Two Sources of Macroeconomic Sluggishness: 
Ultrametric Tree Structures and Entropic Barriers

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Abstract

This paper discusses two sources or mechanisms for macroeconomic sluggishness. The results are stated as two propositions. One has to do with the complexity of organization of agents, measured in terms of the number of levels into which subsets of agents are organized in trees, and of the number of agents in each of the subsets. The other deals with a phenomenon of gradual reduction in the probability of availability of some adjustment rules as economic activities decline, that is, shrinking of sets of feasible decisions in probabilistic sense. This is called an entropic barrier effect since this effect arises even when the cost landscape is flat.

1 Introduction

Aoki and Yoshikawa (2003a) have discussed the role of uncertainty in causing macroeconomic adjustment processes to slow down, and in making macroeconomic policies less effective. In this paper we discuss two additional sources for macroeconomic sluggishness and slowdowns of recovery processes.

One cause of sluggishness is increases in complexity of the ways agents interact among themselves. Suppose we collect agents into clusters according to some criteria of homogeneity or similarity of agents, such as the nature of business, attitudes towards risk and so on, and we organize these clusters into hierarchical form, that is, into trees with leaves corresponding to agents. We measure the complexity of tree structures by the number of nodes and of branches. We also measure 'distance' between clusters in ultrametric terms. The larger the number of levels of hierarchy, that is, the number of levels of branches of trees, and the larger is the sizes of clusters, the slower the adjustment speeds. In other words, dynamics of adjustments of such systems
have larger time constants as the organizations become more complex. This is stated as Proposition 1 in this paper.

The notion of ultrametric trees has been introduced in the physics literature in connection with spin glass, see Ramal and Toulouse (1986), for example, and into economic literature by Aoki (1996, 2002). See also Aoki and Yoshikawa (2003b) for an application of ultrametric distance in labor market model. We claim that this concept is an important ingredient in understanding macroeconomic sluggishness. A toy model makes this point later in this paper.

The other mechanism for slowdowns and sluggishness of adjustments and recoveries from recessions has to do with increases in the probabilities of losing some adjustment mechanisms or paths that are normally available to agents as general level of economic activities declines. Agents stuck in some suboptimal operating conditions are prevented from utilizing better suboptimal or global optimal behavior by barriers caused by this loss of paths or actions. It is called entropic barriers in this paper by borrowing the term from physics literature.

We coin a phrase “economic temperature” to refer to the general level of economic activities in analogy with physical temperature. This economic temperature may become lower as levels of uncertainty pervading in the economy increases, for example.

As economic temperature becomes lower, for whatever reasons, some of the adjustment strategies normal open to agents to improve their performances or lower their costs become closed or the probability for them to be able to follow through some of the adjustment strategies become lower. The lower is economic temperature, the higher is the probability of them being stuck in current conditions.

We use a method known as the Metropolis algorithm in stochastic simulation literature, notably in simulated annealing literature, (Metropolis, et al. (1953), to illustrate this concept, and apply the idea behind the algorithm to explain macroeconomic slowdowns due to entropic barriers in Section 4.

2 Master Equations on Ultrametric Trees

Aoki (1997, Chapter 7) introduced a hierarchical arrangements of economic agents by collecting them into clusters by some criteria, such as the size of firms by numbers of employees, annual sales, type of goods being produced, and so on. An upside-down tree is associated with such classification. At the root, which is at the top level, there is one large cluster of all agents in the model. The root represents a space of all states of some Markov chain. From the root, several branches separate out, which are clusters of agents,

\[1\] Trees are similar to genealogical trees.
each composed of same types of agents, or the set of states is partitioned into subsets of states. Each of the clusters may further subdivide, and so on, until at the bottom level, we find individual agents or states corresponds to leaves. See Fig. 2.3 or Fig. 7.1 in Aoki (1996), or Figure 1 of this paper for examples. Trees generally have several levels. The number of branches from a common branch is the same for symmetric trees, and not the same for asymmetric ones.

The ultrametric distance between two clusters is defined to be the number of levels one has to go up towards the root before a common node of the two branches is found. Transition rate between two clusters, \( i \) and \( j \), is denoted by \( q(i,j) \), and is assumed to depend on the ultrametric distance \( d(i,j) \) between them.

A master equation (backward Chapman-Kolmogorov equation) is a dynamic equation to govern time evolution of the probabilities of being in one of the clusters. Because transition rates are functions of ultrametric distances, the dynamic equation has certain regular structures which allows us to solve the equation analytically in some cases. See Ogieski and Stein (1985), or Shreckenberg (1985). The transition rates are assumed to be decreasing functions of ultrametric distances and of a positive valued parameter denoted as \( \beta \) and an increasing function of economic temperature \( T_e \). We assume

\[
q(i,j) = \exp\{-\beta d(i,j)/T_e\}.
\]

Values of \( \beta \) becomes larger, and the transition rates become smaller as degree of uncertainty prevailing in the economy becomes larger. As \( T_e \) becomes lower, the transition rates become smaller. The time constant of adjustment for the correlations of probabilities, between the initial condition \( P_i(0) = 1 \) at the bottom of a tree, and \( P_i(t) \) for some \( i \) at a branch higher up near the root, say, can be shown to become larger and larger as the number of levels increases, and approach power-laws. This is shown in Sec. 3. Simplified accounts of these papers are in Aoki (1996, p.200, p. 205).

Here, we first give the essence of the phenomenon of slowdown with a simplified model of two states, see Fig. 2, which is similar to the model with two local minima by Kabashima and Shinomoto (1991) discussed in Aoki (1996, p.25). This model is also used to illustrate problems faced by agents when 'cost' or 'utility' landscape is rugged and has many local extrema. See the illustration in Fig. 3. Agents must then employ some strategies such as the Metropolis algorithm in order to avoid being stuck at some local optima. We show later that as general level of economic activities become sluggish, some adjustment paths that are available become more and more inaccessible probabilistically and causes macroeconomic slowdowns.
Two-state Example

To simplify notation we combine $T_e$ with $\beta$ and redefine $\beta/T_e$ as $\beta$. So $\beta$ gets large when $T_e$ becomes small. We consider an example composed of two states, $a, b$ with the transition rates

$$w(a, b) = \exp(-\beta V),$$

and

$$w(b, a) = \exp[-\beta(U + V)],$$

where $\beta$ is a positive parameter. See Fig. 2. (For the moment ignore state c.)

Probabilities $P_a(t)$ and $P_b(t)$ are governed by the master equation

$$\frac{dP_a}{dt} = P_b w(b, a) - P_a w(a, b).$$

Substituting $P_b = 1 - P_a$ out in the above, we rewrite it as

$$\frac{dP_a}{dt} = -\gamma(p_a - \Pi_a),$$

where

$$\gamma = w(a, b) + w(b, a),$$

and

$$\Pi_a = \frac{w(b, a)}{\gamma} = \frac{1}{1 + e^{\beta U}}.$$ 

Suppose that the system is initially in state $a$, i.e., $P_a(0) = 1$. Then, the solution of the master equation gives the expression

$$P_a(t) = e^{-\gamma t} + \Pi_a(1 - e^{-\gamma t}).$$

Substituting this into (1), we rewrite the master equation as

$$\frac{dP_a(t)}{dt} = -\gamma(1 - \Pi_a)e^{-\gamma t} = -\exp(-\beta V - \gamma t).$$

(2)

Suppose we want to change the probability $P_a(t)$ as quickly as possible, and manipulate $\beta/T_e$, which we rename $\beta$ for short, to maximize the right-hand side of this differential equation, or equivalently minimize the exponent with respect to $\beta$

$$V + \frac{\partial \gamma}{\partial \beta} t = 0.$$ 

This leads to the expression

$$t = -\frac{V}{\frac{\partial \gamma}{\partial \beta}} \approx e^{\beta V}.$$
In other words, even when we change $\beta$ to maximize the speed of adjustment by $\beta \approx \ln t/V$, we have

$$P_a(t) = e^{-\gamma t} + \Pi_a(1 - e^{-\gamma t}) \approx e^{-\beta U} \approx t^{-V/U}.$$  

This shows that the probability that the system is in state $a$ reaches the equilibrium value $\Pi_a = 1/(1 + e^{\beta U}) \approx e^{-\beta U}$ at the speed not of exponential function but that of power-law, that is extremely sluggishly.

**Three-state Example**

We next examine the same shape as in the previous example by introducing another state and consider a three state model, with state $\{a, c, b\}$, where state $a$ is a local minimum, $b$ is global minimum, and $c$ is a local maximum. The dynamics have a mode with a very long time constant.

The transition rates are

$$w(a, c) = \exp(-\beta V),$$

$$w(c, a) = w(c, b) = \frac{1}{2},$$

and

$$w(b, c) = \exp[-\beta(V + U)].$$

By substituting out $P_c = 1 - P_a - P_b$, the master equation is

$$\frac{dP_a}{dt} = \frac{1}{2} - [\frac{1}{2} + e^{-\beta V}]P_a - \frac{1}{2}P_b,$$

and

$$\frac{dP_b}{dt} = \frac{1}{2} - [\frac{1}{2} + e^{-\beta(U + V)}]P_b - \frac{1}{2}P_a.$$  

The stationary probabilities are

$$\Pi_a = \frac{1}{1 + e^{\beta U} + 2e^{-\beta V}} \approx \frac{1}{1 + e^{\beta U}} \approx e^{-\beta U},$$

and

$$\Pi_b = \frac{1}{1 + 2e^{-2\beta(U + V)} + e^{-\beta U}} \approx \frac{1}{1 + e^{-\beta U}} \approx 1 - e^{-\beta U}.$$  

Solving the differential equations with the initial condition $P_a(0) = 1$, we obtain

$$P_a(t) = \frac{1}{\lambda_1 - \lambda_2} [\lambda_1 + \frac{1}{2} + e^{-\beta(U + V)}] e^{\lambda_1 t}$$

$$+ \frac{1}{\lambda_2 - \lambda_1} [\lambda_2 + \frac{1}{2} + e^{-\beta(U + V)}] e^{-\lambda_2 t},$$
where $\lambda_i$, $i = 1, 2$ are the roots of the characteristic equation, where

$$\lambda_1 \approx -\frac{1}{2} [e^{-\beta V} + e^{-\beta (U+V)}] \approx -\frac{1}{2} e^{-\beta V},$$

and

$$\lambda_2 \approx -1 - \frac{1}{2} [e^{-\beta V} + e^{-\beta (U+V)}] \approx -1 - \frac{1}{2} e^{-\beta V}.$$

In this example $\lambda_1$ becomes very small, i.e., one of the two time constants associated with this eigenvalue becomes large, and the dynamic mode associated with this eigenvalue is very sluggish, as economic temperature becomes low.

3 Increased Complexity of Interactions and Ultrametric Tree Models: Proposition 1

Example: dynamics of a two-level tree

This tree has three nodes at level 2, that is, three branches come out of the root, $m_2 = 3$. Each node at level 2 separates into three branches, that is $m_1 = 3$. At the end of each branch, there is a leaf. Leaves at the bottom of the tree, that is at level 1, are the states of a continuous-time Markov chain. In this example there are $m_1 m_2 = 9$ states. As Fig. 3 shows, state $a$ and $b$ belongs to the same super cluster $A$, while state $a$ and $f$ belong to different superclusters $A$ and $B$. For simpler notation, name the states as $a, b, c, d, e, f, g, i$. Transition rate between states $a$ and $b$ is denoted by $q_{a,b}$. The expression $q_{a,f}$ means the transition rate between state $a$ and $f$. These two states belong to a different clusters. Transition rates are assumed to be symmetric, i.e., $q_{a,i} = q_{i,a}$, and depends only on the ultrametric distance between the states.

The ultrametric distance between two states is the levels of trees that these states need to go up the tree toward the root to reach the common node, that is node from which both states branch out. If we regard the tree as a genealogical chart with a node representing an anecestor, ultrametric distance 1 means that the states share the same parent, distance 2 grandparent.

Denote the three nodes at level 2 by $A, B, \text{ and } C$.

Therefore, states among the same cluster at level 1 have the same transition rate, $q_1$. Transition rates between nodes at level 2 are the same and denoted by $q_2$. For example, $q_{a,c} = q_1$, but $q_{a,i} = q_2$.

The master equation for state $a$ is

$$\frac{dp_a}{dt} = I_a - O_a,$$
where the inflow of probability flux is

\[ I_a = p_b q_{b,a} + p_c q_{c,a} + p_d q_{d,a} + \cdots p_i q_{i,a}, \]

and

\[ O_a = p_a [(q_{a,b} + q_{a,c}) + (q_{a,d} + \cdots q_{a,i})]. \]

From our discussion of the ultrametrics above we can write these more simply as

\[ I_a = q_1 (p_b + p_c) + q_2 (p_d + \cdots p_i), \]

and

\[ O_a = p_a [(m_1 - 1)q_1 + (m_2 - 1)m_1 q_2]. \]

Noting that \( p_b + p_c = (p_a + p_b + p_c) - p_a = p_A - p_a, \) and that \( p_d + \cdots p_i = p_B + p_C = 1 - p_A, \) we obtain

\[ \frac{dp_a}{dt} = q_2 + (q_1 - q_2)p_A - p_a [m_1(q_1 - q_2) + Nq_2], \]

where \( N = m_1m_2. \)

The master equation for \( p_b \) and \( p_c \) have the same structure. Recall that \( p_a \) is the sum of these three probabilities. Adding the three equations together, we obtain

\[ \frac{dp_a}{dt} = -Nq_2(p_A - \frac{1}{m_2}). \]

The solutions of these equations are

\[ \frac{dp_A(t)}{dt} = \exp(-t/\tau_2)[p_A(0) - \frac{1}{m_2}] + \frac{1}{m_2}, \]

where

\[ 1/\tau_2 = m_1m_2q_2, \]

and

\[ p_a(t) = \exp(-t/\tau_1)[p_a(0) - \frac{1}{m_1}p_A] + \exp(-t/\tau_2)[\frac{1}{m_1}p_A(0) - \frac{1}{m_1m_2}] + \frac{1}{m_1m_2}, \]

with

\[ 1/\tau_1 = m_1m_2q_2 + m_1(q_1 - q_2). \]

**General Case**

:Power-law adjustment behavior

In a \( R \)-level tree with transition rates between two states depending only on the ultrametric distance between them, that is, the difference in levels of the lowest nodes common to the two states, the structure of these solutions slow down holds. Denoting what corresponds to level 2 by \( p_{j_R} \), where \( j_R \)
runs from 1 to $m_R$, which is the number of branches from the root, its
differential equation for the probability at the root is

$$\frac{dp_{jR}}{dt} = -Nq_R(p_{jR} - \frac{1}{m_R}).$$

The expression for the leaves at level 1 is

$$p(j_1, \ldots, j_R) = \sum_{i=1}^{R} \exp(-t/\tau_i)[p(j_i, \ldots, j_R)(0)/N_i - p(j_{i+1}, \ldots, j_R)(0)/N_{i+1}] + 1/N,$$

where

$$\frac{1}{\tau_i} = \sum_{r=i}^{R} (q_r - q_{r+1})N_r,$$

where $N_r = m_1 \cdots m_r$, and $q_{R+1} = 0$.

This sum equals

$$\frac{1}{\tau_i} = \sum_{r=i}^{R} (m/q)^{i-1} = (\lambda^{-i+1} - \lambda^{-R})/(1 - \lambda),$$

where $\lambda = q/m$. With $\lambda$ less than 1 we obtain the usual exponential dy-
namic behavior. With the value of $\lambda$ larger than 1, by letting $\hat{t} = t/\tau_1$ for
convenience, we obtain

$$x_1(t) = \sum m^{-i+1}e^{-\lambda^{-i+1}} + 1/N.$$

The summand is of the form $\exp[(1/m)^i - c\exp(-iln\lambda)]$. Define

$$h(y) = \exp[-\frac{lnm}{ln\lambda}y - ey].$$

Then the sum can be approximated by $\int h(lny)dy$ which can be put into
the form of an incomplete Gamma function. See Ogielski and Stein.

We now discuss general $R$ level trees with large $R$. The time constants
are getting larger as you go up the levels towards the root. At level $i$ it is
given by

$$\frac{1}{\tau_i} = \sum i^R(q_r - q_{r+1})N_r \approx q_iN_i.$$

Then for time $t$ between the two successive time constants, $\tau_{i-1}$ and $\tau_i$, and
closer to the latter, then $t$ is much larger than the former and we have
an approximate expression for $x_1$ in this time interval

$$x_1(t) \sum \frac{\exp(-t/\tau_i)/N_i + 1/N \approx 1/N_{i0}},$$

where $1/\tau_{i0} \approx 1/t \approx q_{i0}N_{i0}$.  

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To present simple cases, assume that \( q_i = q^{-i}, \) and \( m_i = m^i, \) \( i = 1, \ldots, R. \) Then, it is well known that we observe a power-law behavior for the prices at level \( 1. \)

This heuristic argument can be made more precise by evaluating the expression for the sum by approximating it by an integral. See Paladin, Mezard, and de Cominicus (1985) or Ogielski and Stein (1985) for detail. In Appendix we outline the argument. One can use steepest descent argument or change of variables to approximate the sum by an incomplete Gamma function. The conclusion that power-law emerges does not change.

\[
x_1(t) \approx t^{-\theta},
\]

with \( \theta = \frac{\ln m}{\ln(q/m)}. \)

4 Entropic Barriers: Proposition 2

Agents wishing to improve their performances by adjusting some of their decision variables face complicated and hard combinatorial optimization problems. They may find themselves in basins with some local optima, and they are often not sure if there are other basins with better local optima, or global optima. They may also face non-unique choices, because their optimization criteria may be multi-valued, and so on.

Suppose agents conduct exploratory moves and evaluate the results with the Metropolis algorithm. See Ripley (1987) for exposition of the method, for example. In Metropolis method, the move is accepted with probability one if it results in lower cost. In order to move from a local minimum to a better local or global minimum, the move is accepted with probability \( \exp(-\beta \Delta c) \) even when the cost increases by \( \Delta c > 0, \) where \( \beta \) is some positive parameter. A trial move, then, is accepted with probability \( \min\{1, \exp(-\beta \Delta c)\}. \) This algorithm recognizes a possibility that even if a move results in increased cost, the direction of the move may be correct with some positive probability in leaving the basin associated with the current position to a new basin with lower local minimum.

If agents are confident of the directions of move, then they will choose them with high probability, that is, if the value of \( \beta \) is small, namely in economies with high economic temperature. However, if they are uncertain of the moves to improve their conditions, the move will be chosen with small probability, that is \( \beta \) has a large value in case of uncertainty, i.e., of low economic temperature environment. We later analyze an example to illustrate these points.

What is important to recognize is the fact that the parameter \( \beta \) may become very large, especially when macroeconomy is in recession. With a

\[2\text{Note that the relation } x(t) = Ct^\theta, \text{ and } \mu x(\Delta t) = x(t) \text{ implies that } \theta = -\ln \mu/\lambda.\]
very large value of $\beta$, any move not resulting in immediate cost reduction is rejected with very high probability. This results in agents not revising their current operating modes almost with probability one. This will prolong the recessions. Put differently, adjustment paths out of the current states are rarely taken, or become inaccessible when economy is such that $\beta$ is very large.

There are several ways for $\beta$ becoming large. Simple examples are given in Aoki (2002, p.48, p.53; 1996, p.133, p. 179) to illustrate effects of uncertainty associated with consequences of an actions or decision. In these examples, large uncertainty of effects of action $\beta$ becomes very large.

**Entropic Barriers**

This section presents a model that focusses on possibilities of losing adjustment path as $\beta$ becomes large, and not on the aspect of cost minimization. Agents’ adjustment problem is thus cast as a random walk problems in one dimension, that is, as a birth-death process. In this setting we recognize that $\Delta c = 0$. We have a flat cost landscape.

Suppose that agents face ‘cost’ or ‘utility’ surfaces with full of local minima. Agents are not sure whether the minima they find themselves are global minimum or they are stuck in some local minima. They do not know which directions, if any, they should move in order to improve their performances. They must modify steepest descent or some gradient procedure to take account of a possibility that they need to overcome some barriers to leave the current basin of attraction to another one with smaller costs. For this reason we assume that they employ the Metropolis method or some variant of it, Metropolis (1953). This method gives some probability to go in directions which increases the cost in the hope that the path leads to a local peak of the barrier separating the basin from another with smaller local minima. We show that probabilities with which this type of paths is tried become smaller as economic activity levels become lower.

We suppose there are $K$ boxes (cluster, categories, or types) into which agents belong. To be concrete, we may think of the boxes as representing strategies or directions of descents that agents try. In this model economy-wide optimum corresponds to a situation in which all boxes are empty. Presence of an agent in some box indicates that agent has not achieved its optimal operating condition. At a given time we pick an agent at random uniformly and place it in another box at random. If there are $k$ agents in total, one agent is chosen with probability $k/N$ where $N$ is the total number of agents in all the boxes. The chosen agent exits or depart from the box he is in, and will go to one of the remaining box with probability $1/(K - 1)$. Call the box he goes to as the arrival box. Let $S$ denotes (Boltzmann)
entropy of the configuration. The difference
\[ \Delta S = S(n_d - 1, n_a + 1) - S(n_d, n_a), \]
is the change in entropy when one agent departs box \( d \) and goes to box \( a \).
We assume that
\[ \Delta S = -\beta, n_d = 1, n_a \neq 0, \]
\[ \Delta S = \beta, n_d \neq 1, n_a = 0, \]
and \( \Delta S \) is zero in all other cases. In other words, entropy changes only when
the number of empty boxes, that is, successful types of firms is reduced by one \( (n_a = 0, n_d \neq 1) \), which increases \( S \) by \( \beta \), or the number of empty boxes increases by one, \( (n_a \neq 0, n_d = 1) \), which increases the number of empty boxes by one.

To treat the simplest cases we focus on one of the boxes, called box 1,
and let \( n_1(t) \) denotes the number of agents in it,
\[ p_k(t) = \Pr(n_1(t) = k). \]

This and related models have been analyzed by several physicists. We follow
Codrech and Lux 91997). We write the master equation for it as
\[ \frac{dp_k}{d\tau} = (k + 1)p_{k+1} + \lambda(\tau)p_{k-1} - (\lambda(\tau) + k)p_k, k \geq 2, \]
and similar equations for \( k = 2 \) and \( k = 1 \). To simplify equations we change
the time from \( \tau \) to \( t \) by defining
\[ \frac{dt}{d\tau} = \lambda(\tau). \]

With this change of variable we have
\[ \frac{dp_k}{dt} = \frac{k + 1}{\lambda(t)} + p_{k-1} - \left[ 1 + \frac{k}{\lambda(t)} \right] p_k, k \geq 2, \]
\[ \frac{dp_1}{dt} = \frac{2}{\lambda(t)} p_2 + \mu(t)p_0(t) - 2p_1(t), \]
and
\[ dp_0(t) dt = p_1(t) - \mu(t)p_0(t), \]
where
\[ 1/\lambda(t) = 1 + (e^{-\beta} - 1)p_0(t), \]
and
\[ \mu(t) = e^{-\beta} + (1 - e^{-\beta})p_1(t). \]

See Appendix for the derivation of the transition rates.
We easily verify that these $p_k$'s sum to one, and the mean of $k$, $\sum_k k p_k(t) = N/K := \rho$. We take $\rho$ to be 1 for simplicity.

This set of equations can be used to calculate the generating function

$$F(z, t) = \sum_k p_k(t) z^k,$$

with the initial condition assumed to be $F(z, 0) = z$, that is $p_1(0) = 1$ and all other $p_i$s are zero.

This generating function satisfies the partial differential equation

$$\frac{\partial F}{\partial t} = (z - 1)F(z, t) - \frac{z - 1}{\lambda(t)} \frac{\partial F}{\partial z} - (z - 1)Y(t),$$

where $Y(t) = (1 - e^{-\beta}p_0(t)) = 1 - 1/\lambda(t)$. See Aoki (2002, p. 70) for deriving the partial differential equation for the generating function. See also Aoki (2002, App. A.1) for solving the partial differential equation by the method of characteristic curves. The solution is obtained by solving

$$\frac{dt}{1} = \frac{dz}{(z - 1)/\lambda(t)} = \frac{dF}{(z - 1)(F - Y)}.$$

When $\beta = 0$ which corresponds to high level of economic activities and a well-behaved cost curve, the equations is especially simple, because $\lambda(t) = \mu(t) = 1$. We obtain

$$F(z, t) = \{1 + (z - 1)e^{-t}\} \exp\{(z - 1)(1 - e^{-t})\},$$

and from it

$$p_0(t) = (e^{-t} - 1) \exp(e^{-t} - 1) = -\frac{1}{e} + \frac{1}{2e} e^{-2t} + \cdots.$$

This shows that $p_0(t)$ approaches the equilibrium value with time constant $1/2$.

With $\beta$ large Godrech and Lux show that the time constant becomes $e^\beta/\beta^2$, a much larger number, indicating a sluggish approach to the equilibrium.

References


5 Appendix: Error function, Ingber approximation and Gibbs distribution

This appendix is a summary of one way for Gibbs distribution to arise naturally. For fuller account, see Aoki (1996, Sec. 5.6, 2002, Chapt 6).

Suppose two alternative choices confront a collection of agents. Let x be the fraction of agents who have chosen alternative 1, A1 for short. Call them type 1 agents. A type 1 agent anticipates the reward V1(x), while the rest of agents of fraction 1 − x are type 2 agents. A type 2 agent anticipates reward V2(x).

Suppose for the sake of simplicity that the difference V1(x) − V2(x) is a random variable with mean g(x) and variance σ2. (Of course we can make it to depend on x bu supress x). Assume further that the difference is normally distributed. Then the probability that A1 is better than A2 is

\[ \eta(x) = \Pr[V_1(x) - V_2(x) \geq 0] = \frac{1}{2} [1 + erf \{a(x)\}], \]

where erf is the error function, and

\[ a(x) := \frac{g(x)}{\sqrt{2}\sigma}. \]
See Fig. 4 for sketches of $g(x)$, $V_1(x)$, and $V(2)$.

Ingber (1982) discovered that

$$erf(a) := \frac{2}{\sqrt{\pi}} \int_0^a e^{-y^2} dy \approx \tanh(\kappa a),$$

with $\kappa := 2/\sqrt{\pi}$.

The precision of this approximation is such that

$$erf(a) = \kappa \{ a - \frac{a^3}{3} + \frac{a^5}{5} + \cdots \},$$

and

$$\tanh(\kappa a) = \kappa \{ a - \frac{a^3}{2.36} + \frac{a^5}{4.63} + \cdots \},$$

that is,

$$erf(a) - \tanh(\kappa a) \approx \kappa a^3 (0.8 - 0.06a^2 + \cdots).$$

This shows that the approximation of the error function with the arctangent function is good, especially when the coefficient of variation is greater than $1/\sqrt{2}$.

The arctangent approximation yields the Gibbs function expression used in Aoki (1996, 2002).

It is interesting to recall the figure in Aoki and Yoshikawa (2003) which depicts the coefficient of variation of the growth rates of Japanese and US GDP from 1966 to 2002. The coefficient of variation of the Japanese GDP exceeds the critical value $1/\sqrt{2}$ for the approximation of the error function by the Gibbs function from 1974 to 1978, and from 1993 to 2002. This shows that our analysis in terms of Gibbs functions in assessing effectiveness of macroeconomic policy is supported by data.