Leverage and Disagreement

François Geerolf

UCLA

September 15, 2015
In this paper, I develop a model of:

- **Endogenous Leverage**
- Interest Rates on Collateralized Bonds

among competitive investors with **heterogenous beliefs**.

Geanakoplos (1997) and subsequent:

- Only **one** leverage ratio (simplifying assumption on the structure of beliefs / or on the number of agents).
- Counterfactual. **Many** leverage ratios, even for same asset: homebuyers, entrepreneurs, hedge funds, investment banks...

Relaxing the hypotheses leading to one leverage ratio, the model yields two key predictions.
1) When disagreement goes to 0, the **upper tail** of the distribution of leverage ratios goes to a **Pareto** with **endogenous tail coefficient** \[2\], for any smooth and bounded away from zero density of beliefs.

- Cross section of Hedge Funds (TASS Lipper, 2006)

- **Pareto in the upper tail** \([l \in [150, 3000]]\)  
- Point estimate for **tail coefficient**: \(\alpha = 1.95\) (std: 0.2).
Cross-section of homowners’ initial leverage ratios (Dataquick, for example October 1989).

Pareto of leverage ratios found also for:
- Entrepreneurs in the SCF.
- Firms in Compustat.

⇒ Pareto for borrowers’ expected / realized returns, however small belief heterogeneity:
- Pareto Returns to entrepreneurship.
- Pareto Returns to speculation in general.
2) Distribution of interest rates adjusts so that borrowers and lenders are matched assortatively: **interest rates are assignment / hedonic prices**, disconnected from expected and true default probability:

- New determinant for pricing fixed income securities. (⇒ Credit Spread Puzzle? / CDS-Bond Basis)
- Investing in high yield not necessarily risk shifting.
- High customization / fragmentation of the market = Endogenous OTC structure. ⇒ OTC versus exchanges debate.
Model Ingredients:
- Disagreement on mean rather than on default probabilities.

Key Results:
- **Pareto** distributions for leverage ratios / expected and realized returns. Also gives information on:
  - Representativeness of marginal buyer/ Elements of the belief distribution. (⇒ monitoring systemic risk?)
  - Underlying financial structure.
- Credit spreads as **hedonic interest rates**.

Other Theoretical / Methodological contributions:
- Pyramiding Lending Arrangements.
- Endogenous Short-sales:
  - Endogenous rebate rates, without transactions costs / risk aversion.
  - Endogenous short interest.
Literature

- **Heterogeneous Priors.** Miller (1977), Harrison, Kreps (1978), Ofek, Richardson (2003), Hong, Scheinkman, Xiong (2006), Hong, Stein (2007), Hong, Sraer (2012).


- **Credit Spread Puzzle.** Chen, Colling-Dufresne, Goldstein (2009), Buraschi, Trojani, Vedolin (2011), Huang and Huang (2012), Albagli, Hellwig, Tsyvinski (2012), McQuade (2013).

Model with Borrowing Contracts Only

Setup
Equilibrium Definition
Equilibrium Solution
Equilibrium Properties

Extension 1: ”Pyramiding” Lending Arrangements

Extension 2: Short-Sales

Conclusion
Model with Borrowing Contracts Only

Setup
- Equilibrium Definition
- Equilibrium Solution
- Equilibrium Properties

Extension 1: "Pyramiding" Lending Arrangements

Extension 2: Short-Sales

Conclusion
Set-up

- Two Periods: 0 and 1.
Set-up

- Two Periods: 0 and 1.
- Continuum of agents. Measure 1.
Set-up

- Two Periods: 0 and 1.
- Continuum of agents. Measure 1.
- Wealth 1.
Set-up

- Two Periods: 0 and 1.
- Continuum of agents. Measure 1.
- Wealth 1.
- Consume in period 1.
Set-up

▶ Two Periods: 0 and 1.
▶ Continuum of agents. Measure 1.
▶ Wealth 1.
▶ Consume in period 1.

\[ w=1 \]

\[
\begin{array}{c}
0 \\
1 \\
w=1
\end{array}
\]
Storage’s Return $R = 1. \rightarrow \text{Cash.}$


**Borrowing Contracts** collateralized by the Real Asset.

- No-recourse.
- Normalization: 1 unit of Real Asset in Collateral.
- $\phi$: **Face Value** - promised payment in period 1.
- Notation for contract: $(\phi)$.
- Competitive Markets (Anonymous). Price: $q(\phi)$. ”Loan amount”. Implicit interest rate: $r(\phi) = \phi / q(\phi)$.
- Payoff: $\min\{\phi, p_1\}$. 
Beliefs

- Agents agree to disagree on $p_1$.
- Agent $i$: point expectations $p^i_1 \in [1 - \Delta, 1]$.

- Key difference with Geanakoplos (1997), where agents agree on value upon default.
- Generalization:
  - Agents agree on a probability distribution around mean.
  - Risk neutral.
- Density $f(.)$, c.d.f $F(.)$ on $[1 - \Delta, 1]$.
- Exogenously given.
- No learning.
Agents’ Problem

Given \((p, q(.))\), agent \(i\) chooses \((n^i_A, n^i_B(.), n^i_C)\) to max. expected wealth \((W)\) in period 1 under:

- **Budget Constraint (BC).**
- **Collateral Constraint (CC).**

\[
\begin{align*}
\max_{(n^i_A, n^i_B(.), n^i_C)} & \quad n^i_A p^i_1 + \int n^i_B(\phi) \min\{\phi, p^i_1\} d\phi + n^i_C \\
\text{s.t.} & \quad n^i_A p + \int n^i_B(\phi) q(\phi) d\phi + n^i_C \leq 1 \\
\text{s.t.} & \quad \int n^i_B(\phi) \max\{-n^i_B(\phi), 0\} d\phi \leq n^i_A \\
\text{s.t.} & \quad n^i_A \geq 0, \quad n^i_C \geq 0
\end{align*}
\]
Model with Borrowing Contracts Only

Setup
Equilibrium Definition
Equilibrium Solution
Equilibrium Properties

Extension 1: "Pyramiding" Lending Arrangements

Extension 2: Short-Sales

Conclusion
Equilibrium

Definition (Competitive Equilibrium for Economy $\mathcal{E}^B$)

A competitive equilibrium is a price system $(p, q(.))$, and portfolios $(n^i_A, n^i_B(.), n^i_C)$ for all $i$ such that:

- Given $(p, q(.))$, agent $i$ chooses $(n^i_A, n^i_B(\phi), n^i_C)$ maximizing $(W)$ under (BC) and (CC),

- Markets clear:

\[
\int_i n^i_A di = 1, \\
\text{and } \forall \phi, \int_i n^i_B(\phi) di = 0.
\]
Model with Borrowing Contracts Only

Setup
Equilibrium Definition
Equilibrium Solution
Equilibrium Properties

Extension 1: ”Pyramiding” Lending Arrangements

Extension 2: Short-Sales

Conclusion
Agents’ Types

Agents split into three types depending on optimism:

Cash Investors

Lenders

Borrowers

\[ p_i \in [1-\Delta, \xi) \rightarrow \text{Cash Investors.} \]

\[ p_i \in (\xi, p \tau) \rightarrow \text{Lenders.} \]

\[ p_i \in (p \tau, 1) \rightarrow \text{Borrowers.} \]
Agents’ Types
Agents split into three types depending on optimism:

\[ p_1^i \in [\tau, 1] \rightarrow \text{Borrowers ("Homeowners", "Hedge Funds", "Entrepreneurs")}. \]

\[ n_A^i > 0 \quad \exists \phi, \quad n_B^i(\phi) < 0. \]
Agents’ Types

Agents split into three types depending on optimism:

\[ p^i_1 \in [\tau, 1] \rightarrow \text{Borrowers ("Homeowners", "Hedge Funds", "Entrepreneurs")}. \]

\[ n^i_A > 0 \quad \exists \phi, \quad n^i_B(\phi) < 0. \]

\[ p^i_1 \in [\xi, \tau] \rightarrow \text{Lenders ("Banks", "Money-Market Fund").} \]

\[ \exists \phi, \quad n^i_B(\phi) > 0. \]
Agents’ Types

Agents split into three types depending on optimism:

\[ 1 - \Delta \quad \xi \quad p \quad \tau \quad 1 \]

- **Cash Investors**
  - \( p^i_1 \in [1 - \Delta, \xi] \rightarrow \) Cash Investors.
  - \( n^i_C = 1 \).

- **Lenders**
  - \( p^i_1 \in \xi, \tau \rightarrow \) Lenders.
  - \( \exists \phi, \ n^i_B(\phi) > 0 \).

- **Borrowers**
  - \( p^i_1 \in [\tau, 1] \rightarrow \) Borrowers.
  - \( n^i_A > 0 \quad \exists \phi, \ n^i_B(\phi) < 0 \).

- \( n^i_i A > 0 \quad \exists \phi, \ n^i_B(\phi) < 0 \).
Borrowers’ Problem

Lemma

A borrower $p^i_1$ chooses $(\phi)$ s.t.: $\phi = \arg \max_{\phi} \frac{p^i_1 - \phi}{p - q(\phi)}$.

- Coll. Const. binds: 1 Real asset $\Rightarrow$ 1 Borrowing Contract.
- Number: $1/(p - q(\phi))$ of Real assets / Borrowing Contracts.

- Leverage ratio of $(\phi)$: $l(\phi) = p/(p - q(\phi))$.

\[
\frac{1}{p - q(\phi)}(p^i_1 - \phi) = \frac{p^i_1}{p} l(\phi) - \frac{\phi}{q(\phi)} (l(\phi) - 1)
\]

\[
= \frac{p^i_1}{p} + \left( \frac{p^i_1}{p} - r(\phi) \right) (l(\phi) - 1).
\]

- Promise $\phi \uparrow \Rightarrow q(\phi) \uparrow \Rightarrow q'(\phi) > 0 \Rightarrow l'(\phi) > 0$

$\Rightarrow$ Leverage rises with face value $\phi$.

- Trade-off between higher $\phi$ but higher $r(\phi) \Rightarrow r'(\phi) > 0$. 

13/39
Lenders

Lemma

*A lender with beliefs $p^i_1$ chooses contract $(p^i_1)$.*

- For lenders: Face value of the loan $= \text{Beliefs about the Real Asset}$.
  - Why not a higher $\phi$? Default for sure.
    \[
    \text{Return: } \frac{\min\{p^i_1, \phi\}}{q(\phi)} = \frac{p^i_1}{q(\phi)} \searrow \phi.
    \]
  - Why not a lower $\phi$?
    \[
    \text{Return: } \frac{\min\{p^i_1, \phi\}}{q(\phi)} = \frac{\phi}{q(\phi)} = r(\phi) \nearrow \phi.
    \]
- Leverage rises with $\phi$, and $\phi = p^i_1$ of lenders $\Rightarrow$ Leverage rises with beliefs of lenders.
- Lenders think they trade **perfectly safe contracts**.
Positive Sorting

- Supermodularity of Expected Wealth of a Borrower with respect to his Beliefs $p^i_1$ and the face value $\phi$:

$$\frac{p^i_1 - \phi}{p - q(\phi)} = \frac{p^i_1}{p} (1 + l(\phi)) - \frac{\phi}{q(\phi)} l(\phi)$$

$$\Rightarrow \frac{\partial^2}{\partial \phi \partial p^i_1} (.) = \frac{1}{p} l'(\phi) > 0.$$ 

- Complementarity between leverage ($\phi$) and expected return on each asset ($p^i_1$).

- $\phi = p^i_1$ of lenders $\Rightarrow$ Positive Sorting of borrowers and lenders. Empirically: Over-The-Counter (OTC) Markets.

- $\Gamma(.)$: Belief of borrower $\rightarrow$ Belief of lender. Sorting: $\Gamma'(.) > 0$. 

2 first-order ODE for $\Gamma(.)$ and $q(.)$

- $p^i_1 = y$ chooses $\phi$ s.t. lender choosing same $\phi$ is $\Gamma(y)$:

\[
\Gamma(y) = \arg \max_{\phi} \frac{y - \phi}{p - q(\phi)} \Rightarrow q'(\phi) \frac{y - \phi}{p - q(\phi)} = 1
\]

\[
\Rightarrow (y - \Gamma(y)) q'(\Gamma(y)) = p - q(\Gamma(y)).
\]

- Market clearing for contract ($x$):

\[
\int_i n^i_B(x) di = 0 \Rightarrow \frac{f(\Gamma(y))d\Gamma(y)}{q(\Gamma(y))} = \frac{f(y)dy}{p - q(\Gamma(y))}
\]

\[
\Rightarrow (p - q(\Gamma(y))) f(\Gamma(y)) \Gamma'(y) = q(\Gamma(y)) f(y).
\]
- **Unknowns:** $q(.) \equiv r(.)$, $\Gamma(.)$, $\xi$, $p$, $\tau$.

- 2 First-Order ODEs $\Rightarrow$ Need 5 algebraic equations.

- Indifference Cash / Lending: $r(\xi) = 1$. 

![](diagram.png)
Unknowns: $q(.) \equiv r(.)$, $\Gamma(.)$, $\xi$, $p$, $\tau$.

2 First-Order ODEs $\Rightarrow$ Need 5 algebraic equations.

Indifference Cash / Lending: $r(\xi) = 1$.

Indifference Lending / Investing: $r(\tau) = \frac{\tau - \xi}{p - \xi}$. 
Unknwons: \( q(.) \) (\( \equiv r(.) \)), \( \Gamma(.) \), \( \xi \), \( p \), \( \tau \).

2 First-Order ODEs \( \Rightarrow \) Need 5 algebraic equations.

Indifference Cash / Lending: 

\( r(\xi) = 1 \).

Indifference Lending / Investing: 

\( r(\tau) = \frac{\tau - \xi}{p - \xi} \).

Most pessimistic lenders & borrowers: 

\( \Gamma(\tau) \equiv \xi \).
- Unknowns: \( q(.) \) (\( \equiv r(.) \)), \( \Gamma(.) \), \( \xi \), \( p \), \( \tau \).

- 2 First-Order ODEs \( \implies \) Need 5 algebraic equations.

- Indifference Cash / Lending:
  \[ r(\xi) = 1. \]

- Indifference Lending / Investing:
  \[ r(\tau) = \frac{\tau - \xi}{p - \xi}. \]

- Most pessimistic lenders & borrowers:
  \[ \Gamma(\tau) = \xi. \]

- Most optimistic lenders & borrowers:
  \[ \Gamma(1) = \tau. \]
Unknowns: $q(.) \equiv r(.)$, $\Gamma(.)$, $\xi$, $p$, $\tau$.

2 First-Order ODEs $\Rightarrow$ Need 5 algebraic equations.

Indifference Cash / Lending: $r(\xi) = 1$.

Indifference Lending / Investing: $r(\tau) = \frac{\tau - \xi}{p - \xi}$.

Most pessimistic lenders & borrowers: $\Gamma(\tau) = \xi$.

Most optimistic lenders & borrowers: $\Gamma(1) = \tau$.

Market clearing for the real asset: $1 - F(\xi) = p$. 
Model with Borrowing Contracts Only

Setup
Equilibrium Definition
Equilibrium Solution
Equilibrium Properties

Extension 1: "Pyramiding" Lending Arrangements

Extension 2: Short-Sales

Conclusion
Illustrating examples: $f$ uniform, $f$ increasing

- **Uniform**: 2 first-order ODE $\rightarrow$ second-order ODE:
  $$\Gamma'' (\Gamma - x) + \Gamma' + \Gamma'^2 = 0 \quad \Rightarrow \quad \Gamma(x) = -x - a + b\sqrt{x + c}.$$

- **Closed form**: $p$, $\xi$, $\tau$, $r(.)$, $q(.)$, $L(.)$, $a$, $b$, $c$. Example:

  $$p = \frac{1 + \Delta + 2\Delta^2 + 2\Delta^3 - \sqrt{(-1 + \Delta)^2 (1 + 2\Delta^2)}}{2\Delta + \Delta^2 + 4\Delta^3 + 2\Delta^4} = 1 - O(\Delta^2).$$
Cutoffs as a function of $\Delta$ ($f$ uniform)

<table>
<thead>
<tr>
<th>Percentile (Ranked by Degree of Optimism)</th>
<th>Lenders</th>
<th>Borrowers</th>
<th>Cash Investors</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 %ile</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>90 %ile</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>80 %ile</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>70 %ile</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60 %ile</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50 %ile</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40 %ile</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30 %ile</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20 %ile</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 %ile</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 %ile</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Disagreement $\Delta$

- Cutoff $\tau$
- Price of Real Asset $p$
- Cutoff $\xi$

True across bounded away from zero density function:

$$p = 1 - O(\Delta^2), \quad \tau = 1 - O(\Delta^2), \quad \text{and} \quad \xi = 1 - O(\Delta).$$
In uniform case, truncated Pareto with coeff 2:

\[
\frac{p}{p - Q(y)} = \frac{p}{\sqrt{2\xi}} \sqrt{\frac{p - \xi}{\tau - \xi}} \frac{1}{\sqrt{\frac{(p + \xi)\tau - \xi(p - \xi)}{2\xi} - y}}.
\]

Proposition (Limiting Pareto Distribution for Leverage Ratios of Optimists for smooth \(f(.)\))

Let \(f(.)\) differentiable, \(f'\) continuous, \(f(.)\) bounded away from 0. \(G_{\Delta}(.)\) distribution function for the leverage of borrowers for \(f_{\Delta}(.)\):

\[
\exists A_{\Delta}, \quad \|l^2(1 - G_{\Delta}(l)) - A_{\Delta}\|_{\infty}^{[L_{\Delta(1)}/2,L_{\Delta(1)}]} \xrightarrow{\Delta \to 0} 0,
\]

Heuristically:

\[
1 - G_{\Delta}(l) \sim \frac{A_{\Delta}}{l^2}.
\]

Upper tail behavior: not dependent on \(f(.)\).
Pareto Distributions for Leverage Ratios, Uniform Distribution

Coefficient: 2.

Disagreement $\Delta = 10\%$

Disagreement $\Delta = 5\%$

Disagreement $\Delta = 2\%$
Pareto Distributions for Leverage Ratios, Increasing Distribution

Still Coefficient: 2.
**Empirical Counterpart**

**TASS Hedge Fund Database, August 2006.**

![Graph showing the relationship between Log10 Leverage Ratio and Log10 Survivor. The slope of the fitted values is -1.95. The calibration is disagreement ≈ 1.8%.](image)

**Calibration:** disagreement $\approx 1.8\%$. 
Non Bounded away from 0.

- If \( f(x) \sim (1 - x)^\rho \Rightarrow \) Pareto with coefficient \( 2 + \rho \).
- Scale Independence Remains.
Returns to Entrepreneurship?

- Expected Returns are Pareto from envelope condition:

\[
R'(y) = \frac{1}{p - Q(y)} = \frac{\text{Leverage}(y)}{p}.
\]
Hedonic Interest rates $r(\cdot)$ on safe bonds for lenders. Can be substantial. Example with $f(x) = 2(1 - x)/\Delta$.

Corr($r(\cdot), l(\cdot)$) > 0 from disagreement. But: no risk shifting $\Rightarrow$ Different regulatory implications.
Non monotonic relationship between leverage and realized returns of borrowers, because of spreads.
Model with Borrowing Contracts Only
  Setup
  Equilibrium Definition
  Equilibrium Solution
  Equilibrium Properties

Extension 1: "Pyramiding" Lending Arrangements

Extension 2: Short-Sales

Conclusion
Pyramiding Lending Arrangements

▶ Allow Borrowing Contracts to be used as collateral.

▶ Hedonic interest rates ⇒ Lenders want to leverage into them!

▶ Example for houses, loans to SMEs: securitization. Or rehypothecation of collateral, repos of mortgage-backed securities, etc.

▶ Price $p$ increases even more.
Akin to tranching. The lender of type 2 is repaid until $\phi'$, then lender of type 1 is repaid on $\phi - \phi'$, then the borrower gets $p_1 - \phi$. 
Pyramiding Lending Arrangements

- Pareto Coefficients decrease (leverage distributions are multiplied) ⇒ Leverage Ratio distribution shifted to the right.

- Price expresses the opinion of superoptimists.
Empirics

- Leverage Ratios on New Loans. Source: Dataquick.
- \( \approx 100,000 - 500,000 \) new loans per month.
Empirics

- Leverage Ratios on New Loans. Source: Dataquick.
- ≈ 100,000 - 500,000 new loans per month.
Empirics

- Leverage Ratios on New Loans. Source: Dataquick.
- ≈ 100,000 - 500,000 new loans per month.
Empirics

- Leverage Ratios on New Loans. Source: Dataquick.
- \( \approx 100,000 - 500,000 \) new loans per month.
Video: the leverage ratio distribution from 1987 to 2012.
The model allows to recover the corresponding increase in borrowers’ expected returns.

In a model with a little bit of risk aversion: more risk taking?
Model with Borrowing Contracts Only

Setup
Equilibrium Definition
Equilibrium Solution
Equilibrium Properties

Extension 1: "Pyramiding" Lending Arrangements

Extension 2: Short-Sales

Conclusion
Short-Sales

- Unlike existing disagreement models, the model allows the treatment of short-sales.

Price = pessimists’ valuations ⇒ Systematic undervaluation - similar to noise trader risks in De Long et al. (1990), but risk neutrality. Equity premium, discount of closed-end funds, etc.

Endogenous rebate rates - apparent short-selling costs not evidence of constraints: about 100 bps, larger with more disagreement.

Endogenous Short-interest (a few percent).
Endogenous Rebate Rates and Cash Collateral

- No short-selling costs or costs of default.

[Graph showing the relationship between extra return of asset lenders and cash collateral as a fraction of value.]
Endogenous Short Interest

- Only a few percent of stocks are on loan in equilibrium, even though all are potentially available.
Larger Spreads on Bonds, even the safest (AAA)
Model with Borrowing Contracts Only

Setup
Equilibrium Definition
Equilibrium Solution
Equilibrium Properties

Extension 1: "Pyramiding" Lending Arrangements

Extension 2: Short-Sales

Conclusion
Conclusion

- Homeowners / Entrepreneurs’ / Hedge Funds data lend support to a very stylized model.
- New (static) source of Pareto distributions in returns independent from Gibrat’s law/ random growth.
- New intuitions on key financial prices / quantities:
  - Returns on Bonds.
  - Short-selling ”costs”.
  - Short interest

Potential for future work:

- Empirical work on short interest, rebate rates, distributions of leverage ratios to recover disagreement.
- Financial regulation:
  - Costs of moving OTC onto exchanges.
  - Monitoring financial system through ultimate borrowers’ leverage ratio distribution?
Thank you
Leverage Ratios of Entrepreneurs

Slope: -2.02

Log_{10} Leverage Ratio

Log_{10} Survivor