CHAPTER 7: UNCERTAINTY

7.1 Risky Choices 1
   Expected utility rule 4

7.2 Aversion to Risk 11
   Measures of risk aversion 14
   Trading in state claims markets 14

7.3 Complete Market equilibrium 21
   Portfolio Choice 26

7.4 Capital Asset Pricing Model 30
   Capital asset pricing rule 35

7.5 Changes in risk 37
   Stochastic Dominance 38

7.A Answers to Exercises 44
7.1 RISKY CHOICES

If a consumer is uncertain as to the consequences of his action, we can extend the basic consumer choice model as long as the consumer is able to (a) fully characterize the set of possible consequences and (b) assign a probability to each of these consequences. Suppose that there are $S$ possible outcomes or "states." In state $s$ the consequence is $c_s$. Let $\pi_s$ be the probability that the consumer assigns to this state. Then a complete description of the uncertain outcome or "prospect" is the $2 \times S$ vector

$$(\pi : c) = ((\pi_1, \ldots, \pi_S) : (c_1, \ldots, c_S))$$

Under the ordering postulates of consumer choice, there exists a continuous utility function $U(\pi, c)$ over prospects. Since our focus will initially be on the choice over probabilities, it is convenient to fix the $S$ consequences, and simply write the prospect as the probability vector $\pi$. For the case of three states, preferences over prospects $\pi = (\pi_1, \pi_2, \pi_3)$ are illustrated below.

![Fig: 7.1-1: Preferences over lotteries](image)

We assume that $c_1 > c_2 > c_3$. In the figure, only the probabilities of the best and worst state are shown. If $\pi_1 = 1$ the outcome is $c_1$ with probability 1. If $\pi_3 = 1$ the outcome is $c_3$ with probability 1. Since probabilities add up to 1, the set of possible probability vectors is the...
shaded region in the left-hand diagram. In particular at the origin $\pi_1 = \pi_3 = 0$ so this is the certain outcome $c_2$.

Suppose Bev must choose between two prospects $\pi^1$ and $\pi^2$. She is about to choose one of them when she is offered the opportunity to randomize. If she accepts, a spinner will be used that will select prospect 1 with probability $p_1$ and prospect 2 with probability $p_2 = 1 - p_1$. We will write this new "compound prospect" as follows.

$$\hat{\pi} = (p_1, p_2 : \pi^1, \pi^2) = (p_1, p_2 : (\pi^1_1, \ldots, \pi^1_s), (\pi^2_1, \ldots, \pi^2_s))$$

In the figure, this is a convex combination of $\pi^1$ and $\pi^2$ so lies on the line joining these two prospects.

Suppose Bev is indifferent between the prospects $\pi^1$ and $\pi^2$. How will she feel about randomizing between them? When asked this question, respondents overwhelmingly indicate that she will be indifferent between all three prospects. Formally,

$$\pi^1 \sim \pi^2 \Rightarrow \pi^1 \sim (p_1, p_2 : \pi^1, \pi^2), \text{ where } p_1, p_2 > 0 \text{ and } p_1 + p_2 = 1.$$ 

It follows that indifference curves must be linear as depicted. In order for the indifference curves to be parallel, as depicted in the right-hand figure, we need a further assumption. Consider the prospect $q^1 = (1 - \lambda)\pi^1 + \lambda r$, that is, a convex combination of $\pi^1$ and the prospect $r$. Similarly define $q^2 = (1 - \lambda)\pi^2 + \lambda r$. These compound prospects can also be depicted in "tree diagrams" as shown below.

![Tree Diagrams of Compound Prospects](image)

Using the compound prospect notation, $q^t = (1 - \lambda, \lambda : \pi^t, r), \ t = 1, 2$. 

---

© John Riley  7. Uncertainty  28 August 2003
Suppose Bev prefers $\pi^1$ to $\pi^2$. She is then offered the chance to randomize between one of these prospects and a third alternative $r$. That is, she must choose between $q^1$ and $q^2$. When asked what she will do, respondents overwhelmingly conclude that she would prefer $q^1$ over $q^2$. Intuitively, since the outcome $r$ is an independent event, which occurs with probability $\lambda$ in both compound prospects, this should not change the original ranking.

This idea is formalized in the following Axiom.

**Independence Axiom (IA)**

If $\pi^1 \succeq \pi^2$, then for any prospect $r$ and probabilities $p_1, p_2 > 0$ and $p_1 + p_2 = 1$,

$$q^1 = (p_1, p_2 : \pi^1, r) \overset{\text{IA}}{=} (p_1, p_2 : \pi^2, r) = q^2$$

From the geometry of the right-hand diagram in Fig. 7.1-1, it follows that indifference curves are parallel. Note that for every feasible prospect $\pi$, there is a prospect $(v(\pi), 1 - v(\pi))$ on the line $\pi_1 + \pi_3 = 1$ such that

$$\pi \sim (v(\pi), 0, 1 - v(\pi)).$$

This is a prospect in which there is positive probability on only the best and worst outcomes. Thus one possible utility representation of the consumer’s preferences is the “win probability” $v(\pi)$ in this extreme lottery.

For the certain consequence $c_1$ the win probability $v(c_1) = 1$ and for $c_3$, the win probability is $v(c_3) = 0$. For the intermediate consequence the win probability $v(c_2)$ lies between zero and one. As we shall now see, for any prospect $p = (p_1, \ldots, p_5)$, the consumer will be indifferent between $\pi$ and playing the extreme lottery with a win probability of

$$\sum_{s=1}^{s} p_s v(c_s).$$

---

1 Suppose $\pi^1 \sim \pi^2$, so that $\pi^1 \succeq \pi^2$ and $\pi^2 \succeq \pi^1$. Then, by IA $q^1 \succeq q^2$ and $q^2 \succeq q^1$, hence $q^2 \sim q^1$. 

---
Thus we can represent the consumer’s preferences in terms of his expected win probabilities in the extreme lotteries, or expected utility.

**Expected Utility Rule**

Preferences over prospects \((c; p) = (c_1, c_5; p_1, p_5)\) can be represented by the Von Neumann-Morgenstern utility function \(u(p, c) = \sum_{i=1}^{3} p_i v(c_i)\)

To prove this result we appeal to the following equivalent version of the Independence Axiom.

**Independence Axiom (IA')**

If \(\pi^m \succeq \hat{\pi}^m\), \(m = 1, ..., M\), then for any probability vector \(p = (p_1, ..., p_M)\)

\[(p_1, ..., p_M : \pi_1^M, ..., \pi_M^M) \succeq (p_1, ..., p_M : \hat{\pi}_1^M, ..., \hat{\pi}_M^M)\]

Clearly \(\text{IA}'\) implies \(\text{IA}\).²

To derive the expected utility rule, consider any \(S\) consequences where \(c^*\) is the most preferred and \(c_\ast\) the least. For each consequence there must be some probability \(v(c_\ast)\) of winning in the extreme lottery. That is,

\[c_\ast \sim e^r \equiv (v(c_\ast), 1 - v(c_\ast) : c^* , c_\ast)\]

By \(\text{IA}'\), it follows that \((p : c) \sim (p_1, ..., p_S : e^l, ..., e^r)\). The compound prospect is depicted in the tree diagram below. While there are \(2S\) terminal nodes, each of these is either \(c^*\) or \(c_\ast\). Thus the big tree and the little tree in Fig. 7.1-3 are equivalent.

² In Exercise 7.1-1 you are asked to show that the converse is also true.
That is

\[(p : c) \sim (p_1, ..., p_s : e^1, ..., e^s) \sim (u(p, c), 1 - u(p, c) : c_1, c_s)\]

While the assumptions underlying this rule seem very plausible, experiments show that there are systematic deviations from the rule. Perhaps the most famous is an example of Allais.

**Allais Paradox**

Consider the four prospects \(A, B, C\) and \(D\) in the diagram below. Note that the slope of the line \(CD\) is \((1 - ab - (1 - b)) / ab = (1 - a) / a\). This is also the slope of the line \(AB\).

Since indifference curves are parallel lines, if \(D \succ C\) then expected utility must be rising along the line \(AE\). Conversely, if \(D \prec C\), then expected utility must be decreasing along the line \(AE\).
Suppose $a = 0.9$ and $b = 0.1$. Then the four prospects are as follows.

$$A = (0, 1, 0) \quad B = (0.9, 0, 0.1) \quad C = (0, 0.1, 0.9) \quad D = (0.09, 0, 0.91)$$

Following Allais, let the payoffs in the three states be $(c_1, c_2, c_3) = (5, 1, 0)$ where the units are millions of dollars.\(^3\)

Most people, when asked to choose between 1 million for sure (prospect $A$) and a 90% chance of $5$ million (prospect $B$) choose $A$. However, when offered a 10% chance of making $1$ million (prospect $C$) or a 9% chance of making $5$ million (prospect $D$), these same people indicate that they would prefer $D$, violating an implication of the expected utility postulates. Subsequent experiments confirm the robustness of this result.

Given such inconsistencies, there have been many attempts to provide alternatives to the Independence Axiom. One such approach (Machina) allows for indifference curves to be linear but replaces the parallel indifference lines with indifference lines that “fan out” as depicted below. Note that with fanning, it is no longer inconsistent for an individual to prefer $D$ to $C$ and $A$ to $B$.

\(^3\) In his original experiment, Allais used the convex combination $E = (0.9, 0.1: A, B)$ rather than $B$. (Adding up probabilities, it is readily checked that $E = (0.01, 0.89, 0.1)$).
A second famous paradox is due to Ellsberg.

**Ellsberg Paradox**

An urn has 30 black balls and 60 other balls, some red and some green. An individual is invited to choose Black or Green and will be paid $100 if the ball drawn is his color. Typically individuals choose Black. A second experiment offers individuals the chance to pick Red and one other color. If the ball drawn is one of the colors selected, the individual wins $50. Typically individuals choose Red and Green.

If the experiment is conducted with different individuals, it is easy to see why these outcomes might occur. In the first experiment an individual may think that the experimenter will try to minimize his expected loss and therefore have very few green balls. He thus chooses Black. In the second experiment the same logic leads to the Red-Green choice since this guarantees 60 balls. However if the same individual is offered both experiments, the patterns is much harder to explain. For regardless of the number of green balls, if the individual’s probability assessment leads him to choose Black in the first experiment, he must believe that there are at
least as many red balls as green. Then he must believe that there are at least 30 red balls. Then in the second experiment Red-Black is at least as good as Red-Green.

**Rabin Paradox**

Experimenter typically find that individuals appear to exhibit much more aversion to risk in the laboratory than in the market place. Moreover it is easy to construct examples in which subjects’ behavior is inconsistent with the standard expected utility model. Suppose you are offered the opportunity to draw a ball at random from an urn containing 100 balls, some green and some red. If you draw a green ball the payoff is in the left column of the table below. If you draw a red ball the payoff is in the right column. Note that the probability of winning rises as you move to the right along each row. Thus an individual accepting any particular gamble would also accept any other gamble to the right.

<table>
<thead>
<tr>
<th>Payoff if green ball</th>
<th>Number of green balls (out of 100)</th>
<th>Payoff if red ball</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>52 55 60 66 70</td>
<td>-100</td>
</tr>
<tr>
<td>1000</td>
<td>13 20 33 46 57</td>
<td>-100</td>
</tr>
<tr>
<td>5000</td>
<td>7 18 33 46 57</td>
<td>-100</td>
</tr>
<tr>
<td>25000</td>
<td>7 18 33 46 57</td>
<td>-100</td>
</tr>
</tbody>
</table>

Consider the first row. Which of these 5 gambles you would accept. Then look at the other gambles in these same columns. Would you be willing to accept any of these?

For most subjects, there are more acceptable gambles in the bottom rows than in the top rows. Yet expected utility theory implies that this should not be the case.

Let the win probability be $p$ and let the winning payoff be $g$. If the first-row gamble is rejected, it follows that

$$v(w) \geq (1-p)v(w-g) + pv(w+g) \quad (7.1-1)$$

This inequality can be rewritten as follows.

$$v(w+g) - v(w) \leq \frac{1-p}{p} (v(w) - v(w-g)) \quad (7.1-2)$$
We will also assume that there is a range of higher wealth levels over which the ranking would not change. For someone whose wealth is \( w' > w \), it follows that
\[
    v(w' + g) - v(w') \leq \frac{1-p}{p} (v(w') - v(w' - g)) .
\]

Setting \( w' = w + g \),
\[
    v(w + 2g) - v(w + g) \leq \frac{1-p}{p} (v(w + g) - v(w)) .
\]

Substituting from (7.1-2), if follows that
\[
    v(w + 2g) - v(w + g) \leq \left( \frac{1-p}{p} \right)^2 (v(w) - v(w - g)) .
\]

Define
\[
    s_n = 1 + \frac{1-p}{p^2} + \left( \frac{1-p}{p^2} \right)^2 + \ldots + \left( \frac{1-p}{p^2} \right)^n .
\]

Adding (7.1-2) and (7.1-3), it follows that
\[
    v(w + 2g) - v(w - g) \leq s_1 (v(w) - v(w - g)) .
\]

Rearranging this expression it follows that
\[
    v(w) \geq (1 - \frac{1}{s_2})v(w - g) + \frac{1}{s_2}v(w + 2g) .
\]

We can repeat this argument \( n \) times to show that
\[
    v(w) \geq (1 - \frac{1}{s_n})v(w - g) + \frac{1}{s_n}v(w + ng) .
\]

**Exercise 7.1-1:** Equivalence of the Independence Axioms.

For \( M=2 \), show that \( IA \) implies \( IA' \)

(a) Complete the proof by induction. That is, show that if the Proposition it holds for \( M=k-1 \), then it must also hold for \( M=k \).

**Exercise 7.1-2:**

(a) Draw a tree diagram showing that the prospect C can be represented as a compound gamble between A and (0,0,1). Draw another tree diagram showing that the prospect D can be represented as a compound gamble between B and (0,0,1), where the probability of (0,0,1)
is the same.
(b) Show that the ranking of A and B must be the same as the ranking of E and D. Hence establish the original version of the Allais Paradox.

**Exercise 7.1-3:**
(a) Show that if preferences can be represented by the von Neumann utility function \( v(\cdot) \), they can also be represented by any affine transformation \( Av(\cdot) + B, \ A > 0 \).

(b) Alex has an expected utility function \( u(p,c) = \sum p_s \ln c_s \). Bev has a utility function \( U(p,c) = c_1^{p_1} c_2^{p_2} ... c_5^{p_5} \). Is she an expected utility maximizer? Do her preferences satisfy the Independence Axiom?

**Exercise 7.1-4:**
(a) You are offered either $100 for sure or one of the following risky alternatives. Which would you accept?

<table>
<thead>
<tr>
<th>Payoff if green ball</th>
<th>Number of green balls (out of 100)</th>
<th>Payoff if red ball</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>52  55  60  66  70</td>
<td>0</td>
</tr>
<tr>
<td>1100</td>
<td>13  20  33  46  57</td>
<td>0</td>
</tr>
<tr>
<td>5100</td>
<td>7   18  33  46  57</td>
<td>0</td>
</tr>
<tr>
<td>25100</td>
<td>7   18  33  46  57</td>
<td>0</td>
</tr>
</tbody>
</table>

(b) Compare your answers to the answers you gave above. Are the differences consistent with expected utility maximization?
7.2 Aversion to Risk

Thus far, our entire focus has been on the ranking of probability vectors. We now switch the focus to the choice of consumption bundles. Suppose that there are two states. In state 1 the Republican candidate wins the election, while in state 2 the Democratic candidate wins. Let $\pi_s$ be the probability of state $s$ and $c_s$ be consumption in state $s$. Bev’s preferences are represented by the expected utility function

$$U(c_1, c_2) = \pi_1 v(c_1) + \pi_2 v(c_2)$$

If $v(\cdot)$ is a concave function

$$U(c_1, c_2) = \pi_1 v(c_1) + (1 - \pi_1) v(c_2) \leq v(\bar{c}),$$

where $\bar{c} = \pi_1 c_1 + (1 - \pi_1) c_1 = \pi \cdot c$.

That is, Bev prefers the certain consumption $\bar{c}$ to any other risky bundle $(c_1, c_2)$ with the same expected value. This is depicted in the left-hand diagram below.

In geometric terms, the consumption bundles with equal expected value are the points on the line through $(\bar{c}, \bar{c})$ with steepness $\pi_1 / \pi_2$. The steepness of the indifference curve

$$\left. \frac{\partial U}{\partial c_1} \right|_{u=\sigma} = \frac{\pi_1 v'(c_1)}{\pi_2 v'(c_2)} \quad \text{or} \quad MRS(c_1, c_2) = \left. \frac{dc_2}{dc_1} \right|_{u=\sigma} = \frac{\partial U}{\partial c_1} \bigg/ \frac{\partial U}{\partial c_2} = \frac{\pi_1 v'(c_1)}{\pi_2 v'(c_2)}.$$ (7.2-1)
With concave preferences \( c_2 > c_1 \Rightarrow v'(c_1) \leq v(c_2) \) thus above the 45° line the indifference curve is steeper than the budget line. Below the 45° line the indifference curve is less steep. Thus the certainty point \((\bar{c}, \bar{c})\), is strictly preferred to any other point on the line \( \pi \cdot c = \bar{c} \).

The right-hand diagram depicts the indifference curve if preferences are convex. In this case

\[ U(c_1, c_2) = \pi_1 v(c_1) + (1-\pi_1) v(c_2) \geq v(\bar{c}) , \]

so the individual prefers all risky bundles to \((\bar{c}, \bar{c})\). Indeed, since the indifference curve is flatter above the 45° line and steeper below it, the most preferred bundle is on one of the axes. Such extreme gambles are sufficiently rare that we will henceforth assume that the utility function is concave. Then individuals will be “risk averse” in the sense that a certain consumption bundle will always be preferred to a risky bundle with the same expected value.

While our argument is for the two state case, it is easily generalized to \( S \) states.

**Jensen’s Inequality**

For any probability vector \( \pi \) and consumption vector \( c \), if \( v(\cdot) \) is concave then

\[ \sum_{s=1}^{S} \pi_s v(c_s) \leq v(\bar{c}) \text{ where } \bar{c} = \sum_{s=1}^{S} \pi_s c_s \]

If \( v(\cdot) \) is continuously differentiable, this follows immediately from the second definition of concavity. For each \( c_s \)

\[ v(c_s) \leq v(\bar{c}) + v'(\bar{c})(c_s - \bar{c}) \]

Multiplying both sides by \( \pi_s \) then summing over \( s \),

\[ \sum_{s=1}^{S} \pi_s v(c_s) \leq v(\bar{c}) + v'(\bar{c}) \sum_{s=1}^{S} \pi_s (c_s - \bar{c}) = v(\bar{c}) , \text{ where } \bar{c} = \sum_{s=1}^{S} \pi_s c_s . \]

In Fig. 7.2-2 below we depict indifference curves through \((\bar{c}, \bar{c})\) for Alex and Bev. The points on or above these indifference curves are the set of acceptable gambles for each consumer. Since Bev’s set of acceptable gambles lies is a subset of the corresponding set for Alex, we will describe Bev as being more risk averse. Since both indifference curves have the same slope on the certainty line, a necessary condition for Bev to be more risk averse is that,
moving around the indifference curve, Bev’s marginal rate of substitution changes more rapidly. Taking the logarithm of (7.2-1) and differentiating by $c_1$,

\[
\ln M = \ln v'(c_1) - \ln v'(c_2) + \ln \frac{\pi_1}{\pi_2}
\]

Hence,

\[
\frac{d}{dc_1} \ln M(c_1, c_2) = \frac{1}{M} \frac{dM}{dc_1} = \frac{\partial M}{\partial c_1} \frac{dc_1}{dc_2} \bigg|_{\pi=\pi}
\]

But,

\[
\frac{\partial}{\partial c_1} \ln M = \frac{v''(c_1)}{v'(c_1)} \quad \text{and} \quad \frac{\partial}{\partial c_2} \ln M = -\frac{v''(c_2)}{v'(c_2)} \tag{7.2-2}
\]

Therefore,

\[
\frac{1}{M} \frac{dM}{dc_1}(c_1, c_2) = \frac{v''(c_1)}{v'(c_1)} + \frac{v''(c_2)}{v'(c_2)} \frac{dc_2}{dc_1} \bigg|_{\pi=\pi}
\]

On the certainty line the steepness of the indifference curve is equal to the odds $\pi_1/\pi_2$. Thus, if $(c_1, c_2) = (\bar{c}, \bar{c})$,

\[
\frac{1}{M} \frac{dM}{dc_1} = \frac{v''(c_1)}{v'(c_1)} + \frac{v''(c_2)}{v'(c_2)} \frac{\pi_1}{\pi_2} = -\frac{v''(\bar{c})}{v'(\bar{c})} \left(1 + \frac{\pi_1}{\pi_2}\right)
\]
Then a necessary condition for Bev’s indifference curves to bend more rapidly is that

\[
-\frac{v_B''(\bar{c})}{v_B'(\bar{c})} \geq -\frac{v_A''(\bar{c})}{v_A'(\bar{c})},
\]

(7.2-3)

**Definition: Measures of Risk Aversion**

- **Absolute Risk Aversion**
  \[ A(c) = -\frac{v''(c)}{v'(c)} \]

- **Relative Risk Aversion**
  \[ R(c) = -\frac{cv''(c)}{v'(c)} \]

Thus condition (7.2-3) is the requirement that Bev should have an everywhere higher absolute (or relative) risk aversion.

It is easy to see that this condition is not only necessary but also sufficient for Bev’s set of acceptable gambles to be a subset of Alex’s set. Compare the logarithm of the marginal rate of substitution at \((c_1, c_2)\) with that at \((c_1, c_1)\) steepness of the two indifference curves at \(E\) with the steepness at \(F\).

\[
\ln M(\alpha, \beta) - \ln M(\bar{c}, \bar{c}) = \int_{c_2}^{\beta} \frac{\partial}{\partial c_2} \ln M(\alpha, c_2) dc_2.
\]

Substituting from (7.2-2),

\[
\ln M(\alpha, \beta) - \ln M(\bar{c}, \bar{c}) = \ln M(\alpha, \beta) - \ln \pi_1 / \pi_2 = \int_{\alpha}^{\beta} -\frac{v''(c)}{v'(c)} dc.
\]

Thus if Bev has a higher degree of absolute risk aversion, then

\[ c_2 > c_1 \Rightarrow M_B(c_1, c_2) \geq M_A(c_1, c_2) \text{ and } c_2 < c_1 \Rightarrow M_B(c_1, c_2) \leq M_A(c_1, c_2). \]

**Trading in markets for state claims**

Suppose Bev will have a higher income in state 1 than state 2. Before the election she can trade in the Iowa Presidential futures market. Let \(p_1\) be the current price of 1 unit of consumption if the Republican candidate wins and let \(p_2\) be the corresponding price if the Democrat wins. Her budget constraint is

\[
p_2(c_2 - y_2) = p_1(y_1 - c_1), \text{ or rearranging, } p_1c_1 + p_2c_2 = p_1y_1 + p_2y_2
\]
Her budget set is depicted below. The steepness of the indifference curve on the certainty line is equal to Bev’s odds of a Republican victory. Thus if the market price ratio is equal to her subjective odds, Bev will fully ensure. As depicted, the price is less favorable so Bev partially insures against a Democratic victory.

How would the riskiness of her optimal consumption bundle change if her wealth were to increase. As we shall see, this depends on how her degree of absolute risk aversion changes with wealth. If it is constant, then her new consumption point is on the dotted line parallel to the certainty line. Thus she chooses the same absolute risk. If the degree of absolute risk aversion \(-v''(c)/v'(c)\) declines as \(c\) increases, then Bev will take on a greater absolute risk.

Consider the effect on the marginal rate of substitution as consumption rises from \((c_1, c_2)\) to \((c_1 + x, c_2 + x)\), that is, to a point on the line parallel to the 45° line.

\[
\ln M(c_1 + x, c_2 + x) = \ln v'(c_1 + x) - \ln v'(c_1 + x) + \ln \pi_1 / \pi_2.
\]

Hence

\[
\frac{1}{M} \frac{dM}{dx} = \frac{v''(c_1 + x)}{v'(c_1 + x)} - \frac{v''(c_2 + x)}{v'(c_2 + x)}.
\]
It follows immediately that if the degree of absolute risk aversion is constant, the marginal rate of substitution is unchanged. If the degree of absolute risk aversion decreases with \( c \), then below the 45° certainty line

\[
-\frac{v''(c_1 + x)}{v'(c_1 + x)} < -\frac{v''(c_2 + x)}{v'(c_2 + x)}.
\]

Thus in this case the marginal rate of substitution increases. The optimal consumption bundle is therefore farther from the certainty line. So in this case greater wealth leads to a consumption bundle exhibiting greater absolute risk.

Intuitively, the degree of absolute risk aversion must be closely related with the degree of concavity of the utility function. Consider the utility functions for Bev and Alex as shown below. For each consumption level we can plot their utility levels and thus the mapping \( f(\cdot) : v_A \rightarrow v_B \). Clearly this mapping is monotonic. We now show that Bev has a greater degree of absolute risk aversion for all \( c \) if and only if the mapping is concave.

Since \( v_B(c) = f(v_A(c)) \), \( v_B'(c) = f'(v_A)c v_A'(c) \). Taking the logarithm and differentiating,

\[
-\frac{v''(c)}{v'(c)} = -\frac{f''(c)}{f'(c)}v_A'(c) - \frac{v_A''(c)}{v_A'(c)}.
\]
Thus Bev is more risk averse if and only $f''(\cdot) \leq 0$.

Finally, we consider a simple portfolio decision. Alex must decide how much to invest in a riskless asset which has a gross yield of $r_1$ and how much in a risky asset (or mutual fund) with gross yield $1+r_2$. Intuitively, if Alex is risk averse he will need to be paid some premium to invest in the risky asset. That is, unless the expected yield for the risky asset is sufficiently greater than for the riskless asset, he will choose not to take any risk. As we shall see, this intuition is not quite correct.

Converting to the state claim formulation, let the gross yield of the risky asset is $1+r_{2s}$ in state $s$, and let the probability of state $s$ be $\pi_s$, $s=1,\ldots,S$. Suppose that Alex invests $x$ dollars in the risky asset and his remaining wealth $W-x$ in the riskless asset. His final consumption in state $s$ is then

$$c_s = (W-x)(1+r_1) + x(1+r_{2s}) = W(1+r_1) + x\theta_s \text{ where } \theta_s = r_{2s} - r_1 \tag{7.2-4}$$

Substituting from (7.2-4), Alex has an expected utility of

$$U(x) = \sum_{s=1}^{S} \pi_s v(W(1+r_1) + x\theta_s).$$

Then his marginal gain from increasing $x$ is

$$U'(x) = \sum_{s=1}^{S} \pi_s v'(W(1+r_1) + x\theta_s) \theta_s = \sum_{s=1}^{S} \pi_s v'(c_s) \theta_s. \tag{7.2-5}$$

Differentiating again, it is easily checked that the second derivative is negative thus there is a single turning point. Moreover, at $x = 0$,

$$U'(0) = v'(W(1+r_1)) \sum_{s=1}^{S} \pi_s \theta_s > 0 \iff \sum_{s=1}^{S} \pi_s \theta_s > 0.$$

Thus unless an individual is infinitely risk averse, he will purchase some of the risky asset.

To understand why this must be the case, consider equation (7.2-5). Alex evaluates marginal claims in each state by his marginal utility in that state and then takes the expectation of marginal utilities. But, when he is taking no risk, each of the marginal utility weights is the same. Thus the decision whether to invest at all is the same as the decision of a risk neutral individual. Of course for any positive investment in the risky asset the marginal utility weights change. Intuitively, this must depend on the degree of risk aversion. We now
show that this is correct. If a second individual, Bev, is everywhere more risk averse she will invest less in the risky asset.

We have seen that if Bev is more risk averse than Alex then her utility function $v_B(c)$ is a concave function of Alex’s utility function, that is $v_B(c) = f(v(c))$ where $f$ is concave. Let $x^*$ be optimal for Alex and define $c^*_s = W(1 + r_s) + \theta_s x^*$. Then

$$U_A'(x^*) = \sum_{s=1}^{S} \pi_s v'(c^*_s)\theta_s = 0.$$ 

If we can show that, for Bev, the expected marginal utility of increasing $x$ is negative at $x^*$ it will follow that Bev will invest less in the risky asset.

Suppose we order states so that $\theta_1 > \theta_2 > ... > \theta_S$. Let $t$ be the largest state for which $\theta_s$ is positive. Then $v(c^*_s) \geq v(c^*_t)$, $s = 1, ..., t$ and $v(c^*_s) < v(c^*_t)$, $s = t + 1, ..., S$.

$$U_B'(x^*) = \sum_{s=1}^{S} \pi_s v'_B(c^*_s)\theta_s = \sum_{s=1}^{S} \pi_s f'(v(c^*_s))v'(c^*_s)\theta_s$$

$$= \sum_{s=1}^{t} \pi_s f'(v(c^*_s))v'(c^*_s)\theta_s - \sum_{s=t+1}^{S} \pi_s f'(v(c^*_s))v'(c^*_s)(-\theta_s).$$

Each term in the two summations is non-negative. For each of the terms in the first summation, the concavity of $f(\cdot)$ implies that $f'(v(c^*_s)) \leq f'(v(c^*_t))$. For each of the terms in the second summation, $f'(v(c^*_s)) \geq f'(v(c^*_t))$.

Hence

$$U_B'(x^*) \leq \sum_{s=1}^{t} \pi_s f'(v(c^*_s))v'(c^*_s)\theta_s - \sum_{s=t+1}^{S} \pi_s f'(v(c^*_s))v'(c^*_s)(-\theta_s)$$

$$= f'(v(c^*_s)) \sum_{s=1}^{S} \pi_s v'(c^*_s)\theta_s = f'(v(c^*_s))U_A'(x^*)$$

**Exercise 7.2-1:** Derive Jensen’s Inequality without appealing to the differentiability of $v(\cdot)$. 
Hint: Suppose that Jensen’s inequality holds for $S = k-1$. (Given concavity, it holds for $S = 2$.) First show that $\sum_{s=1}^{k-1} \frac{x_s}{1-x_s} v(c_s) \leq v(c^*)$, where $c^* = \sum_{s=1}^{k-1} \frac{x_s}{1-x_s} c_s$. Since $v(\cdot)$ is concave, it is also the case that $(1-\pi_k) v(c^*) + \pi_k v(c_k) \leq v((1-\pi_k)c^* + \pi_k c_k)$.

**Exercise 7.2-2:**
For the two state case, show that if Bev has a higher degree of absolute risk aversion, her final consumption bundle will lie closer to the certainty line.

**Exercise 7.2-3:**
An individual has a wealth $c$. He is offered the following prospect. $(\frac{1}{2} + z, \frac{1}{2} - z; c + x, c - x)$. Suppose that $z$ is chosen so that he is indifferent between accepting and rejecting the gamble. Show that if $x$ is small, to a first approximation $z = \frac{1}{2} x (-v''(c)/v(c))$.

**Exercise 7.2-4:** Relative risk aversion
An individual with a strictly concave preference scaling function $v(c)$ is offered a gamble $x = (w(1+\theta), w(1-\theta), 0.5 + \beta, 0.5 - \beta)$. Suppose that she is just indifferent between accepting and rejecting the gamble, that is $(0.5 + \beta)v(w(1+\theta)) + (0.5 - \beta)v(w(1-\theta)) = v(w)$. If $x$ is small so that the quadratic Taylor approximation of $v(\cdot)$ is a good approximation, show that $\beta$ is proportional to the degree of relative risk aversion $R(w) = -w v''(w)/v'(w)$.

**Exercise 7.2-5:** Relative Risk Aversion
The degree of relative risk aversion $RRA(c) = -cv''(c)/v'(c)$.

(a) Show that if an individual exhibits constant relative risk aversion, his marginal rate of substitution $M(c_1, c_2)$ is constant along a ray from the origin. Thus the risk he will take on rises proportionally to his wealth.

(b) Show that if $v'(c) = c^{-\sigma}$, $\sigma > 0$, the individual exhibits constant relative risk aversion.

(b) It is usually argued that individuals exhibit increasing relative risk aversion and constant
absolute risk aversion. What does this imply about the shape of wealth expansion paths?

**Exercise 7.2-6: Risky choices with 2 commodities**
A consumer purchases commodities $x$ and $y$. If his von Neumann Morgenstern expected utility is $v(x, y) = x^\alpha y^\beta$ solve for the consumer’s indirect utility function $V(p, I)$.

(a) Under what assumptions is the consumer’s utility function concave in income (so that the marginal utility of income decreases with income?)

(b) Under what assumptions is utility convex in prices?

(c) Does it follow that in an economy with Cobb-Douglas preferences, all individuals would favor price uncertainty over mean preserving price stabilization?

*Exercise 7.2-7 Wealth Effects*
An individual with wealth $W$ must decide how much to invest in a riskless asset with yield $1 + r_1$ and a risky asset with yield $1 + \tilde{r}_2$ where $E(\tilde{r}_2) > r_1$.

(a) If the individual exhibits constant absolute risk aversion, show that his investment in the risky asset is independent of his wealth.

*(b) If the individual exhibits decreasing absolute risk aversion, show that he will invest more as his wealth increases.*

*Exercise 7.2-8: Wealth effects on asset shares.*
Suppose that the individual invests a fraction $z$ of his wealth in the risky asset.

Show that expected utility can be written as

$$U(x) = E[V(W(1 + r_1) + Wz\tilde{\theta})]$$

(a) Assume constant relative risk aversion and show that the asset share is independent of wealth.

(b) Show that if relative risk aversion increases with wealth, the asset share $z^*$ will decline with wealth.
7.3 Complete Market Equilibrium

As a first step in developing general principles, consider a very simple 2 person economy. Alex owns a farm to the East and Bev to the West of a volcano. The volcano will explode for sure and which farm is damaged will depend upon the way the wind is blowing. In “state 1” the wind is from the West and so Alex loses some of his output. In “state 2” the wind is from the East and Bev loses some of her output. Thus each farmer has an endowment which is “state dependent.” Let \( \omega_h = (\omega_h^A, \omega_h^B), \ h \in \{A, B\} \) be the endowment of agent \( h \). Initially we will assume that the loss is the same in each state, that is \( \omega_h^A = \omega_h^B - \omega_h^A \) and \( \omega_h^B = \omega_h^B - \omega_h^B \).

Then the aggregate endowment is the same in each state. Let the probability of state \( s \) be \( \pi_s \).

Then if agent \( h \) is a VNM expected utility maximizer, and is allocated a final consumption bundle \( c_h = (c_h^A, c_h^B) \), his expected utility is

\[
U^h(c_h) = \sum_{s=1}^{2} \pi_s v^h(c_s^h), \ h \in \{A, B\}.
\]

Agent \( h \) has a marginal rate of substitution

\[
MRS_{12}^h = \frac{\partial U^h / \partial c_{2}^h}{\partial U^h / \partial c_{1}^h} = \frac{\pi_2 v'_h(c_2^h)}{\pi_1 v'_h(c_1^h)} = \frac{\pi_2}{\pi_1} \text{ along the } 45^\circ \text{ line.}
\]

With no aggregate uncertainty, the Edgeworth box is square as depicted below.

Fig. 7.3-1: Efficiency with no aggregate risk
Note that the diagonal is the 45° line for each individual. Thus the indifference curves are tangential along this line. It follows that the Pareto Efficient allocations must lie along the diagonal. As long as each agent is holding risk, both gain from shedding some of that risk. Setting this in a market context, Alex would like to trade away some of his state 2 goods for additional claims in state 1. Bev, on the other hand, would like to trade some of her state 2 claims for additional claims to state 1 output. Each seeks to insure against the possible loss.

Let $p_s$ be the market price of a claim to a unit of output in state $s$. If individual $h$ is free to trade in these claims, she seeks to solve the following problem.

$$\max_c \left\{ \sum_{s=1}^{2} \pi_h(c_s^b) \left| p \cdot c_s^b \leq p \cdot \omega^b \right. \right\}.$$  

Except for the interpretation, this problem is completely standard. By the first welfare theorem, the Walrasian equilibrium must be Pareto efficient. But we have just argued that for efficiency $c_1^b = c_2^b$. Thus the equilibrium price ratio

$$\frac{p_1}{p_2} = \frac{\pi_h(c_1^b)}{\pi_h(c_2^b)} = \frac{\pi_1}{\pi_2}.$$  

That is, the equilibrium price ratio is equal to the probability ratio or “odds”.

Suppose next that the loss is bigger in state 2. Then the aggregate endowment is larger in state 1.
Note that below Alex’s certainty line $MRS_{12}^A < \frac{\pi_1}{\pi_2}$ and below Bev’s certainty line $MRS_{12}^B > \frac{\pi_1}{\pi_2}$. Thus no efficient allocation lies below both certainty lines. An identical argument establishes that no efficient allocation lies above both certainty line either. Thus the PE allocations lie in the region depicted below, between the certainty lines.

There are two immediate implications. First, at any efficient allocation, Both Alex and Bev are allocated more state 1 claims than state 2 claims. Thus the aggregate risk is shared. Second, it follows that for all PE allocations,

$$MRS_{12}^h = \frac{\pi_1 v(h)(c_1^h)}{\pi_2 v(h)(c_2^h)} < \frac{\pi_1}{\pi_2}.$$ 

Hence the market equilibrium price of state claims $\frac{p_1}{p_2} = MRS_{12}^h = \frac{\pi_1 v(h)(c_1^h)}{\pi_2 v(h)(c_2^h)} < \frac{\pi_1}{\pi_2}$. Thus the market prices of state claims reflect the relative shortage of state 2 output.

Production

As long as there is a price for each state, the value to a consumer of a state contingent dividend stream from a firm is simply the market value of this stream. Thus a price-taking the shareholder’s interests are served if the firm chooses a production plan which maximizes the value of the dividend stream using these prices.

Example: Walrasian equilibrium with production

Alex owns a firm with a state contingent output of (140,80). Bev owns a firm with output $(80 - \frac{z^2}{20}, z)$. The two states are equally likely. Each individual has a VNM utility function

$$U^h(c_1^h, c_2^h) = \pi_1 \ln(c_1^h) + \pi_2 \ln(c_2^h).$$

In principle we could solve this problem as follows. First solve for Bev’s profit maximizing outputs given state claims prices $p = (p_1, p_2)$. This yields the aggregate supply vector $y(p)$. Then compute Bev’s profit and the value of Alex’s endowment at these prices. We can then
solve for the consumption choice of each consumer at these prices and hence solve for aggregate demand \( x(p) \). Finally choose a price ratio which equates supply and demand.

Instead there is a convenient short-cut. We note that Alex and Bev have the same homothetic preferences. Then we can treat the economy as a Robinson Crusoe economy with just one individual. Aggregate supply is \((220 - z^2 / 20, 80 + z), \ z \geq 0\). The expected utility of the aggregate individual is therefore

\[
U^R = \frac{1}{2} \ln(220 - \frac{z^2}{20}) + \frac{1}{2} \ln(80 + z).
\]

It is readily checked that this is maximized at \( z^* = 20 \) and hence that aggregate supply is \((200, 100)\). To solve for the equilibrium prices we seek prices such that \( z^* \) is indeed the profit maximizing plan. Iso-profit lines and the production possibility frontier of Bev’s firm are depicted below.

Along the frontier we have \((y_1(z), y_2(z)) = (80 - z^2 / 20, z)\). Thus the slope of the frontier at \( z^* \) is

\[
\frac{dy_2}{dy_1} = \frac{y_2'(z^*)}{y_1'(z^*)} = -\frac{z^*}{10} = -2
\]
Thus the slope of the iso-profit line and the frontier are equal at $z^*$ if the price ratio $p_1 / p_2 = 1/2$.

Suppose we set the price of state 1 claims equal to 1. Then $p_2 = 2$. The value of Alex’s firm is $P^A = (1, 2) \cdot (140, 80) = 300$ and the value of Bev’s firm is $P^B = p \cdot y(z) = 100$.

Thus far we have focused on trading in state claims markets. Suppose instead that the two individuals trade shares in the two firms. Given her initial shareholding of 100% of the shares in firm 2, Bev’s wealth is 100. If she does no trading, her final consumption is the output of the firm (60,20). Alternatively she could sell her firm and buy a fraction 300/100 = 1/3 of firm 1. This would give her one third of the total dividend stream from firm 1. These consumption bundles associated with these two non-diversified portfolios are depicted below.

A third alternative is to purchase one quarter of the shares of firm 1 at a cost of 75. This leaves Bev holding shares worth 25 in firm 2, that is, a 25% holding. Her final dividend is then one quarter of the total output in each state, that is (50,25). Thus by trading in the asset markets, Bev is able to achieve the same outcome as by trading in state claims markets.

This illustrates a general proposition. As long as there are as many linearly independent asset returns as states, any allocation achievable in a state claims market equilibrium, is also achievable by trading assets. To see this, let $z_{is}$ be the output of firm $i$ in state $s$. Given state claims equilibrium prices, $p = (p_1, \ldots, p_S)$, the equilibrium prices of the $S$
assets must be as follows.

\[(P^n_1, ..., P^n_s) = p'[z_{ts}] = p'Z\]

As long as the asset returns are independent we can invert and write

\[p' = P^n'Z^{-1}\]

Thus associated with the S asset prices is an implicit vector of state claims prices. Suppose that the consumption vector \(c\) is feasible for individual \(h\) at these implicit prices. That is

\[p'c \leq W^h.\]

Substituting,

\[P^n'Z^{-1}c^h \leq W^h.\]

Thus the individual can obtain the state dependent bundle \(c^h\) by purchasing an asset vector \(q = Z^{-1}c^h\).

To illustrate, the optimal consumption bundle in the example is \(c^* = (50, 25)\). The matrix of returns,

\[
\begin{bmatrix}
140 & 60 \\
80 & 20
\end{bmatrix}
\]
hence

\[
\begin{bmatrix}
\frac{1}{100} & -\frac{1}{100} \\
-\frac{1}{4} & \frac{4}{100}
\end{bmatrix}
\]

and so Bev chooses a shareholding of

\[q = Z^{-1}c = \begin{bmatrix}
\frac{1}{100} & -\frac{1}{100} \\
-\frac{1}{4} & \frac{4}{100}
\end{bmatrix}\begin{bmatrix}50 \\ 25\end{bmatrix} = \begin{bmatrix}\frac{1}{4} \\ \frac{1}{4}\end{bmatrix}.
\]

**Exercise 7.3-1: Uncertainty in an economy with CES preferences**

Each individual in an economy in which there are two states has expected utility function

\[u(c^h, \pi) = \pi_1(1 - \frac{1}{\sigma})c_1^{(1 - \frac{1}{\sigma})} + \pi_2(1 - \frac{1}{\sigma})c_2^{(1 - \frac{1}{\sigma})}, \quad \sigma > 0, \sigma \neq 1.\]
The aggregate endowment vector is \((\omega_1, \omega_2)\), where \(\omega_1 > \omega_2\).

(a) Show that these preferences are homothetic. Hence, or otherwise, solve for the equilibrium state claims prices.

(b) What is the degree of relative risk aversion in this economy?

(c) How does the equilibrium price ratio vary with the degree of risk aversion?

(d) What happens as the degree of risk aversion goes to infinity? Can you explain this result.

**Exercise 7.3-2: Asset Market Equilibrium**

Each individual in an economy in which there are two states has expected utility function

\[ u(c^h, \pi) = \pi_1 c_1^{\frac{1}{2}} + \pi_2 c_2^{\frac{1}{2}} \]

There are two assets in the economy. Asset a has a return \(\omega_a = (100, 100)\), while asset b has a return \(\omega_b = (100, 300)\). The two states are equally likely

(a) Solve for the equilibrium state claims prices. Hence solve for the ratio of asset prices

(b) Suppose that there are no state claims markets. Asset a has a price of $500. What will be the equilibrium price of asset b?

(c) In this economy, if someone with an income of 500 wanted to hold only claims to state 1, what would they do?

**Exercise 7.3-3: Variations on a logarithmic theme**

Suppose consumer \(h, h = 1, \ldots H\) has a logarithmic utility function

\[ u(x) = \sum_{i=1}^{n} \alpha_i \ln x_i. \]
(a) Show that if this is an endowment economy with aggregate endowment \((\omega_1, \ldots, \omega_n)\), the Walrasian equilibrium prices satisfy

\[
\frac{p_i}{p_1} = \frac{\alpha_i \omega_1}{\alpha_1 \omega_i}, \quad i = 2, \ldots, n
\]

(b) Consider a three period economy in which each individual has a utility function

\[
U(c) = \ln c_1 + \delta \ln c_2 + \delta^3 \ln c_3
\]

The aggregate endowment in the three periods is \((\omega_1, \omega_2, \omega_3) = (k, (1 + \theta)k, (1 + \theta)^2 k)\).

Solve for the Walrasian equilibrium prices. Hence show that the equilibrium interest rate between period 1 and period 2 is the same as the interest rate between period 2 and period 3.

(c) Consider a one period economy with three states. Each individual has a utility function

\[
u(x) = \sum_{s=1}^{3} \pi_s \ln c_s.
\] The aggregate supply of state claims is \((100, 200, 300)\). Solve for the state claims equilibrium prices.

**Exercise 7.3-4: Asset Market equilibrium**

Continue with the data of the previous question.

(a) Suppose that the "endowment" is really two assets. The riskless asset a has a yield of 100 in each state. Risky asset b yields nothing in state 1, 100 in state 2 and 200 in state 3. What will be the prices of these two assets?

(b) Explain why, in general, trading in asset markets alone cannot be a perfect substitute for trading in state claims markets.

(c) Suppose there are two individuals. Individual 1 initially owns asset a while individual 2 own asset b. If the probabilities of the three states are each 1/3, what is the state claims equilibrium allocation? If there were no state claims markets, and the asset market prices were the same as in part (d), could each individual trade to his allocation in the state claims equilibrium?

(d) Draw a conclusion about the efficiency of the asset market equilibrium in this case.
Exercise 7.3-5: Time, uncertainty and production

Preferences for each individual in a two period, two state model are as follows:

\[ U(c_1, c_{21}, c_{22}) = \ln c_1 + \delta (\pi_1 \ln c_{21} + \pi_2 \ln c_{22}) . \]

The aggregate endowment in period 1 is \( \omega_1 \). In period 2 the endowment is \( \omega_{21} \) if state 1 (rain) occurs, otherwise it is \( \omega_{22} \).

(a) Solve for the equilibrium spot and contingent futures prices.

Henceforth assume that \( \delta = 1 \).

(b) Suppose that there is also a linear production technology. Each unit of input of the commodity in period 1 yields 2 units in period 2 if it rains, otherwise half a unit. That is, the production vector is \((-x, 2x, 0.5x)\). For simplicity suppose that the first period endowment is 100, and the second period endowment is 0 in both states. If the price of the commodity in period 1 is 1, what can be said about \( p_{21} \) and \( p_{22} \)?

Hint: Use the representative individual and the fact that his expected utility is \( U(x) = \ln(100 - x) + \pi_1 \ln 2x + \pi_2 \ln(0.5x) \) and then solve for the optimal \( x \) and hence the optimal supply in each state. Equilibrium prices must equate supply and demand.

Exercise 7.3-6: Equilibrium gambling

UCLA fans think that the Bruins will beat the USC Trojans with probability 0.9. Trojan fans think that the Bruins only have an equal chance of winning. The preference scaling function of each fan is logarithmic (\( v(c) = \ln c \)). If the total wealth of UCLA and Trojan fans in the same, show that the equilibrium state claims price ratio will be 7/3. How much would a fan with a wealth \( W \) actually bet?
7.4 Asset Pricing

We have seen that if there are as many linearly independent asset returns as states, asset markets are a perfect substitute for markets in state claims. But what if there are fewer asset markets? In general, trading in asset markets will yield a Pareto inferior outcome. Moreover it is generally not possible to characterize the equilibrium asset prices.

Here we consider a special case in which the number of states is infinite and individuals trade a finite number of assets. By introducing the strong simplifying assumptions of normally distributed returns and constant absolute risk aversion, we are able to characterize prices in terms of the variance and covariance of asset returns.

Capital Asset Pricing Model

Suppose all assets are normally distributed. Individuals trade in assets so that consumption $\tilde{c}$ is normally distributed. Let $\mu$ and $\sigma^2$ be the mean and variance of consumption for individual $h$. Then $\tilde{c}$ has density function

$$f(\tilde{c}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(\frac{\tilde{c}-\mu}{\sigma})^2} \quad (7.4-1)$$

If individual $h$ exhibits constant absolute risk aversion, so that $-\frac{v''(c)}{v'(c)} = A^h$ his utility function (up to a linear transformation) is $v(c) = e^{-A^hc}$. His expected utility is therefore

$$u = E\{v(\tilde{c})\} = \int_{-\infty}^{\infty} e^{-A^hc} f(c)dc \quad (7.4-2)$$

Substituting from (7.4-1), expected utility is

$$u = E\{v(\tilde{c})\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(2A^h + \frac{\tilde{c} - \mu}{\sigma})^2} dc.$$  

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-A^h\mu} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(2A^h+\frac{\tilde{c} - \mu}{\sigma})^2} dc \quad (7.4-3)$$

Completing the square,
\[
\begin{align*}
u &= \frac{1}{\sigma \sqrt{2\pi}} e^{-A^b (\mu - A^b \sigma^2)} \int_{-\infty}^{\infty} e^{\frac{1}{2} \frac{1}{2} (A^b \sigma)^2 + 2A^b \sigma (\frac{-\mu}{\sigma}) + (\frac{-\mu}{\sigma})^2} dc \\
\text{Hence} \\
\begin{align*}
u &= e^{-A^b (\mu - A^b \sigma^2)} e^{\frac{1}{2} \frac{1}{2} (A^b \sigma)^2 + 2A^b \sigma (\frac{-\mu}{\sigma}) + (\frac{-\mu}{\sigma})^2} dc \\
\text{But the integrand is a normal density function. Hence} \\
\begin{align*}
u &= e^{-A^b (\mu - A^b \sigma^2)} \\
\text{Then} \\
U(\mu, \sigma) &= \mu - \frac{1}{2} A^b \sigma^2 \tag{7.4-4}
\end{align*}
\end{align*}
\]

is a representation of the individual’s preferences.

Suppose that the individual must choose between a riskless asset with a gross yield 1 + r and a risky asset which has a price of \( P_a \) and a normally distributed return \( \bar{z}_a \). Let \( \mu_a \) be the mean return and \( \sigma_a^2 \) be the variance of the return. Let \( q_a \) be the number of units purchased of the risky asset. The individual’s remaining wealth \( W^h - q_a P_a \) is invested in the riskless asset. Final consumption is then

\[
\tilde{c} = (W^h - q_a P_a)(1 + r) + q_a \bar{z}_a \tag{7.4-5}
\]

Expected final consumption is

\[
\mu = E\{\tilde{c}\} = (W^h - q_a P_a)(1 + r) + q_a \mu_a. \tag{7.4-6}
\]

Hence \( \tilde{c} - E\{\tilde{c}\} = q_a (\bar{z}_a - \mu_a) \) and so \( \text{var}(\tilde{c}) = (q_a \sigma_a)^2 \). Hence,

\[
\sigma(\tilde{c}) = q_a \sigma_a \tag{7.4-7}
\]

Thus as \( q_a \) increases, both the mean return and the standard deviation of the portfolio increase linearly. Substituting for \( q_a \) using (7.4-6) and (7.4-7), we then obtain the opportunity line,

\[
\mu = W^h (1 + r) + (\mu_a - (1 + r))q_a \frac{\sigma}{\sigma_a}
\]

This is depicted in Figure 7.4-1. The point \( N_0 \) is the individual’s final consumption if he only purchases the riskless asset. The point \( N_1 \) is his final consumption if the individual
chooses the other non-diversified portfolio and invests all his wealth in the single risky asset. All points on the line in between represent diversified portfolios.

![Figure 7.4-1: Choosing an optimal portfolio](image)

We now solve for the optimal portfolio $D$. From above we know that

$$E(\tilde{c}) = (W^h - q_a P_a)(1 + r) + q_a \mu_a \quad \text{and} \quad \sigma(\tilde{c}) = q_a \sigma_a$$

(7.4-8)

Substituting these expressions into (7.4-4), individual $h$ has utility

$$U = q_a (\mu_a - (1 + r) P_a) + W^h (1 + r) - \frac{1}{2} A^h (q_a \sigma_a)^2$$

(7.4-9)

Differentiating, individual $h$’s demand for the risky asset must satisfy the first order condition

$$\frac{\partial U}{\partial q_a} = (\mu_a - (1 + r) P_a) - A^h q_a \sigma_a^2 = 0.$$  

(7.4-10)

Rearranging, we have, at last individual $h$’s demand for the risky asset.
$q_a = \frac{1}{A_h} \frac{\left( \mu_a - (1 + r) P_i \right)}{\sigma_a^2}.$ \hspace{1cm} (7.4-11)

Note that demand is independent of the individual’s wealth. This follows from the assumption of constant absolute aversion to risk.

**The Representative Individual (Aggregation of preferences)**

Suppose next that there are $n$ risky assets all normally distributed. Asset $i$ has a return of $\bar{z}_i$ and a price $P_i$. There is also a riskless asset with gross yield $1+r$. The portfolio constraint of individual $h$ is then

$$q_0^h + \sum_{i=1}^n q_i^h P_i = W^h,$$

where $q_0$ is the number of units purchased of the riskless asset. The individual’s consumption is then

$$\tilde{c} = q_0^h (1 + r) + \sum_{i=1}^n q_i^h \bar{z}_i$$

Substituting for $q_0$ this can be rewritten as

$$\tilde{c} = \sum_{i=1}^n q_i^h (\bar{z}_i - (1 + r) P_i) + W^h (1 + r)$$

The mean return and variance are then as follows.

$$\mu(\tilde{c}) = \sum_{i=1}^n q_i^h (\mu_i - (1 + r) P_i) + W^h (1 + r) \quad \text{var}(\tilde{c}) = \sum_{i=1}^n \sum_{j=1}^n q_i^h \sigma_{ij} q_j^h,$$

where $\sigma_{ij}$ is the covariance between assets $i$ and $j$.

Substituting into (7.4-4), utility is

$$U = \sum_{i=1}^n q_i^h (\mu_i - (1 + r) P_i) + W^h (1 + r) - \frac{1}{2} A_h \left( \sum_{i=1}^n \sum_{j=1}^n q_i^h \sigma_{ij} q_j^h \right) \hspace{1cm} (7.4-12)$$

Differentiating by $q_i$, the first order condition for utility maximization is
\[ \mu_i - (1+r)P_i = A^h \left( \sum_{j=1}^{n} \sigma_{ij} q_j^h \right), \ i = 1,...,n, \ i = 1,...,n \]  

(7.4-13)

Expressing this in matrix form we obtain

\[
\begin{bmatrix}
\sigma_{ij} \\
\vdots \\
\sigma_{nj}
\end{bmatrix} \begin{bmatrix}
q_i^h \\
\vdots \\
q_n^h
\end{bmatrix} = \frac{1}{A^h} \begin{bmatrix}
\mu_1 - (1+r)P_1 \\
\vdots \\
\mu_n - (1+r)P_n
\end{bmatrix}
\]

Summing over all \( H \) consumers, the market demand for the \( n \) assets must satisfy

\[
\begin{bmatrix}
\sigma_{ij} \\
\vdots \\
\sigma_{nj}
\end{bmatrix} \begin{bmatrix}
q_i \\
\vdots \\
q_n
\end{bmatrix} = \frac{1}{A} \begin{bmatrix}
\mu_1 - (1+r)P_1 \\
\vdots \\
\mu_n - (1+r)P_n
\end{bmatrix}
\]

where \( \frac{1}{A} = \sum_{h=1}^{H} \frac{1}{A^h} \)  \( (7.4-14) \)

Note that these \( n \) conditions are exactly the first order conditions for a single individual whose degree of absolute aversion to risk is \( A \). It follows that market demand can be represented by a single individual with utility function

\[ U^M = \mu - \frac{1}{2} A \sigma^2 \]

Pricing the market Portfolio

Since market demand can be represented by a single representative individual with constant absolute risk aversion, it is an easy matter to solve for the equilibrium price of the market portfolio. The reason is that the representative individual must end up holding the market portfolio. Thus if units are defined so that the supply of each asset is 1 unit, the supply of the market portfolio is 1 unit. The representative agent must, in equilibrium, hold the riskless asset and a mutual fund consisting of all the risky assets. We can then appeal to the analysis of the single individual choosing between a riskless asset and a single risky asset (the mutual fund.) Appealing to (7.4-11), demand for the risky market portfolio is

\[ q_M = \frac{1}{A} \left( \frac{\mu_M - (1+r)P_M}{\sigma_M^2} \right) \]

In equilibrium, the market must clear and so \( q_M = 1 \). Therefore

\[ \mu_M - (1+r)P_M = A_M \sigma_M^2 \]  \( (7.4-15) \)
We have therefore succeeded in pricing the market portfolio.

**Pricing each Asset (CAPM)**

In equilibrium, the representative individual’s optimal portfolio decision (acting as a price taker) is to hold the market portfolio. Thus consider his decision when he must choose between holding units of the market portfolio and units of asset a. Let \( q_a \) and \( q_M \) be his purchases of the two assets. From (7.4-13),

\[
\mu_a - (1 + r)P_a = A(\sigma_{aq} q_a + \sigma_{aqM} q_M) \quad \text{and} \quad \mu_M - (1 + r)P_M = A(\sigma_{aM} q_a + \sigma_{aMM} q_M)
\]

In equilibrium the individual purchases only the market portfolio. Thus \((q_a, q_M) = (0, 1)\). Hence

\[
\mu_a - (1 + r)P_a = A\sigma_{aM} \quad \text{and} \quad \mu_M - (1 + r)P_M = A\sigma_{aMM}
\]

Since \( A \) is not directly observable we combine the above expressions to obtain the following result.

**Capital Asset Pricing Rule**

\[
\mu_a - (1 + r)P_a = \frac{\sigma_{aM}}{\sigma_{M}^2} (\mu_M - (1 + r)P_M) \quad (7.4-16)
\]

If we convert the above expression to yields, we obtain

\[
E\{\tilde{r}_i\} - r = \frac{\text{cov}(\tilde{r}_i, \tilde{r}_M)}{\text{var}(\tilde{r}_M)} (E\{\tilde{r}_M\} - r).
\]

Investment houses run regressions of each stock’s yield on the market yield and report the “beta” of the stock. Suppose two listed firms have the same expected yield but the return of the first firm has a higher beta (is more highly correlated with the market portfolio.) Then the equilibrium expected yield on the first firm must be higher since it offers less of an opportunity to spread risk.
Exercise 7.4-1: Buying on Margin
Suppose the individual borrows funds from his broker and purchases even more of the risky asset. Explain carefully why this allows the individual to move further up the portfolio line in Figure 7.4-1.

Exercise 7.4-2: Valuing independent assets
There are \( n \) risky assets in an economy. They are normally distributed, independent and each has the same mean and variance.

(a) Suppose all \( H \) individuals have the same degree of absolute risk aversion \( A \).  
Hint: Start with \( n = 1 \), then \( n = 2, \ldots \).

(b) Holding the number of firms constant, what happens as the number of individuals becomes large.

(c) What if the number of firms and individuals increase at the same rate?

Exercise 7.4-3: Mutual Fund Theorem

(a) Appealing to (7.4-13), show that each individual will wish to purchase risky assets in the same proportions as the representative individual.

(b) Hence explain why each individual optimizes by purchasing only the riskless asset and shares in a single mutual fund which holds the market portfolio.
7.5 Changes in Risk

Suppose that an individual’s uncertain consumption depends on some choice of the consumer, \(x\) and some underlying characteristic of the economy, that is,
\[
\bar{c} = c(x, \bar{\theta})
\]
It is useful to be able to characterize parametric shifts in the distribution of consumption that are favorable (or unfavorable) to the consumer. Given such a characterization we can then ask how a consumer will respond to a parametric shift. Initially we fix the consumer’s choice and consider two random variables,
\[
\bar{c}_1 = c(x, \bar{\theta}_1) \text{ and } \bar{c}_2 = c(x, \bar{\theta}_2).
\]
Let the associated distribution functions for consumption be \(F_1(\cdot)\) and \(F_2(\cdot)\). For expositional ease we assume that each random variable has support \([\alpha, \beta]\) and that the distribution functions are twice differentiable.

Suppose that consumption is observed to be in the interval \([c, c + \Delta c]\). For the first distribution, the probability of this event is \(F_1'(c)\Delta c\) if \(\gamma = \gamma_1\) and for the second distribution the probability is \(F_2'(c)\Delta c\). Thus the “odds” that the underlying distribution is the second rather than the first are
\[
\frac{F_2'(c)\Delta c}{F_1'(c)\Delta c} = \frac{F_2'(c)}{F_1'(c)}.
\]

If this “likelihood ratio” is increasing the density function of \(\bar{c}_2\) is weighted more towards higher values.

**Definition: Monotone Likelihood Ratio Property**
The random variable \(\bar{c}_2\) dominates \(\bar{c}_1\) according to the MLR property if the likelihood ratio, \(F_2'(\cdot)/F_1'(\cdot)\) is increasing.

If the MLR property holds then for any \(x, y\) and \(c\) such that \(x \leq c \leq y\),
\[
\frac{F_2'(x)}{F_1'(x)} \leq \frac{F_2'(c)}{F_1'(c)} \leq \frac{F_2'(y)}{F_1'(y)}. \quad (7.5-1)
\]

From the first inequality,
\[
F_2'(x) \leq \frac{F_2'(c)}{F_1'(c)} F_1'(x) \text{ for all } x \leq c.
\]

Integrating with respect to \( x \) over the interval \([\alpha, c]\) and rearranging,
\[
\frac{F_2(c)}{F_1(c)} \leq \frac{F_2'(c)}{F_1'(c)}.
\]

Making a symmetric argument using the second inequality it follows that
\[
\frac{F_2'(c)}{F_1'(c)} \leq \frac{1 - F_2(c)}{1 - F_1(c)}.
\]

Combining these two results it follows that \( F_2(c) \leq F_1(c) \). This weaker condition places greater weight in every left tail.

**Definition: (First order) Stochastic Dominance**
\( \tilde{c}_2 \) exhibits stochastic dominance over \( \tilde{c}_1 \) if, for all \( c \) in \([\alpha, \beta]\), \( \Pr\{\tilde{c}_2 \leq c\} \leq \Pr\{\tilde{c}_1 \leq c\} \).

It is intuitively clear that the first distribution with its larger left tail must have a lower mean.

This follows almost immediately since, integrating by parts,
\[
\mu_i = E(\tilde{c}_i) = \int_{\alpha}^{\beta} cF_i'(c)dc = \beta - \int_{\alpha}^{\beta} F_i(c)dc \quad (7.5-2)
\]

Then if the second distribution stochastically dominates, \( \mu_2 \geq \mu_1 \).

Next define the functions
\[
T_i(c) = \int_{a}^{c} F_i(x)dx, \quad i = 1, 2 \quad (7.5-3)
\]

**Definition: Second order Stochastic Dominance**
\( \tilde{c}_2 \) exhibits second order stochastic dominance over \( \tilde{c}_1 \) if, for all \( c \) in \([\alpha, \beta]\),
\[
T_i(c) \geq T_2(c)
\]
From (7.5-2)
\[ \mu_i = \beta - T_i(\beta) \]
Then with second order stochastic dominance \( \mu_2 \geq \mu_1 \).
In the limiting case of second order stochastic dominance the means of the two distributions are the same.

**Definition: Mean Preserving spread**

\( \tilde{c}_2 \) is a mean preserving spread of \( \tilde{c}_1 \) if the two distributions have the same mean and the latter exhibits second order stochastic dominance over the former.

From our previous definitions,
\[
T_i(c) - T_2(c) = \int_{\alpha}^{c} (F_i(x) - F_2(x))dx \geq 0 \quad \text{and} \quad T_i(\beta) - T_2(\beta) = 0.
\]
If, as depicted below, the two distribution functions cross only once, then the two shaded regions must be equal. Moreover for low values of \( c \) \( F_1(c) > F_2(c) \) and for high values the inequality is reversed. Thus the two distributions have the same mean and the first distribution has more weight in each tail. Hence the first distribution is a “mean preserving spread” of the second.

![Fig. 7.5-1: Mean Preserving Spread](image-url)
Choice over risky prospects

Let the consumer have utility function $v(c)$. Then expected utility

$$U_i = \int_a^\beta v(c) dF_i(c)$$

$$= v(\beta) - \int_a^\beta v'(c) F_i(c) dc$$

$$= v(\beta) - v'(\beta) T_i(\beta) + \int_a^\beta v''(c) T_i(c) dc.$$  \hspace{1cm} (7.5-4)

(7.5-5)

The following proposition is readily derived from (7.5-4) and (7.5-5).

**Proposition: Ranking Distributions**

Let $\tilde{c}_1$ and $\tilde{c}_2$ be two consumption prospects. Expected utility is higher for the second distribution if

(i) $\tilde{c}_2$ exhibits (first order) stochastic dominance over $\tilde{c}_1$ and $v(\cdot)$ is an increasing function.

(ii) $\tilde{c}_2$ exhibits second order stochastic dominance over $\tilde{c}_1$ and $v(\cdot)$ is an increasing, concave function.

(iii) $\tilde{c}_1$ is a mean preserving spread of $\tilde{c}_2$ and $v(\cdot)$ is a concave function.

Increasing risk and its effect on optimal actions

Suppose that an agent faces a risk represented by the random variable $\Theta$ and must make some decision $x$. For example, in section 7.2, an investor was deciding how much to invest in a risky asset. As we saw there, if the investor spent $x$ on the risky asset, his expected utility was

$$U(x) = E[v(w + x\Theta)] = \int_a^\beta v(w + x\Theta) dF(\Theta)$$

More generally, we can write expected utility as follows.

$$U(x) = E[v(c(x, \tilde{\Theta}))] = \int_a^\beta u(x, \theta) dF(\Theta).$$
We now consider how his optimal choice is affected as the distribution of the random variable changes. Initially we consider the case of a mean preserving increase in risk. Let the initial c.d.f of $\theta$ be $F(x)$ and the final c.d.f. be $F(x)$.

Consider the expected marginal gain from increasing $x$.

$$U'_i(x) = \int_a^\beta \frac{\partial u}{\partial x} dF_i(\theta), \quad i = 1, 2.$$ 

If $\frac{\partial u}{\partial x}$ is a concave function of $\theta$, and the final distribution is a mean preserving spread of the initial distribution, then by the Ranking Proposition

$$U'_1(x) \leq U'_2(x).$$

We thus have the following result.

**Proposition: Responding to a mean preserving increase in risk**

If $\tilde{\theta}_1$ is a mean preserving spread of $\tilde{\theta}_2$ and $\frac{\partial u}{\partial x}$ is a concave function of $\theta$, $x_1^* \leq x_2^*$. In general, we should expect this to be a very strong condition since mean preserving spreads are a combination of income and substitution effects.

Suppose that the new distribution is such that, holding the action constant, expected utility is unchanged (no income effect.) Then the effect on the optimal action is the pure substitution effect.

**Proposition: Substitution effect**

Define $\tilde{v}_1 = u(x, \tilde{\theta}_1)$ and $\tilde{v}_2 = u(x, \tilde{\theta}_2)$. Suppose that these two distributions have the same mean and that $u(x, \theta)$ increases with $\theta$. If $\tilde{v}_2$ exhibits second order stochastic dominance over $\tilde{v}_1$ and

$$\frac{\partial^2}{\partial \theta \partial x} \ln \frac{\partial u(x, \theta)}{\partial \theta} \leq 0,$$

then $x_1^* \leq x_2^*$.

To prove this proposition, note that expected utility is
\[ U_i(x) = \int_a^\beta u(x, \theta)dF_i(\theta) \]  

(7.5-6)

Since \( u(x, \theta) \) is increasing in \( \theta \), we can invert and define \( \theta = h(x, v) \). Then (7.5-6) can be rewritten as

\[ U_i(x) = \int_a^\beta u(x, h(x, v))dG_i(v) \]

where \( G_i(v) \) is the c.d.f. of \( v \).

Suppose that the two distributions of utility have the same mean \( (U_1(x) = U_2(x)) \) and the second exhibits second order stochastic dominance over the first.

Differentiating (7.5-6)

\[ U_i'(x) = \int_a^\beta \frac{\partial u}{\partial x}dF_i(\theta). \]

Substituting for \( \theta = h(x, v) \)

\[ U_i'(x) = \int_a^\beta \frac{\partial u}{\partial x}(x, h(x, v))dG_i(v), \]  

(7.5-7)

To apply part (iii) of the Ranking Proposition, we need to show that

\[ \phi(v) = \frac{\partial u}{\partial x}(x, h(x, v)) \]  

(7.5-8)

is a concave function of \( v \).

Again inverting we have

\[ \phi(u(x, \theta)) = \frac{\partial u}{\partial x}(x, \theta) \]

Differentiating this expression by \( \theta \),

\[ \phi'(u) \frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta} \frac{\partial u}{\partial x}(x, \theta). \]

Hence
\[
\phi'(u) = \frac{\partial}{\partial \theta} \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \frac{\partial u}{\partial \theta} = \frac{\partial}{\partial x} \ln \frac{\partial u}{\partial \theta}.
\]

Differentiating again yields the result.

**Exercise 7.5-1:**
For the 2 asset problem, suppose that absolute risk aversion is decreasing and relative risk aversion is non-decreasing. Show that a mean utility preserving increase in risk will lead to less spending on the risky asset.

**Exercise 7.5-2:**
If the first distribution is a mean preserving spread of the second, integrate by parts to show that \( \sigma_1^2 \geq \sigma_2^2 \).

**Exercise 7.5-3:**
Derive the Ranking Proposition

**Exercise 7.5-4 The MLR Property and Conditional Stochastic Dominance**
The random variable \( \tilde{c}_2 \) dominates \( \tilde{c}_1 \) according to the Conditional Stochastic Dominance Property if for all \( s, t \), \( s < t \),

\[
\frac{F_2(s)}{F_2(t)} \leq \frac{F_1(s)}{F_1(t)},
\]

If the MLR property holds, then for any \( x < c \) and any \( y \in [c, \hat{c}] \)

\[
\frac{F_2'(x)}{F_1'(x)} \leq \frac{F_2'(c)}{F_1'(c)} \leq \frac{F_2'(y)}{F_1'(y)}
\]

(a) Use the first inequality to show that \( \frac{F_2(c)}{F_1(c)} \leq \frac{F_2'(c)}{F_1'(c)} \).

(b) Use the second inequality to show that \( \frac{F_2'(c)}{F_1'(c)} \leq \frac{F_2(\hat{c}) - F_2(c)}{F_1(\hat{c}) - F_1(c)} \).

(c) Hence establish that the MLR property implies Conditional Stochastic Dominance.
CHAPTER 7 ANSWERS

Section 7.1

Exercise 7.1-1:

Suppose \( \pi^m \succeq \hat{\pi}^m \), \( m = 1, ..., M \)

(a) For \( M = 2 \), it is easy to show that IA implies IA'. From IA, for any probability vector \((p_1, p_2)\),

\[
\pi^1 \succeq \hat{\pi}^1 \Rightarrow (p_1, p_2 : \pi^1, \pi^2) \succeq (p_1, p_2 : \hat{\pi}^1, \hat{\pi}^2)
\]

and

\[
\pi^2 \succeq \hat{\pi}^2 \Rightarrow (p_1, p_2 : \pi^1, \pi^2) \succeq (p_1, p_2 : \hat{\pi}^1, \hat{\pi}^2)
\]

Therefore \( \pi^1 \succeq \hat{\pi}^1 \) and \( \pi^2 \succeq \hat{\pi}^2 \Rightarrow (p_1, p_2 : \pi^1, \pi^2) \succeq (p_1, p_2 : \hat{\pi}^1, \hat{\pi}^2) \)

(b) Define \( \pi^* = \left( \frac{p_1}{1-p_1}, ..., \frac{p_k}{1-p_k} : \pi^1, ..., \pi^{k-1} \right) \) and \( \hat{\pi}^* = \left( \frac{p_1}{1-p_1}, ..., \frac{p_k}{1-p_k} : \hat{\pi}^1, ..., \hat{\pi}^{k-1} \right) \).

By hypothesis the equivalence holds for \( M = k - 1 \). Thus \( \pi^* \succeq \hat{\pi}^* \).

Next note that

\[
(p_1, ..., p_k : \pi^1, ..., \pi^k) = (1 - p_k, p_k : \pi^*, \pi^k)
\]

and

\[
(p_1, ..., p_k : \hat{\pi}^1, ..., \hat{\pi}^k) = (1 - p_k, p_k : \hat{\pi}^*, \hat{\pi}^k)
\]

Since IA' holds for \( M = 2 \), it follows that

\[
(1 - p_k, p_k : \pi^*, \pi^k) \succeq (1 - p_k, p_k : \hat{\pi}^*, \hat{\pi}^k)
\]

Section 7.2

Exercise 7.2-7: Wealth Effects

From section 7.2, if an individual invests \( x \) in the risky asset his marginal expected utility is
\[ U'(x) = \sum_{s=1}^{S} \pi_s \theta_s v'(w + x \theta_s), \] where \( \theta_s = r_{2,s} - r_1 \) and \( w = (1 + r_1)W \)

We then ask what is the effect of an increase in wealth on the marginal utility of investing in the risky asset.

\[
\frac{d}{dw} U'(x) = \sum_{s=1}^{S} \pi_s \theta_s v'(w + x \theta_s) \tag{7.A-1}
\]

Let \( x^* \) be the optimal investment. If \( \frac{d}{dw} U'(x^*) > 0 \), increasing wealth raises the marginal utility of investing in the risky asset. Then, at the higher wealth level, the individual invests more in the risky asset.

(a) If absolute risk aversion is constant so that \( v''(c_j) = -A v'(c_j) \). Substitute this into the above equation,

\[
\frac{d}{dw} U'(x) = -A \sum_{s=1}^{S} \pi_s \theta_s v'(w + x \theta_s)
\]

At the optimum it follows from the first order condition that the right hand side is zero. Thus

\[
\frac{d}{dw} U'(x^*) = 0
\]

and so a change in wealth has no effect on the optimal investment.

(b) The trick to analyzing (7.A-1) is to make use of our assumption that absolute risk aversion is diminishing. We index states so that the yield on the risky asset is lower in higher states. We also define state \( t \) to be the highest state for which \( \theta_s > 0 \).

Substituting \( A(c) = -\frac{v''(c)}{v'(c)} \) into (7.A-1),

\[
\frac{d}{dw} U'(x) = \sum_{s=1}^{S} \pi_s \theta_s v'(w + x \theta_s) = \sum_{s=1}^{t} \pi_s (-\theta_s)A(w + x \theta_s)v'(w + x \theta_s) - \sum_{s=t+1}^{S} \pi_s \theta_s A(w + x \theta_s)v'(w + x \theta_s)
\]

If \( \theta_s > 0 \), \( A(w + x \theta_s) < A(w) \). Hence

\[
\sum_{s=1}^{S} \pi_s \theta_s A(w + x \theta_s)v'(w + x \theta_s) < \sum_{s=1}^{S} \pi_s \theta_s A(w)v'(w + x \theta_s)
\]
Also, if \( \theta_j < 0 \), \( A(w + x\theta_j) > A(w) \). Hence

\[
\sum_{i=1}^{t-1} \pi_j(-\theta_j)A(w + x\theta_j)v'(w + x\theta_j) > \sum_{i=1}^{t-1} \pi_j(-\theta_j)A(w)v'(w + x\theta_j)
\]

Combining these results, it follows that

\[
\frac{d}{dw} U'(x) > \sum_{i=1}^{s} \sum_{i=1}^{t-1} -\pi_j(-\theta_j)A(w)v'(w + x\theta_j)
\]

Since \( A(w) \) is independent of the state, we can take it outside the summation. Then

\[
\frac{d}{dw} U'(x) > A(w)U'(x)
\]

In particular, at \( x^* \),

\[
\frac{d}{dw} U'(x^*) > A(w)U'(x^*) = 0.
\]

Thus investment in the risky asset rises with wealth.