Bayesian Nash Equilibrium

Ichiro Obara

UCLA

February 1, 2012
Bayesian Game

We like to model situations where each party holds some private information. For example,

- A bidder does not know other bidders’ values in auction.
- A trader has some insider information about a recent technological innovation by some firm.
- ...

...
A **Bayesian Game** consists of

- **player**: a finite set $N$
- **state**: a set $\Omega$
- **action**: a set $A_i$ for each $i \in N$
- **type**: a set $T_i$ for each $i \in N$
- **belief**: a function $p_i : T_i \to \Delta \left( \Omega \times \prod_{j \neq i} T_j \right)$ for each $i \in N$
- **payoff**: a function $u_i : A \times \Omega \to \mathbb{R}$ for each $i \in N$. 
Bayesian Game

Interpretation:

- $\Omega$ is a set of possible \textbf{states of nature} that determine all physical setup of the game (payoffs).
- $T_i$ is the set of $i$’s private \textbf{types} that encode player $i$’s information/knowledge (ex. private signal).
- $p_i$ is player $i$’s \textbf{interim belief} about the state and the other players’ types.

We are already familiar with the basic idea of BG: correlated equilibrium and variety of interpretations of mixed strategy (Harsanyi’s purification argument etc.) are some special cases. BG introduces incomplete information into games in a very flexible way.
Bayesian games are often described more simply by eliminating $\Omega$ as follows.

- Interim belief on $T_{-i}$: $p_i(t_{-i}|t_i) := \sum_{\omega \in \Omega} p_i(\omega, t_{-i}|t_i)$
- Payoff on $A \times T$: $u_i(a, t) := \sum_{\omega} u_i(a, \omega)p_i(\omega|t)$

where $p_i(\omega|t) = \frac{p_i(\omega, t_{-i}|t_i)}{\sum_{\omega \in \Omega} p_i(\omega, t_{-i}|t_i)}$.

- In this formulation, type encodes both payoff and belief.
Consider the standard model of first price auction where each bidder only knows her value and have a belief about the other bidders’ values. This is a simple Bayesian game where

- the set of players (bidders) is \( N \)
- the set of states is \( V_1 \times \ldots, \times V_n \)
- the set of actions for bidder \( i \) is \( A_i = \mathbb{R}_+ \)
- the set of types for bidder \( i \) is \( V_i \)
- bidder \( i \)'s interim belief is \( p_i(v_{-i}|v_i) \).
- bidder \( i \)'s payoff is \( u_i(b, v) = 1(b_i \geq \max_{j \neq i} b_j)(v_i - b_i) \).
Common Prior Assumption (CPA)

- We almost always assume that all the interim beliefs are derived from the same prior. That is, we assume common prior.

Common Prior Assumption

A Bayesian Game $(N, \Omega, (A_i), (T_i), (p_i), (u_i))$ satisfies the common prior assumption if there exists $p \in \Delta(\Omega \times \prod_{i \in N} T_i)$ such that $p_i(\omega, t_{-i}|t_i), t_i \in T_i, i \in N$ are all conditional distributions derived from $p$.

- This assumption is not entirely convincing, but it is a useful one.
Bayesian Nash Equilibrium

- Player \( i \)'s strategy \( s_i \) is a mapping from \( T_i \) to \( A_i \). Let \( S_i \) be the set of player \( i \)'s strategies. It is like a contingent plan of actions.

- Given \( s = (s_1, ..., s_n) \), player \( i \)'s interim expected payoff for type \( t_i \) is

\[
E \left[ u_i \left( (s_i(t_i), s_{-i}(\tilde{t}_{-i})) , \tilde{\omega} \right) | t_i \right]
\]

\[
:= \sum_{\omega \in \Omega} \sum_{t_{-i} \in T_{-i}} u_i \left( (s_i(t_i), s_{-i}(t_{-i})) , \omega \right) p_i(\omega, t_{-i} | t_i)
\]

- A strategy profile \( s = (s_1, ..., s_n) \) is a **Bayesian Nash Equilibrium** if for every \( i \in N \), \( s_i \) assigns an optimal action for each \( t_i \) that maximizes player \( i \)'s interim expected payoff.
Here is a formal definition.

**Bayesian Nash Equilibrium**

$s^* = (s_1^*, \ldots, s_n^*) \in S$ is a Bayesian Nash Equilibrium if

\[
E \left[ u_i \left( (s_i^*(t_i), s_{-i}^*(\tilde{t}_{-i})), \tilde{\omega} \right) \mid t_i \right] 
\geq E \left[ u_i \left( (a_i, s_{-i}^*(\tilde{t}_{-i})), \tilde{\omega} \right) \mid t_i \right]
\]

holds for every $a_i \in A_i$ and $t_i \in T_i$, for every $i \in N$. 
Comments.

- A Bayesian Nash equilibrium can be regarded as a Nash Equilibrium of some appropriately defined strategic game.
  - One interpretation is to regard each type as a distinct player and regard the game as a strategic game among such $\sum_i |T_i|$ players (cf. definition in O&R). Then a BNE can be regarded as a NE of this strategic game.
  - Suppose that there exists common prior $p$. Let $U_i(s) = E \left[ u_i((s_i(t_i), s_{-i}(\tilde{t}_{-i})), \tilde{\omega}) \right]$ be player $i$’s ex ante expected payoff given $s \in S$. Then a BNE $s^*$ is a NE of strategic game $(N, (S_i), (U_i))$. 
Comments.

- Suppose that there are finite actions and finite types for each player. In this case, the whole game can be regarded as a finite strategic game (in either interpretation). In this setting, we can allow each type to randomize over actions as we did in mixed strategy NE. Then we can define a mixed strategy BNE and it follows immediately from the existence of MSNE that there exists a MSBNE.
Example 1: Second Price Auction

- Suppose that \( v = (v_1, \ldots, v_n) \) is generated by common prior \( p \in \Delta([0, 1]^n) \) in second price auction.

- Since it is a dominant action to bid one’s true value, bidders’ beliefs are not relevant for their decision. Hence \( b^* = (b_1^*, \ldots, b_n^*) \), where \( b_i^*(v_i) = v_i \), is a BNE.
Example 2: First Price Auction

- Assume that $v_i$ is i.i.d. across bidders and follow the uniform distribution on $[0, 1]$.

- We look for a symmetric BNE $(b^*, ..., b^*)$ in first price auction.

- We use “guess and verify method”: we assume $b(v) = \theta v$ for some $\theta$, then verify that this strategy is in fact optimal against itself for some $\theta$. 
Example 2: First Price Auction

- Bidder $i$’s expected payoff with value $v_i \in [0, 1]$ and bid $b_i \in [0, 1]$ is

$$\Pr(win|b_i)(v_i - b_i) = \Pr\left(\max_{j \neq i} v_j \leq \frac{b_i}{\theta}\right)(v_i - b_i)$$

$$= \left(\frac{b_i}{\theta}\right)^{n-1} (v_i - b_i)$$

- The first order condition is:

$$\frac{n - 1}{\theta} \left(\frac{b_i}{\theta}\right)^{n-2} (v_i - b_i) - \left(\frac{b_i}{\theta}\right)^{n-1} = 0$$

- $\frac{n-1}{n} v_i$ maximizes the payoff given $v_i$ (independent of $\theta$).

- Hence $b^*(v) = \frac{n-1}{n} v$ is the optimal bid for each bidder when all the other bidders are using it. That is, it is a symmetric BNE for every bidder to follow $b^*(v) = \frac{n-1}{n} v$. 
Example 2: First Price Auction

Next we consider general cumulative distribution function $F$ (still i.i.d. is assumed).

Assume that the symmetric BNE $b^*$ is strictly increasing and differentiable.

The expected payoff for type $v$ bidder is

$$\Pr(\text{win}|b)(v - b) = F(b^{*-1}(b))^{n-1}(v - b)$$

The type $v$ bidder’s optimal bid $\hat{b}(v)$ is obtained from the following first order condition:

$$\frac{(n - 1)f(b^{*-1}(\hat{b}(v)))F(b^{*-1}(\hat{b}(v)))^{n-2}}{b'^*b^{*-1}(\hat{b}(v))}(v - \hat{b}(v)) - F(b^{*-1}(\hat{b}(v)))^{n-1} = 0$$
Example 2: First Price Auction

Since \( \hat{b}(v) = b^*(v) \) in equilibrium, this differential equation simplifies to

\[
\frac{(F(v)^{n-1})'}{b'(v)}(v - b(v)) - F(v)^{n-1} = 0.
\]

Solving this, we obtain

\[
b^*(v) = \int_0^v \frac{(F(x)^{n-1})'}{F(v)^{n-1}} xdx.
\]

or, by integration by parts,

\[
b^*(v) = v - \frac{\int_0^v F(x)^{n-1} dx}{F(v)^{n-1}}.
\]

We can verify that (1) this is in fact strictly increasing and differentiable and (2) second order condition is satisfied.
Example 2: First Price Auction

It is a Bayesian Nash equilibrium for every bidder to follow the strategy

\[ b(v) = v - \frac{\int_0^v F(x)^{n-1} dx}{F(v)^{n-1}} \]

for the first price auction with i.i.d. private value.
Example 3: Cournot Competition with Private Cost

Consider a Cournot model where each firm’s cost is private information and drawn from $[0, 1]$ according to the same CDF $F$ independently. Let $\bar{c}$ be the average cost.

Assume that the inverse demand function is $p(Q) = 3 - Q$.

Let’s try to find a symmetric BNE $(q^*, ..., q^*)$. 
Example 3: Cournot Competition with Private Cost

- Firm $i$’s expected profit when its cost is $c_i$ and $q_i$ is produced is
  \[
  \pi_i(q_i, q^*) = E \left[ (3 - q_i - (n - 1)q^*(\tilde{c}) - c_i) q_i \right]
  \]

- From FOC, we obtain
  \[
  q_i(c_i) = \frac{3 - (n - 1)E[q^*(\tilde{c})] - c_i}{2}
  \]

- Taking the expectation and imposing symmetry, we have $E[q^*(\tilde{c})] = \frac{3 - c}{n+1}$.

- Hence $(q^*, ..., q^*)$, where $q^*(c) = \frac{3 - c}{n+1} - \frac{c - \tilde{c}}{2}$, is the symmetric BNE.
Example 4: Double Auction (Chatterjee and Samuelson 1983)

- One buyer and one seller.
- Buyer’s value and and seller’s value is independently and uniformly distributed on \([0, 1]\).
- Buyer’s payoff is \(v_b - p\) when trading at price \(p\), 0 otherwise. Seller’s payoff is \(p - v_s\) when trading at price \(p\), 0 otherwise.
- Buyer’s strategy is \(p_b : [0, 1] \to [0, 1]\) (price offer) and seller’s strategy \(p_s : [0, 1] \to [0, 1]\) (asking price).
- Trade occurs at price \(\frac{p_b + p_s}{2}\) only if \(p_b \geq p_s\). Otherwise no trade.
Example 4: Double Auction

- Look for a linear BNE $p^*_b(v_b) = a_b + c_b v_b$, $p^*_s(v_s) = a_s + c_s v_s$.
- Given this linear strategy by the seller, the buyer’s problem is

$$\max_{p_b} \left[ v_b - \frac{1}{2} \left\{ p_b + \frac{a_s + p_b}{2} \right\} \right] \frac{p_b - a_s}{c_s}$$

- From the first order condition, we obtain

$$p_b(v_b) = \frac{2}{3} v_b + \frac{1}{3} a_s$$
Example 4: Double Auction

- Seller’s problem is

\[
\max_{p_s} \left[ \frac{1}{2} \left( p_s + \frac{p_s + a_b + c_b}{2} \right) - v_s \right] \frac{a_b + c_b - p_s}{c_b}
\]

- From the first order condition, we obtain

\[
p_s(v_s) = \frac{2}{3} v_s + \frac{1}{3} (a_b + c_b)
\]
Example 4: Double Auction

- Matching coefficients, we get $a_b = 1/12$, $a_s = 1/4$, $c_b = c_s = 2/3$.
- Hence we find one BNE:

$$p_b^*(v_b) = \frac{2}{3}v_b + \frac{1}{12}$$

$$p_s^*(v_s) = \frac{2}{3}v_s + \frac{1}{4}$$
Remark

- An allocation is efficient if trade occurs whenever $v_b \geq v_s$. So this BNE does not generate an efficient allocation.
- In this uniform distribution environment, however, this BNE is the most efficient one. Thus it is impossible to achieve the efficient allocation by any BNE in double auction.
- This inefficiency result is very general. The efficient allocation cannot be achieved by ANY Bayesian Nash Equilibrium in ANY mechanism.
Example 5: Global Game (Carlson and van Damme 1993)

- There are two investors $i = 1, 2$. Each investor chooses whether to purchase risky asset (RA) or safe asset (SA).
- Each investor’s payoff depends on the state of economy $\theta \in \mathbb{R}$ and the other investor’s decision.

**Information Structure:**

- $\theta$ is “uniformly distributed” on $\mathbb{R}$.
- Investor $i$ observes a private signal $t_i = \theta + \epsilon_i$, where $\epsilon_i$ follow $N(0, \sigma)$.
- Given $t_i$, investor $i$ believes that $\theta_i$ is distributed according to $N(t_i, \sigma)$ and $t_{-i}$ is distributed according to $N(t_i, \sqrt{2}\sigma)$.
Example 5: Global Game

- The payoff matrix given each realization of $\theta$ is as follows.

<table>
<thead>
<tr>
<th></th>
<th>RA</th>
<th>SA</th>
</tr>
</thead>
<tbody>
<tr>
<td>RA</td>
<td>$\theta,\theta$</td>
<td>$\theta-1,0$</td>
</tr>
<tr>
<td>SA</td>
<td>$0, \theta-1$</td>
<td>$0,0$</td>
</tr>
</tbody>
</table>

- Note that, if $\theta$ is publicly known,
  - RA is a dominant action when $\theta > 1$.
  - SA is a dominant action when $\theta < 0$.
  - Both ($RA, RA$) and ($SA, SA$) is a NE when $0 \leq \theta \leq 1$. 
Example 5: Global Game

- Let $b(x)$ be a solution for $t - \Phi \left( \frac{x-t}{\sqrt{2}\sigma} \right) = 0$ given $x$. The interpretation of $b(x)$ is that $b(x)$ should be an investor’s optimal cutoff point (i.e. choose SA if $t$ is below $b(x)$ and RA if $t$ is above $b(x)$) when the other investor’s cutoff point is $x$.

- $b(x)$ looks as follows. It is strictly increasing and crosses the 45 degree line at $x = 0.5$. 

![Graph showing the function $b(x)$ with a 45 degree line at $x = 0.5$.]
Example 5: Global Game

We can show that this is the unique BNE by applying iterated elimination of strictly dominated actions for each type as follows.

- For $t_i < 0, i = 1, 2$, RA is strictly dominated. So delete RA for every $t_i$ strictly below 0.
- Then we can delete RA for every $t_i < b(0) (> 0)$ for $i = 1, 2$ (why?).
- We can delete RA for every $t_i$ below $b^2(0) < b^3(0) < \ldots$.
- So we can delete RA for every $t_i < 0.5$ in the limit.
- Similarly we can delete SA for every $t_i$ above $b^2(0) > b^3(0) > \ldots$, hence for every $t_i > 0.5$ in the limit.