), h = 1, ..., H\) be the equilibrium allocation. Expanding equation (B.1),

\[
\Pi^f(p_1, p_2) = \max_{y_1^f, y_2^f} \{ p_1 \cdot y_1^f + p_2 \cdot y_2^f \mid (y_1^f, y_2^f) \in Y^f \}.
\]

If we set the first period production vector at its profit maximizing level \((y_1^f = \bar{y}_1^f)\) it follows that

\[
p_2 \cdot \bar{y}_2^f = \max_{y_2^f} \{ p_2 \cdot y_2^f \mid (y_1^f, y_2^f) \in Y^f \}.
\]  \hspace{1cm} (B.4)

Also, the choice of consumer \(h\), \((\bar{x}_1^h, \bar{x}_2^h)\), is the solution to the following problem.

\[
\max_{x_1^h, x_2^h} \{ U^h(x_1^h, x_2^h) \mid p_1 \cdot x_1^h + p_2 \cdot x_2^h \leq p_1 \cdot \bar{x}_1^h + p_2 \cdot \bar{x}_2^h \}
\]

Setting \(x_1 = \bar{x}_1^h\), it follows immediately that \(\bar{x}_2^h\) solves

\[
\max_{x_2^h} \{ U^h(\bar{x}_1^h, x_2^h) \mid p_2 \cdot x_2^h \leq p_2 \cdot \bar{x}_2^h \}.
\]  \hspace{1cm} (B.5)

Suppose that markets unexpectedly reopen in period and that the period 2 spot price vector \(p_2^*\) is proportional to the futures price vector, that is,

\[
p_2^* = (1 + r) p_2
\]  \hspace{1cm} (B.6)

Substituting (B.6) into (B.4) and (B.5), it follows that

\[
p_2^* \cdot \bar{y}_2^f = \max_{y_2^f} \{ p_2^* \cdot y_2^f \mid (\bar{y}_1^f, y_2^f) \in Y^f \}.
\]

and that \(\bar{x}_2^h\) solves

\[
\max_{x_2^h} \{ U^h(\bar{x}_1^h, x_2^h) \mid p_2^* \cdot x_2^h \leq p_2^* \cdot \bar{x}_2^h \}.
\]
Then the planned period 2 consumption remains an equilibrium if period 2 spot markets unexpectedly reopen.

**Rational expectations**

Taking the argument further, suppose that all agents expect markets to reopen in period 2. An individual can either purchase period 2 goods in the futures market or save and then make purchases in the period 2 spot market. A unit of commodity $j$ to be delivered in period 2 costs $p_{j2}$ in the futures market. If instead the funds are saved and the interest rate is $r$ the consumer has $(1+r)p_{j2}$ tomorrow. For consumers (and firms) to be indifferent between purchasing in period 1 and period 2 it follows that the anticipated future spot price vector must be $p'_2 = (1+r)p_2$. Equivalently, the futures price vector is the present discounted value of the anticipated future spot price vector.

$$p_2 = \frac{p'_2}{1+r}$$

Given such beliefs, suppose all trade were to take place in period 1. We know from the discussion about the unexpected reopening of markets that no one would wish to make any further trades in period 2. Thus again we have an equilibrium.

Trading in period 1 is just one of a continuum of possible exchanges in financial markets which lead to the same final allocation of commodities. Take the simplest case of two consumers. Alex has a relatively low endowment of kiwifruit today and Bev has a relatively high endowment today. One possibility is for Alex to sell a futures contract to deliver kiwifruit to Bev in period 2 and purchase kiwifruit from her to be delivered immediately. Equivalently, Bev sells her kiwifruit in period 1, saves the money she makes and uses this to purchase from Alex in period 2. Alex borrows from the bank in period 1 to make his kiwifruit purchases and repays his loan out of his sales in period 2.

The reason for the continuum of alternative financial outcomes is that we have introduced more markets than are necessary for equilibrium. We have the full set of $2 \times n$ markets in period 1, borrowing and lending at the interest rate $r$ and $n$ further period 2 spot markets. Conceptually this does not create any difficulties. However, no market is really costless to operate. In particular, trading in futures markets requires confirmation that the individual promising to deliver goods next period will be in a position to carry out his commitment. It is therefore natural to focus attention on financial arrangements which economize on these costs.

**Financial Intermediaries**

We will now see how financial intermediaries can substitute for the futures markets. Suppose that only spot markets open in period 1. There are no futures markets.
However, firms and consumers can borrow and lend via a banking system at the market rate of interest. In order to make decisions about how much to save or borrow, an individual must have beliefs about what prices will prevail in the future. Suppose that the first period spot prices are the equilibrium prices that would prevail with all the markets open. Suppose, also, that everyone believes that the period 2 spot price vector will be \( p^s_2 = (1 + r) p_2 \), that is, the gross return on saving times the equilibrium futures prices if all the markets were open. Then if firms maximize present value, firm \( f \) will choose a production plan to solve

\[
\Pi^f = \max_{y^f} \left\{ p^s_1 \cdot y^f_1 + \frac{1}{1 + r} p^s_2 \cdot y^f_2 \mid y^f \in Y^f \right\}
\]

(B.7)

If the vector of period 1 dividends is \( d_1 = (d^1_1, ..., d^F_1) \), and consumer \( h \) saves \( S^h \), his first period budget constraint is

\[
p_1 \cdot x^h + S^h \leq \sum_{f=1}^{F} \theta^{hf} d^f
\]

(B.8)

The ex-dividend value of firm \( f \) in period 1 is \( \Pi^f - d^f \). The future ex-dividend value of this firm is therefore \( (1 + r)(\Pi^f - d^f) \). Consumer \( h \) thus has an anticipated period 2 budget constraint,

\[
p^s_2 \cdot x^h \leq (1 + r) \left[ \sum_{f=1}^{F} \theta^{hf} (\Pi^f - d^f) + S^h \right].
\]

(B.9)

Dividing the last inequality by \( 1 + r \) and then adding inequality (B.8), it follows that the consumer has a life cycle budget constraint,

\[
p^s_1 \cdot x^h_1 + \frac{1}{1 + r} p^s_2 \cdot x^h_2 \leq \sum_{f=1}^{F} \theta^{hf} \Pi^f
\]

(B.10)

Comparing (B.7) and (B.10), with (B.1) and (B.2), it follows immediately that, as long as

\[
p^s_2 = (1 + r) p_2,
\]

an equilibrium with all markets open in period 1 is also an equilibrium in which only spot markets and financial markets open in period 1. Thus, in a world of certainty, trading in spot markets plus the introduction of a banking system fully substitutes for the futures markets.
However there in one important caveat. In the financial market equilibrium, individuals trade in a single futures market - the market for future cash. In order to sustain this equilibrium, agents must correctly forecast equilibrium prices in all future spot markets. For commodities where intertemporal changes are small, this may not be too heroic an assumption. However if technological differences between periods are important, so that relative price changes are likely to be significant, price expectations are most readily formed if a futures market is open in the earlier period. Our equilibrium analysis thus suggests that a hybrid model is likely to best explain trading over time. For some commodities there will be opportunities to trade directly in futures markets, while for others, financial institutions substitute for such direct trading.

**An example**

Suppose that consumers all have the same logarithmic preferences,

\[ U = u(x_1) + \frac{1}{2} u(x_2) \quad \text{where} \quad u(x_t) = 2 \ln x_{1t} + \ln x_{2t} \quad (B.11) \]

Since such preferences are homothetic, it follows that we can analyze equilibrium as if there were a single aggregate individual, holding all the endowment and shares.

In period 1 there are two commodities which can be either consumed or invested. Each unit of commodity \( a \) invested yields 2 units of commodity \( a \) in period 2. Each unit of commodity \( b \) invested yields 4 units of commodity \( b \) in period 2. Let \( z_{1i} \) be the input of commodity \( i \) in period 1 and let \( q_{2i} \) be period 2 output, \( i = a,b \). The production functions can then be written as

\[ q_{a2} = 2z_{a1} \quad \text{and} \quad q_{b2} = 4z_{b1}. \]

Let \( x_t = (x_{at}, x_{bt}) \) be period \( t \) consumption. The first period endowment is \( \omega = (120,120) \). There is no period 2 endowment.

Then

\[ x_{a1} = \omega_{a1} - z_{a1}, \quad x_{b1} = \omega_{b1} - z_{b1}, \quad x_{a2} = 2z_{a1} \quad \text{and} \quad x_{b2} = 4z_{a1} \quad (B.12) \]

Substituting (B.12) into (B.11) we obtain

\[ U = 2 \ln(120 - z_{a1}) + 2 \ln(120 - z_{b1}) + \ln 2z_{a1} + \ln 4z_{b1} \]

We can then solve for the optimal period 1 investments. It is readily confirmed that the utility maximizing input vector \((z_{a1}^*, z_{b1}^*) = (40,40)\) and hence that the optimal consumption vector is \( \bar{x} = (80,80,80,160) \).
Since the technology and preferred sets are convex, we know that there are Walrasian equilibrium prices (Lagrange multipliers) which support this optimum. Let the prices for delivery in period 1 be \( p_1 \) and the prices for delivery in period 2 (futures prices) be \( p_2 \). Then if the value of the consumer’s wealth is \( W \), he chooses his consumption vector \( x \) to solve

\[
\text{Max}_{x} \{ U(x) \mid p \cdot x = p_1 \cdot x_1 + p_2 \cdot x_2 = W \}
\]

The first order conditions are as follows.

\[
\frac{\partial U}{\partial x_{a1}} = \frac{\partial U}{\partial x_{b1}} = \frac{\partial U}{\partial x_{a2}} = \frac{\partial U}{\partial x_{b2}} = \lambda
\]

For our example, with Cobb-Douglas preferences, this can be rewritten as

\[
\begin{align*}
\frac{2}{p_{a1}} &= \frac{1}{p_{b1}} = \frac{1}{p_{a2}} = \frac{1}{p_{b2}} = \lambda
\end{align*}
\]

Then for the representative consumer to choose the (Pareto) optimum, \( \bar{x} = (80,80,80,160) \) we require,

\[
\begin{align*}
\frac{2}{p_{a1}} &= \frac{1}{p_{b1}} = \frac{1}{p_{a2}} = \frac{1}{p_{b2}} = \frac{1}{2}
\end{align*}
\]

Suppose we make commodity \( a \) the numeraire commodity and set the price of commodity \( a \) in period 1 equal to 1. Then the equilibrium price vector is

\[
p = (p_{a1}, p_{b1}, p_{a2}, p_{b2}) = (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{8})
\]

Note once again that these prices are all period 1 prices. The first two are prices for immediate delivery (“spot prices”) while the second two are prices for delivery in the future (“futures prices.”)

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**Exercise B.1:**

Continuing with the example, suppose that commodity \( a \) is the "numeraire commodity" with a spot price of \( a \) and that the market rate of interest is zero.

(a) What are the equilibrium future spot prices?
(b) What must the interest rate be if the future spot price and spot price of the numeraire commodity are both 1?
Exercise B.2: Own rates of return and the "real" interest rate
Continuing with the example, the "own rate of return" on a commodity is the number of units that can be consumed in period 2 if one withholds a unit of consumption in period 1.
(a) What is the equilibrium own rate of return on each commodity?
(b) There is no "real" interest rate in this economy. True or false? Explain.

Exercise B.3: Rational expectations equilibrium
Suppose that the example is modified so that each consumer has an endowment of \((\omega_a, \omega_b) = (120,120)\) in period 2 as well as in period 1. There is a banking sector but there are no futures markets. What are the equilibrium spot prices in each period if the interest rate is zero?

Exercise B.4: Worthless Technology?
Suppose that the example is modified so that the second period endowment \(\omega_2 = (600,600)\). In period 1 there are both spot and futures markets for each commodity.
(a) If the technology is destroyed so that only the endowments remain, what will be the equilibrium prices?
(b) What are the equilibrium prices with the technology?
(c) For what period 2 endowments is the technology worthless?

C. LIFE CYCLE CONSUMPTION AND SAVINGS
We now impose strong assumptions on preferences in order to provide a more detailed analysis of consumer choice over time. Since the focus is on capital accumulation, we simplify and assume that there is a single consumption good. An individual chooses a consumption sequence \(\{c_t\}_{t=1}^T\). We also allow for the possibility that he is committed to leaving a bequest \(B_{T+1}\) in the period after he dies.\(^4\) We assume that preferences can be expressed in terms of the following additively separable concave utility function.

\[
U(c_1, \ldots, c_T) = \sum_{t=1}^{T} \delta^{t-1} u(c_t)
\]

where \(u(c)\) is a strictly increasing concave function. Unless otherwise noted, it will be assumed that \(u'(0) = \infty\), thereby ensuring that consumption in all periods is strictly positive.

\(^4\) A more general analysis would include the bequest in the utility function \(U(c_1, \ldots, c_T, B_{T+1})\). We can think of the problem solved here as the first stage of a two-stage optimization problem. In the first stage we solve for maximized utility \(U^*(B_{T+1})\), holding the bequest fixed. In the second stage the bequest is chosen to maximize \(U^*(B_{T+1})\).
The assumption that preferences can be represented by an additively separable function ensures that the marginal rate of substitution of period $t$ for period $t+1$ consumption depends only on consumption in these two periods. This greatly simplifies the analysis. A further strong simplification which we shall sometimes make is that this marginal rate of substitution depends only on the ratio of consumption levels in the two periods.

$$\frac{\partial U}{\partial c_{t+1}} = \frac{u'(c_t)}{u'(c_{t+1})} = \frac{1}{\delta} \left( \frac{c_{t+1}}{c_t} \right)^{1/\sigma}, \quad \sigma > 0 . \quad (C.1)$$

The higher the elasticity of substitution $\sigma$, the more slowly the marginal rate of substitution changes as the consumption ratio increases and so the more willing an individual is willing to substitute consumption between any two periods.

Initially we shall assume that the individual can borrow or lend at the interest rate $r$. His income in period $t$ is $y_t$. Let $K_t$ be the capital stock that the individual has available in period $t$. He then decides how much to consume and how much to leave for the future. His budget constraint is thus

$$K_{t+1} = (1+r)(K_t + y_t - c_t) . \quad (C.2)$$

Since utility is strictly increasing in consumption, this constraint must be binding at the optimum. Rearranging, the growth in capital must satisfy

$$K_{t+1} - K_t = (1+r)(\frac{r}{1+r} K_t + y_t - c_t) . \quad \text{capital accumulation equation} \quad (C.3)$$

All debt must be repaid so the individual faces the terminal constraint $K_{T+1} \geq B_{T+1}$.

The individual chooses his optimal consumption plan to maximize utility subject to the capital accumulation constraints given by (C.2). That is $\{c_t, K_{t+1}\}_{t=1,...,T}$ solves

$$\max \{ \sum_{t=1}^{T} \delta^{t-1} u(c_t) \mid c_t \geq 0, \quad K_{T+1} \geq B_{T+1}, \quad K_{t+1} \leq (1+r)(K_t + y_t - c_t), \quad t = 1,...,T \} .$$

Note that there is no restriction that capital must be positive throughout the life-cycle. The individual may go into debt if he wishes.

To characterize the optimal path we write down the Lagrangian.

$$L = \sum_{t=1}^{T} \delta^{t-1} u(c_t) + \sum_{t=1}^{T} \lambda_t [(1+r)(K_t + y_t - c_t) - K_{t+1}]$$
We now suppose that the sequence \( \{c_t, K_{t+1}\}_{t=1}^T \) is optimal. Then the following first order conditions must hold.\(^5\)

\[
\frac{\partial L}{\partial c_i} = \delta^{t-1} u'(c_i) - \lambda_t = 0, \quad t = 1, \ldots, T
\]

\[
\frac{\partial L}{\partial K_t} = (1 + r) \lambda_t - \lambda_{t-1} = 0, \quad t = 2, \ldots, T
\]

From these two conditions it follows that

\[
MRS(c_t, c_{t+1}) = \frac{u'(c_t)}{u'(c_{t+1})} = 1 + r. \quad \text{necessary condition (C.4)}
\]

An even more basic way of seeing why this condition is necessary is to consider changes in consumption levels only in periods \( t \) and \( t+1 \). Let \((\bar{c}_t, \bar{c}_{t+1})\) be the optimal consumption bundle and let \((c_t, c_{t+1})\) be some feasible alternative. Suppose \( c_t < \bar{c}_t \). Then the individual can take the difference and save it until next period. He can used the increased savings \((1 + r)(\bar{c}_t - c_t)\) to increase his consumption in period \( t+1 \). Thus

\[
c_{t+1} - \bar{c}_{t+1} = (1 + r)(\bar{c}_t - c_t).
\]

Rearranging,

\[
c_t + \frac{c_{t+1}}{1 + r} = \bar{c}_t + \frac{\bar{c}_{t+1}}{1 + r}.
\]

The feasible alternatives and the individual’s preferences are depicted in Figure C.1. That equation (C.4) is a necessary condition then follows immediately.

To complete the analysis, we need to find a path satisfying both the growth equation and the first order condition, which also satisfies the terminal condition \( K_{T+1} \geq B_{T+1} \). In the discussion that follows, we will assume that there is no bequest commitment. Then the period \( T+1 \) capital stock, \( K_{T+1} = 0 \). To fully characterize the optimal path we first note that once we choose a first period consumption, there is a unique path satisfying the capital accumulation equation (C.3) and the necessary condition. Given an initial capital stock \( K_1 \) and a choice of first period consumption \( c_1 \), the entire consumption sequence is determined by the necessary condition, (C.4). Moreover, given \( K_t \) and \( c_t \), capital

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\(^5\) Since the constraints are linear, the feasible set is convex. Since the maximand is concave, it follows that first order conditions are also sufficient for a maximum.

\(^6\) By hypothesis, \( u'(0) = \infty \), so consumption in each period is strictly positive.
invested for period $t+1$, $K_{t+1}$, is determined by (C.3). We now show that a sequence $\{c_t, K_{t+1}\}_{t=1}^{T}$ which starts with lower consumption must, for every period, have lower consumption and higher capital.

![Diagram of consumption and capital dynamics](image)

**Fig:** C-1: Necessary Condition

**Path Comparison:**

Given an initial capital $K_1$, consider two sequences $\{\bar{c}_t, \bar{K}_{t+1}\}_{t=1}^{T}$ and $\{c_t, K_{t+1}\}_{t=1}^{T}$ which satisfy the growth equation (C.2) and the necessary condition (C.4). If $\bar{c}_t < c_t$ then,

$$\bar{c}_t < c_t \quad \text{and} \quad \bar{K}_{t+1} - K_{t+1} > (1 + r)(\bar{K}_t - K_t) > 0 \quad \text{for all} \quad t > 1, \quad (C.5)$$

To derive this result, first note that if $\bar{c}_t < c_t$, then $u'(\bar{c}_t) > u'(c_t)$. From the necessary condition it follows that $u'(\bar{c}_{t+1}) > u'(c_{t+1})$ and hence that $\bar{c}_{t+1} < c_{t+1}$. Thus the first inequality holds. From (C.2)

$$\bar{K}_{t+1} - K_{t+1} = (1 + r)[(\bar{K}_t - K_t) - (\bar{c}_t - c_t)] \quad (C.6)$$

$$> (1 + r)(\bar{K}_t - K_t), \quad \text{since} \quad \bar{c}_t < c_t$$

$$> 0, \quad \text{since} \quad \bar{K}_1 = K_1$$

Thus the gap between the two capital sequences rises faster than exponentially over time.

**Increasing consumption:** $(1 + r)\delta > 1$
Initially we will focus on the case in which \((1 + r)\delta > 1\). That is, the discount factor is more than offset by the interest factor. From the necessary condition, and the fact that \(u(\cdot)\) is concave, it follows that \(c_{t+1} > c_t\). The intuition should be clear. One unit saved rather than consumed in period \(t\) has a marginal disutility of \(u'(c_t)\). The \(1 + r\) units available next period yield a discounted marginal utility of \(\delta(1 + r)u'(c_{t+1})\). Then, starting from a situation in which \(c_t = c_{t+1}\), it pays to decrease \(c_t\) and increase \(c_{t+1}\) if and only if \(\delta(1 + r) > 1\).

**The Phase Diagram**

Given our assumptions, we have seen that consumption increases over time. The analysis of capital accumulation is a bit more subtle. We begin with the special case in which the individual has a constant income. The capital accumulation equation thus becomes

\[
K_{t+1} = (1 + r)(K_t + y - c_t).
\]

If \(K_t < K_{t+1}\), the right-hand side is less than \((1 + r)(K_{t+1} + y - c_t)\). Then capital accumulates if and only if \(K_{t+1} < (1 + r)K_{t+1} + y - c_t\). Rearranging, it thus follows that

\[
K_t < K_{t+1} \text{ if and only if } c_t < y + \frac{r}{1 + r}K_{t+1}
\]

![Phase Diagram](image)

*Figure C.2: Phase Diagram with \((1 + r)\delta > 1\)*
In the diagram, capital accumulates if and only if consumption in period \( t \) lies below the heavy line bounding the shaded region. Since consumption is always increasing, the arrows show the time path of capital and consumption in each region. The first point in the sequence \( \{c_t, K_{t+1}\}_{t=1}^T \) lies on the line

\[
K_2 = (1 + r)(K_1 + y - c_1).
\]

This is the dotted "starting line" depicted in the figure.

We shall now argue that, if the time horizon is sufficiently long, the life-cycle can be divided into two phases. Over the initial phase, the individual is in the shaded region, accumulating capital. In a second phase he consumes his capital until, in the final period, his capital is entirely spent. For any starting capital \( K_1 \) and first period consumption choice \( c_1 \), there is a unique path \( \{c_t, K_{t+1}\}_{t=1}^T \).

Three such paths are depicted below.

Figure C.3: Optimal path

If initial consumption is small, the individual accumulates a lot of capital in the initial phase. In an intermediate case, there is less capital accumulation. And if initial consumption is large, capital declines throughout. Consider the path \( A'F' \) in Fig. C.3. Appealing to the path comparison result, (C.5), the path from \( A \) must have the property that, for all \( t \), \( K_t > K_t' \) and \( c_t < c_t' \). Thus, at the time the sequence from \( A' \) reaches the
final bequest, $K_{T+1}$, the sequence from A must be at some point like $F$ with higher capital stock and lower consumption. It follows that the A-sequence takes longer to reach the terminal capital stock.\footnote{This same argument also implies that the total time before capital starts to decrease is strictly greater for the A-sequence.} Thus, for any initial capital stock and terminal date $T$, there is a unique path satisfying (i) the capital accumulation equation, (ii) the necessary condition and (iii) the terminal condition. Moreover, the longer the time horizon the smaller is initial consumption and the greater the length of time spent accumulating capital.

**Decreasing consumption** - - $\delta(1+r) < 1$.

If the discount factor outweighs the interest factor,

$$\delta MRS(c_t, c_{t+1}) = \frac{u'(c_t)}{u'(c_{t+1})} = \delta(1+r) < 1.$$  

It follows that marginal utility rises over time and hence that the consumption path must be decreasing. Consider the special case in which there is no bequest motive. The new phase diagram is depicted below.

The three paths depicted are all consistent with the growth equation and the necessary condition. For sufficiently high initial consumption, the individual eventually borrows...
and continues to go into debt. For sufficiently low initial consumption the individual retains positive capital throughout and accumulates for large $t$. For intermediate levels of initial consumption, the individual becomes a borrower and then works off his debt again.

In the special case in which there is no bequest, final capital must be zero so the actual path must be one of these intermediate cases. Two such paths are depicted in Figure C.5. Both are chosen so that the terminal capital $K_{T+1} = 0$. For the path beginning at $A$, first period consumption is higher (and hence second period capital is lower) than for the path beginning at $A'$. We have already seen that a path which has a lower capital stock in the first period has a lower capital stock in all future periods. Thus the number of periods along the path through $A$ must be strictly longer. Both paths are therefore optimal paths but for different time horizons.

Figure C.5: Phase Diagram with $\delta(1+r) < 1$
**Exercise C.1: Maximum indebtedness**

Suppose that $\delta(1+r)<1$, income is $y$ in each period and the time horizon is very long.

(a) Characterize as completely as you can the optimal consumption and capital stock sequences.

Hint: Even with an infinite horizon, what is the most that any creditor would lend to this individual?

(b) With a very long finite horizon, is it necessarily the case that the individual spends most of the time in debt?

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**Total Wealth Accumulation**

The analysis thus far, utilizes the simplifying assumption of constant income over time. As we shall now see, we can avoid this assumption by focussing on total wealth (financial and human) rather than financial capital. To simplify the exposition, we assume that there is no bequest commitment. Then the optimal terminal total wealth,

$$ W_{T+1} = K_{T+1} = 0. $$

In period $t$, if an individual has accumulated financial capital $K_t$, his total wealth is

$$ W_t = K_t + y_t + \frac{y_{t+1}}{1+r} + \ldots + \frac{y_T}{(1+r)^{T-t}}. $$

(C.7)

Suppose that the individual were to convert all of his human capital into financial capital by borrowing. Then all his future income would be owed to his creditor's account but he would have total financial assets of $W_t$. If he were to spend nothing, his financial assets would rise next period to $(1+r)W_t$. However if he consumes $c_t$ in period $t$ his wealth in period $t+1$ is

$$ W_{t+1} = (1+r)(W_t - c_t). \quad \text{wealth accumulation equation (C.8)} $$

If $W_t < W_{t+1}$ it follows from this equation that $W_{t+1} < (1+r)(W_{t+1} - c_t)$. Rearranging,

$$ W_t < W_{t+1} \quad \text{if and only if} \quad c_t < \frac{r}{1+r}W_{t+1}. $$

We can therefore analyze the path of consumption and total wealth in exactly the same manner as consumption and financial capital. Indeed Figure C.6 can be viewed as a special case of Figure C.2 with the phase boundary line intercepting the vertical axis at the origin.
Total wealth accumulation with a long finite horizon

For the rest of this section we will consider the CES case. Then the individual has a marginal rate of substitution,

$$MRS(c_t, c_{t+1}) = \frac{u'(c_t)}{\delta u'(c_{t+1})} = \frac{1}{\delta} \left( \frac{c_{t+1}}{c_t} \right)^{1/\sigma}.$$

We will assume that $\delta(1 + r) > 1$ so that consumption grows over the life cycle. Equating the marginal rate of substitution with the interest rate, it follows that

$$\frac{c_{t+1}}{c_t} = (\delta(1 + r))^\sigma = 1 + \gamma. \quad \text{necessary condition} \quad (C.9)$$

That is, consumption grows at a constant rate $\gamma$. Note that as long as the elasticity of substitution, $\sigma$, does not exceed 1,

$$1 + r > \delta(1 + r) > (\delta(1 + r))^\sigma = 1 + \gamma.$$

Thus the consumption growth rate is less than the interest rate.

Combining the wealth accumulation equation and necessary condition, it follows that

$$\frac{W_{t+1}}{c_t} = (1 + r)(\frac{W_t}{c_t} - 1) = \frac{1 + r}{1 + \gamma} \frac{W_t}{c_{t-1}} - (1 + r).$$

Hence
\[
\frac{W_{t+1} - W_t}{c_t} = (1+r)(\theta \frac{W_t}{c_{t-1}} - 1) \quad \text{where} \quad \theta = \frac{r - \gamma}{(1+\gamma)(1+r)} \quad \text{(C.10)}
\]

Figure C.5 is reproduced below with the added line \( c_t = \theta W_{t+1} \).

Suppose that, for some \( t \), the path is in the shaded region so that, \( \frac{W_t}{c_{t-1}} > \frac{1}{\theta} \). From (C.10),

\[
\frac{W_{t+1} - W_t}{c_t} > 0 \quad \text{and hence} \quad \frac{W_{t+1}}{c_t} > \frac{1}{\theta}.
\]

It follows immediately that if the optimal path were ever to fall below the line \( c_t = \theta W_{t+1} \), it would remain there ever after. Thus wealth would rise throughout the life cycle. But this is inconsistent with the terminal condition, \( W_{T+1}^* = 0 \). Thus, with a finite horizon and no bequest motive, the optimal path must lie everywhere above the more heavily shaded region.

Fig. C.6: Phase Diagram for the C.E.S. case
From equation (C.10), note also that if \( \frac{W_t - 1}{c_{t-1}} \) is small, \( \frac{W_{t+1} - W_t}{c_t} \) is also small and thus \( \frac{W_{t+1} - 1}{c_t} \) is small. Then if the optimal path begins close to the boundary line, it moves away from this line very slowly. Thus the consumption-wealth ratio changes very slowly.

As already noted earlier in this section, the longer the time horizon, the lower the initial consumption and the greater the wealth accumulation in early periods. From the above argument it follows that with a sufficiently long finite horizon \( T \), there is some time \( T^* \) such that for all \( t < T^* \),

\[
c_t \approx \theta W_{t+1} = \frac{r - \gamma}{(1 + \gamma)(1 + r)} W_{t+1} \approx \frac{r - \gamma}{1 + r} W_t.
\]

But wealth \( W_t \) is financial capital plus discounted future earnings. Thus

\[
c_t \approx \frac{r - \gamma}{1 + r} (K_t + y_t + \frac{y_{t+1}}{1 + r} + \ldots + \frac{y_{T-r}}{(1 + r)^{T-r}})
\]

Note that the marginal propensity to consume out of current income \( \frac{\partial c_t}{\partial y_t} \approx \frac{r - \gamma}{1 + r} \).

Thus, as first emphasized by Milton Friedman, the life-cycle model predicts a low marginal propensity to consume out of current income for all individuals with many periods yet to live.\(^8\)

In the limit as \( T \) goes to infinity, the optimal path must approach the boundary line

\[
c_t = \theta W_{t+1}.
\]

This suggests that the solution with an infinite horizon must be the boundary line. We shall show in the next section that this is indeed the case.

\[\boxphantom{Exercise C.2.}\]

**Exercise C.2:**

(a) Obtain an expression for total wealth in period \( t+1 \) as a function of \( K_{t+1} \) and the future income stream.

(b) Use the fact that \( K_{t+1} = (1 + r)(K_t + y_t - c_t) \) to show that

\[\boxphantom{As Friedman emphasized, this result is inconsistent with Keynes assumption of a marginal propensity to consume close to 1.}\]
\[ W_{t+1} = (1 + r)(K_t + y_t + \frac{y_{t+1}}{1 + r} + \ldots + \frac{y_T}{(1 + r)^{T-t}} - c_t) \].

Hence obtain the wealth accumulation equation (C.8)

**Exercise C.3: Optimal Wealth Accumulation**

(a) Show that if \((1 + r)\delta > 1\) and the number of periods is small, total wealth must decline over the life-cycle.

(b) Show also that if the horizon is long, total wealth rises initially.

(c) Suppose income is \(y\) in each of the first \(T\) periods and is zero thereafter. Characterize the optimal path of wealth and capital.

**Exercise C.4: Growth path with a high elasticity of substitution**

Suppose that the elasticity of substitution is sufficiently high that the consumption growth rate \(\gamma > r\).

(a) Draw the phase diagram in this case.

(b) What happens to first period consumption as the time horizon becomes large?

HINT: Consider first the limiting case in which \(\sigma = \infty\) and so the utility function is linear.

**Exercise C.5: Solving for the optimal consumption path**

A consumer lives for \(T\) periods. The interest rate is \(r\) and in period \(t\) he earns an income \(y_t\). His utility function is

\[ U(c) = \sum_{t=1}^{T} \delta^{t-1} v(c_t), \quad \text{where} \quad v'(c) = \left(\frac{1}{c}\right)^{1/\sigma}, \quad \sigma > 0 \]

(a) Show that the capital accumulation constraints can be combined into a single life-cycle budget constraint,

\[ \sum_{t=1}^{T} \frac{c_t}{(1 + r)^{t-1}} = W_t, \]

where \(W_t\) is total wealth in period 1.

(b) Set up the Lagrangian and hence (or otherwise) confirm that

\[ \frac{c_{t+1}}{c_t} = 1 + \gamma \equiv (\delta(1 + r))^\sigma. \]

(c) Hence show that the present value of the consumption stream can be written as

\[ PVC = c_1(1 + \frac{\gamma r}{1 + r} + \left(\frac{\gamma r}{1 + r}\right)^2 + \ldots + \left(\frac{\gamma r}{1 + r}\right)^{T-1} = \frac{\gamma r}{r - \gamma} c_1 \left(1 - \left(\frac{\gamma r}{1 + r}\right)^T\right) \]

Hence solve for first period consumption.
(d) If $\gamma < r$, show that in the limit, as $T \to \infty$, $c_t \to \frac{r - \gamma}{1 + r} W_t$.

(e) Suppose $\gamma > r$. What is the consumption path when $T$ is very large?

Exercise C.6: A debtor who saves for retirement

Suppose an individual earns an income $y$ in each of the first $\overline{T}$ periods and must then retire. He lives off his savings for the remaining $T - \overline{T}$ periods of his life. He has a discount factor $\delta$ and the interest rate is $r$ where $\delta(1 + r) < 1$. Use two phase diagrams to describe his life-time path of consumption and asset accumulation.

Hint: Assume that he wishes to begin his retirement with asset level $K_{T+1}$. Analyze the first $T_1$ periods with this terminal condition. Then appeal to the discussion above to characterize the path in the remaining periods.

Life cycle analysis with different borrowing and lending rates

As noted in section A, individuals typically face much higher rates for borrowing than for lending. As we shall see, this too can be readily incorporated into the $T$ period model.

We assume that income $y$ is the same in each period. Since it is the most interesting case, we also that

$$\frac{1}{1 + \rho} < \delta < \frac{1}{1 + r}$$

With $K_{T+1} < 0$, the consumer is a borrower and thus faces the interest rate $\rho$. The phase diagram is as in Fig. C.2, except that the phase boundary line is

$$c_t = (1 + \rho)\left(\frac{\rho}{1 + \rho}K_t + y\right).$$

With $K_{T+1} > 0$, the consumer is a saver and thus faces the interest rate $r$. The phase diagram is as in Fig. C.4. Combining the left half of Fig. C.2 with the right half of Fig. C.4 yields the new phase diagram below. Note that there are now four phases. In phase I, capital and consumption both decrease. In phase II, capital decreases and consumption increases. In phase III both are increasing and in phase IV capital increases while consumption decreases.

Note that the consumer will never enter phase II since his terminal capital stock cannot be negative. Moreover, if there is no bequest motive, the terminal capital stock is zero this the consumer will never enter phase IV. Thus if the consumer starts with some initial capital he must remain in phase I forever, ending on the vertical axis. In the limiting case, with no initial capital, he simply consumes his income each period.
**Exercise C.7:**
Suppose that an individual has the same income $y$ in each period. His utility function is

$$U(c_1, \ldots, c_T, K_{T+1}) = \sum_{t=1}^{T} \delta^{t-1} u(c_t) + \delta^T V(K_{T+1})$$

Use a phase diagram to analyze the optimal path

(a) when $(1+r)\delta > 1$  
(b) $(1+\rho)\delta < 1$  
(c) $\frac{1}{1+\rho} < \delta < \frac{1}{1+r}$.

**Exercise C.8: Linear preferences**

Suppose that preferences are linear and that $\delta(1+r) > 1$. What is the solution for the finite horizon problem? Is there a solution if the horizon is infinite?

**Exercise C.9: Long Finite Horizon**
Suppose that \( \frac{1}{1+\rho} < \delta < \frac{1}{1+r} \) and the time horizon is very long. The consumer has an initial capital stock \( K_i > 0 \), a constant wage income \( y \) and no bequest motive.

(a) Explain why the optimal path must, except near the beginning, be close to the stationary point \( S \) in Fig. C.7.

(b) How will your answer change if the consumer has a bequest motive? In particular, is it still the case that the path will be close to \( S \) most of the time?