We study a class of strategic games called supermodular game, which is useful in many applications and has a variety of nice theoretical properties.

A game is supermodular if the marginal value of one player’s action is increasing in the other players’ actions.

We are also interested in supermodularity between actions and exogenous parameters.
Lattice

We need to introduce a few mathematical concepts first. Let’s start with lattice.

- For each $x, y \in \mathbb{R}^k$, define $x \land y, x \lor y \in \mathbb{R}^k$ as follows.
  
  \begin{itemize}
  
  \item $(x \land y)_i := \min \{x_i, y_i\}$ ("meet")
  \item $(x \lor y)_i := \max \{x_i, y_i\}$ ("join")
  \end{itemize}

- A set is lattice if it includes the join and the meet of any pair in the set.

**Lattice**

$X \subset \mathbb{R}^k$ is a **lattice** if $x \land y, x \lor y \in X$ for every $x, y \in X$. 
Lattice

Remark.

- We are restricting our attention to a special class of lattices (sublattices of $\mathbb{R}^k$). The theory can be much more general.
- $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^m$ is a lattice if and only if $X \times Y \subset \mathbb{R}^{k+m}$ is a lattice.
Lattice

Examples

- Interval \([0, 1]\).
- A set of \(x \in \mathbb{R}^k\) such that \(x_i \geq x_{i+1}\) for \(i = 1, \ldots, k - 1\).
Greatest and Least Element of Lattice

- Notion of greatest and least:
  - $x^* \in X$ is a **greatest** element in $X$ if $x^* \geq x$ for any $x \in X$.
  - $x_* \in X$ is a **least** element in $X$ if $x^* \leq x$ for any $x \in X$.

- Nonempty compact lattice $A$ has the greatest element and the least element (why?). We denote them by $\bar{A}$ and $\underline{A}$ respectively.
Increasing Differences

A function $f(x, y)$ satisfies **increasing differences** if the marginal gain from increasing $x$ is larger when $y$ is larger.

Let $X, Y \subset \mathbb{R}^k$ be lattices. A function $f : X \times Y \rightarrow \mathbb{R}$ satisfies **increasing differences** in $(x, y)$ if

$$f(x', y') - f(x, y') \geq f(x', y) - f(x, y)$$

for any $x' \geq x$ and $y' \geq y$.

- $f$ satisfies **strictly increasing differences** in $(x, y)$ if the inequality is strict for any $x' > x$ and $y' > y$.
- This formalizes the notion of **complementarity**.
A closely related concept is **supermodularity**.

**Supermodularity**

Let $X \subset \mathbb{R}^k$ be a lattice. A function $f : X \to \mathbb{R}$ is **supermodular** if

$$f(x \lor x') + f(x \land x') \geq f(x) + f(x')$$

for every $x, x' \in X$. 
When is $f$ supermodular?

- It is easy to see that a function $f$ on lattice $X \times Y$ satisfies increasing differences in $(x, y)$ if and only if $f$ satisfies increasing differences for any pair of $(x_i, y_j)$ given any $x_{-i}, y_{-j}$.
- Similarly a function $f$ on lattice $X$ is supermodular if and only if $f$ satisfies increasing differences with respect to any pair of variables $(x_i, x_j)$ given any $x_{-i,j}$ (show it).
- When $f$ is twice continuously differentiable on $X = \mathbb{R}^k$, $f$ is supermodular if and only if $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$ for any $x_i, x_j$.

Note. It is sometimes useful to work with $\log f$ instead of $f$ (log supermodularity).
Supermodularity

Example

- In the simplest Bertrand game with \( n \) firms, each firm’s profit is \( \pi_i(p) = (p_i - c_i)(a - p_i + b \sum_{j \neq i} p_j) \). Hence \( \pi_i(p) \) is supermodular in \( p \).

- For Cournot game with \( n \) firms, \( \pi_i(q) \) is not supermodular. However, when \( n = 2 \), it is supermodular with respect to firm 1’s production and the negative of firm 2’s production: each firm’s profit function \( \pi_i(q_1, -q_2) \) satisfies increasing differences in \( (q_1, -q_2) \).
Monotone Comparative Statics

- We prove an important preliminary result: **Monotone Comparative Statics**.

- When there is a **complementarity** between choice variable $x$ and parameter $t$, we often show that the optimal solution increases in $t$ by using the implicit function theorem as follows.
  - **FOC**: $f_x(x,t) = 0$. Then $x'(t) = -\frac{f_{x,t}(x,t)}{f_{x,x}(x,t)}$.
  - **SOC**: $f_{x,x} < 0$. Then $x'(t) \geq 0$ if and only if $f_{x,t} \geq 0$.

- We prove the same thing without any differentiability.
Let $X \subset \mathbb{R}^k$ be a compact lattice and $T \subset \mathbb{R}^m$ be a lattice. Suppose that $f : X \times T \rightarrow \mathbb{R}$ is supermodular and continuous on $X$ for each $t \in T$, and satisfies increasing differences in $(x, t)$. Let

$$x^*(t) = \arg\max_{x \in X} f(x, t)$$

be the set of the optimal solutions given $t$. Then

- $x^*(t) \subset X$ is a nonempty compact lattice
- $x^*(t)$ is increasing in strong set order, i.e.
  $$x \in x^*(t) \& x' \in x^*(t') \Rightarrow x \vee x' \in x^*(t') \text{ and } x \wedge x' \in x^*(t) \text{ when } t' > t.$$
- In particular, $\bar{x}^*(t') \geq \bar{x}^*(t)$ and $\underline{x}^*(t') \geq \underline{x}^*(t)$ when $t' > t$. 

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Proof.

- $x^*(t)$ is nonempty and compact for each $t$ by Weierstrass theorem.

- For any $x, x' \in x^*(t)$, $f(x \land x') + f(x \lor x') \geq f(x) + f(x')$. Since $X$ is a lattice, it must be the case that $f(x \land x') = f(x \lor x') = f(x) = f(x')$. Hence $x^*(t)$ is a lattice.

- For any $x \in x^*(t), x' \in x^*(t')$, we have $f(x, t) - f(x \land x', t) \geq 0$. By ID and SM, $f(x \lor x', t') - f(x', t') \geq 0$. This means $x \lor x' \in x^*(t')$. Thus $x \leq x \lor x' \leq \overline{x}^*(t')$ for any $x \in x^*(t)$, hence $\overline{x}^*(t') \geq \overline{x}^*(t)$.

- By the same proof, $x^*(t') \geq x^*(t)$. 


Monotone Comparative Statics

- If $f$ satisfies strictly increasing differences, then $x^*(t)$ is increasing in the sense that $x' \geq x$ for any $x' \in x^*(t')$ and $x \in x^*(t)$ when $t' > t$.
  - This means that any selection from $x^*(t)$ such as $\overline{x}^*(t)$ is nondecreasing.

- The above proof works even when the choice set $X(t)$ is increasing in strong set order.
Supermodular Game

What is the implication of all these to strategic games?

A strategic game $G = (N, (A_i), (u_i))$ is supermodular if

- Each $A_i \subset \mathbb{R}^k$ is a compact lattice
- $u_i$ is continuous and supermodular on $A_i$ for every $a_{-i} \in A_{-i}$, and satisfies increasing differences in $(A_i, A_{-i})$ for each $i \in N$. 
Theorem

There exists a pure strategy Nash equilibrium for any supermodular game.

Remark.

- Note that no concavity assumption is imposed, no continuity is assumed with respect to $a_i$ and no mixed strategy is needed.
- For a finite strategic game, $u_i$ is automatically continuous and $A_i$ is automatically compact. So we just need $A_i$ to be a lattice and $u_i$ to satisfy supermodularity/increasing differences.
Proof.

For this proof, we assume that $u$ is continuous.

For any $a_{-i} \in A_{-i}$, $B_i(a_{-i})$ is a nonempty compact lattice by MCS. Hence $B(a) = (B_1(a_{-1}), ..., B_n(a_{-n}))$ is nonempty compact lattice.

Let $a^*(0) \in A$ be the greatest action profile in $A$. Let $a^*(t)$, $t = 0, 1, 2, ...$ be a sequence such that $a^*(t + 1) = \overline{B}(a^*(t))$. Then $a^*(t)$ is a decreasing sequence by MCS. Since a decreasing sequence in a compact set in $\mathbb{R}^k$ converges within the set. There exists $a^* \in A$ such that $a^* = \lim_{t \to \infty} a^*(t)$.

We show that $a^*$ is a NE. For any $i$ and $a_i \in A_i$,

$$u_i(a^*_i(t + 1), a^*_{-i}(t)) \geq u_i(a_i, a^*_{-i}(t)).$$

Then $u_i(a^*_i, a^*_i) \geq u_i(a_i, a^*_i)$ by continuity.
Comments

- We can find the least NE if we start from the least action profile.

- Consider a parametrized strategic game, where \( u_i(a, t) \) depends on some exogenous parameter \( t \). If each \( u_i \) satisfies increasing difference in \((a_i, t)\) (in addition to all the other assumptions), then it follows from the above proof that the greatest NE and the least NE is increasing in \( t \).

- To drop continuity, use Tarski’s fixed point theorem.

Tarski’s Fixed Point Theorem

Let \( X \in \mathbb{R}^k \) be a compact lattice and \( f : X \rightarrow X \) be a nondecreasing function. Then there exists \( x^* \in X \) such that \( f(x^*) = x^* \).
Example

- Consider a Bertrand competition model with \( n \) firms, where firm \( i \)'s demand function \( q_i(p, \theta) \) depends on every firm's price and market condition \( \theta \). Firm \( i \)'s cost function is \( c_i q_i(p) \).
  - The logarithm of firm \( i \)'s profit function satisfies increasing differences in \((p_i, c_i)\) for \( p_i \geq c_i \).
  - Also suppose that firm \( i \)'s profit function satisfies increasing differences in \((p_i, (p_{-i}, \theta))\) (when is this the case?).

- Assuming that there is a natural upper bound on \( p_i \), the following results follow without assuming any explicit functional form for \( q_i \):
  - There exists the highest equilibrium price vector and the lowest equilibrium price vector.
  - The highest equilibrium price vector and the lowest equilibrium price vector increase when \( c_i \) increases for any \( i \) or when \( \theta \) increases.
Supermodularity and Rationalizability

- Let $A_i(t) = \{ a_i \in A_i : a_i \leq a^*(t) \}$.

- Every $a_i \not\leq a_i^*(t)$ is strictly dominated by $a_i \land a^*(t + 1)$ in strategic game $(N, (A_i(t)), (u_i))$, because for any $a_{-i} \in A_{-i}(t)$,

$$0 < u_i(a^*(t + 1), a_{-i}(t)) - u_i(a_i \lor a^*(t + 1), a_{-i}(t))$$

$$\leq u_i(a^*(t + 1), a_{-i}) - u_i(a_i \lor a^*(t + 1), a_{-i}) \quad \text{(by ID)}$$

$$\leq u_i(a_i \land a^*(t + 1), a_{-i}) - u_i(a_i, a_{-i}) \quad \text{(by SM)}$$

- This means that no action above $a^*$ survives IESDA, hence no action above $a^*$ is rationalizable.

- The largest rationalizable action profile and the largest NE (hence the largest MSNE) coincide for supermodular games with continuous $u$. 

References