New Frameworks for Macroeconomic Modelings:
Some Illustrative Examples

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Abstract

Microeconomic behavior is fundamentally stochastic and we need statistical methods to study macroeconomy composed of a large number of stochastically behaving agents. For this reason, we reject the standard approach to microfoundation of macroeconomics as misguided, since use of intertemporal optimization formulation for representative agents are entirely inadequate.

Given that economies are composed of many agents of different types, a fundamentally different approach is needed. This paper illustrates our proposed approaches by brief descriptions of examples drawn from four separate problem areas.

1 Introduction

In cooperation with some like-minded macroeconomists and physicists, Aoki has been using modeling approaches to macroeconomics that are substantially different from the procedures commonly used by mainline macroeconomists. Briefly put, we construct continuous-time Markov chains for several types of interacting economic agents to study macroeconomic problems. Stochastic dynamics are described by master (Chapman-Kolmogorov) equations, and stochastic dynamic behavior of clusters of agents of various types are used to draw macroeconomic results. Instead of the usual notion of equilibrium as a deterministic concept, we use stationary distributions as our definition of stochastic equilibria.

By now we have produced several new results and insights on macroeconomic behavior that are different, more informative, or not available in the mainline macroeconomic or financial literature. We also have some new perspectives on some macroeconomic phenomena.\footnote{The author gratefully acknowledges several important insights he obtained as the results of many conversations with H. Yoshikawa.} This paper surveys some of them, mostly drawn from Aoki (1996, 2002), and Aoki and Yoshikawa (2002, 2003), and Aoki, Nakano, and Yoshida (2004).
This paper presents them loosely grouped into four problem areas listed below. 2

1. Policy ineffectiveness: A new reason for macroeconomic policy ineffectiveness is described. We focus on uncertainties of consequences of decisions which surround decision-making agents.

2. Sluggish behavior of macro-price indices: In macroeconomy, in which clusters of agents are situated as leaves of trees with many levels of branches, responses of price indices become sluggish, due mainly to stochastic spreads of effects of exogenous shocks throughout the trees. This type of lags is different from that discussed by Taylor (1980).

3. New approach to labor market dynamics: Without using the notion of matching functions, we model labor market dynamics to explain Okun’s law and Beveridge curves. A new notion of distance, called ultrametric distance, is used to model different probabilities of unemployed being rehired, depending on their human capitals, types of jobs, durations of unemployment, and so forth. Distances between separate clusters of unemployed are modeled as ultrametric distance of multi-level trees by organizing clusters of workers with different characteristics as separate leaves of trees.

4. New Schumpeterian perspective on long-run behavior in industry: Interaction between innovation and imitation processes in a long-run are established by solving a model of two-sector economy with innovative and imitative sectors. Explicit stationary solutions of the master equations are obtained using cumulant generating functions for dynamically interacting two sectors.

The first has to do with effects on policies of degree of uncertainty which surrounds consequences of economic decisions by agents. 3

The second arises from slow stochastic responses to stochastic exogenous shocks by economic units. We examine stochastic dynamics of how effects of shocks in one sector of economy propagate throughout the model, and show that the expected speed of spread depends on the forms of tree graphs of agent connections. As the number of levels of tree increases, the responses tend to be more sluggish, and in the limit exhibit power-law like responses rather than exponentially decaying responses.

The slowness of responses come from sluggish probability spillovers from the origin of shocks to other sectors of economy, and is called information lag for short in this paper to distinguish it from the dynamic lags.

The third arises from our view that aggregate outputs respond to aggregate demands. In standard arguments, unequalized productivities across sectors imply unexploited profit opportunities, and contradict the notion of equilibrium. Heterogeneous stochastic agents behave differently. All agents

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2 There are other results not included in this list. See Yoshikawa (2003), and Aoki and Yoshikawa (2004).

3 Yoshikawa coined the term "uncertainty trap" to distinguish it from liquidity trap.
and production factors do not move instantaneously and simultaneously to the sector with the highest productivity. Their moves are governed by the transition rates of the continuous-time Markov chains.

We have used a simple quantity adjustment model to draw new perspectives on business cycles, in Aoki (2002, Sec. 8.6) and Aoki and Yoshikawa (2002). This model is extended here to include labor dynamics.

In the fourth category we examine long-run behavior of innovation and imitation processes in the Schumpeterian spirit. Explanations of this approach and some simple illustrative examples are found in Aoki (1996, 2002). Initial results are reported in Aoki, Nakano, and Yoshida (2004).

2 Policy Ineffectiveness: Uncertainty Trap

2.1 Introductory Remarks

The Japanese economy is now seemingly coming out of a long period of stagnation. Many explanations have been offered as to the causes of the long stagnation. Similarly many suggestions have been offered to bring the economy out of the stagnation, such as by Krugman (1998), Bernanke (2000), Blanchard (2000), Girardin and Horsewood (2001), among others.

We describe two probable sources for macroeconomic sluggishness which so far have not received the attention of economists in general, and those who specialize in Japanese economic performances in particular in the latest decades or so. 4

One is what we call uncertainty traps in Section 2. This effect has been pointed out in Aoki, Yoshikawa, and Shimizu (2003).

The other source of sluggish behavior by Japanese economy is due to sluggish responses of dynamics when economies are structured as multi-level tree forms. This effect is discussed in Section 3. It is due to the ways clusters of agents of various types interact. When effects of shocks to one sector of economy stochastically spread throughout the whole economy, we find that expected speed of this process, that is, the spread of shocks depends on the ways agents are interconnected. Agents of similar nature are collected as clusters or sectors and are arranged hierarchically as leaves of tree graphs. Tree graphs have several levels of nodes, from which branches spread ending up with leaves, which corresponds to clusters. Probabilities of disturbance at these leaves are initially zero, except at the cluster where the shock originates. Eventually the effects of this shock reach all leaves, that is all sectors of economy are affected, but the expected times of the shocks are shown to to be very long if trees have many levels, and if transition probabilities of shocks are functions of ultrametric distances between clusters. In our model, transition-rates of continuous time Markov chains, with leaves of trees as states, are assumed to be functions of 'distances' between the origin and the leaves.

Macroeconomists are aware that economies may be organized into hierarchical structures. However, they do not effectively differentiate clusters

4 The sources we discuss in this paper are not necessarily specific to Japanese economy and could affect other macroeconomies when conditions described in this paper are present.
in such hierarchy. It is as if clusters are all at the same distance from each other. We explicitly introduce a notion of distance between clusters or sectors, called ultrametric distance between clusters. When macroeconomies contain many clusters organized into many levels of a tree-shaped graph, exogenous (policy) shocks to any one of the sectors or clusters can spread to other clusters at speed which depends on the distances. They reach some clusters very slowly and takes a long time to affect the whole economy. Hence, expected change in macroeconomic price indices after such a shock change very sluggishly.

There exists a large body of literature devoted to labor markets, some theoretical and some empirical, such as Blanchard and Diamond (1989, 1992), Mortensen (1989), Pissarides (2000) and Davis and Haltiwanger (1992). Importance of flows of job creations and job destructions as determinants of labor market dynamic behavior is illustrated by Davis, Haltiwanger and Schuh (1996).

Existing models of labor dynamics, however, do not successfully explain these empirical findings. For one thing, differences in geographical locations, human capitals, jobs experiences and the like are stressed in verbal discussions of labor markets, but are not incorporated into models convincingly, if at all. Unemployed workers are differentiated at most by their reservation wages in search models, or by the length of spell of unemployment.

In Sec. 4, we treat clusters of different types of unemployed workers as forming a tree structure, and use ultrametrics to measure similarities of workers in different clusters. When a sector hires a worker it does so randomly from a pool of workers composed of different clusters which are suitably weighted by the ultrametric distances as we show in a later section. See Aoki (1996, Sec.2.5) on ultrametrics.

In Aoki (2002, Sec. 8.6) we describe a model in which GDP responds to changes in demand patterns on the sectors of economy. This continues to hold for our model in this section.

Two new features are explained. One is on the Okun's law. The Okun's law refers to a stable empirical relation between unemployment rates and rates of GDP changes: one percent increase (decrease) in GDP corresponds to x percent decrease (increase) in unemployment, where the value of x varies from country to country and from period to period, see Hamada and Kurosaka (1984) for example. It was about 4 in US when Okun announced this relation. With a Cobb-Douglas production function x is about 1. To obtain the value 4 we need some other effects such as increasing marginal product of labor or some other nonlinear effects. See Yoshikawa (2000). We report on the Okun's law based on a small scale simulation runs. In the simulation studies of our new model we obtain numbers larger than 4, even though our model use linear production functions for all sectors. See Yoshikawa (2000, 2003) on discussions of traditional literature on the Okun's law, and on the role of demands in macroeconomics.

The other feature is Beveridge curve for the relation between vacancies and unemployment rates.
2.2 Uncertainty Trap

To explain this notion concretely, suppose that there are $N$ agents (firms) in the economy. We keep $N$ fixed for simpler exposition. Each agent has two choices, choice 1 and choice 2, in selecting its production level. Choice $i$ means production at the rate $y_i$, $i = 1, 2$, where $y_1 > y_2 \geq 0$.

Let $n$ be the number of firms with choice 1. The number of firms with choice 2 is $N - n_1$.

The total output of the economy, or GDP, is $Y = ny_1 + (N - n)y_2$. We express this in terms of the fraction $x = n/N$ of agents with choice 1 as

$$Y = N[y_1x + (y_2(1 - x))].$$

Note that $x$ is a random variable between 0 and 1, since the number of firms with choice 1 is random.

We analyze how agents change their choices over time by modeling the process of changes in $n$ as a continuous-time Markov chain (also called jump Markov chain). Firms can change their mind any time. They constantly evaluate the two present values, $V_i$, $i = 1, 2$ associated with the two choices, where $V_i$ be the random discounted present value for a firm with choice $i$, conditional on the number of fraction $x$. During a small time interval only one firm may change its production rate. Agents’ stochastic switching between the two choices are described in terms of two transition rates which uniquely determine the stochastic process involves here. See Breiman (1968, Chapt. 15) for example.

Define the probability that choice 1 is better than choice 2, given fraction $x$ as

$$\eta(x) := \Pr(V_1 - V_2 \geq 0|x).$$

In Appendix we show that when the random variable $\eta(x)$ is approximated by normal distribution with mean

$$g(x) := E[\eta(x)],$$

and variance $\sigma^2$ which is assumed constant for simpler explanation, it can be approximately expressed as

$$\eta(x) = \frac{e^{\beta g(x)}}{X},$$

where $\beta = \sqrt{2}/\pi\sigma$, and

$$X = e^{\beta g(x)} + e^{-\beta g(x)}.$$

Let the expected value of $x$ be denoted by

$$\phi := E(x).$$

It is shown in Aoki (1996, p. 136), Aoki (1998), and Aoki (2002, p. 44) that this mean value is governed by the differential equation

$$\frac{d\phi}{dt} = (1 - \phi)\eta(\phi) - \phi[1 - \eta(\phi)].$$

Its stationary solution is obtained by setting its left-hand side to zero, and noting that $\eta(\phi)/(1 - \eta(\phi))$ is equal to $e^{2\beta g(\phi)}$, it is the solution of
\[ g(\phi) = \frac{1}{2\beta} \ln\left( \frac{\phi}{1 - \phi} \right). \]

With little uncertainty about the consequences of choices, that is, with small \( \sigma \), the value of the parameter \( \beta \) is large, and the above equation is approximately equal to

\[ g(\phi) = 0. \]

This shows that the expected value of the fraction of firms with choice 1 is the zero of the \( g(\cdot) \) function, that is, critical points of \( g \),

\[ g(\tilde{\phi}) = 0. \]

This \( \tilde{\phi} \) is a locally stable equilibrium if \( g' \) is negative at \( \tilde{\phi} \).

In this case, \( x \) varies randomly in neighborhood of \( \tilde{\phi} \). Accordingly, GDP fluctuates randomly but mostly in neighborhood of \( \tilde{Y} = N((y_1 - y_2)\tilde{\phi} + y_2) \). Standard comparative analysis holds with no problem here. If policy makers find this \( \tilde{Y} \) value too low, they can raise it by increasing \( \tilde{\phi} \) by shifting up \( g(\cdot) \). In circumstances with small uncertainty about the relative merits of alternative choices, the zeros of \( g \) function basically determines the stationary \( Y \) values.

The situation is quite different when uncertainty is large, that is, when value of parameter \( \beta \) is close to zero. We turn to this case next.

To understand the model with small values of \( \beta \) we obtain the stationary distribution of \( x \), not just the mean as above. Using \( n(t) \) as the basic variable, we solve the (backward) Chapman-Kolmogorov equation for it for the probability \( P(n(t) = k) \), written as \( P(k, t) \). Over a small interval of time \( (t, t + \delta t) \), the number of firms with choice one increases by one at the rate

\[ r_k := \lambda N(1 - k/N)\eta(k/N), \]

and decreases by one at the rate

\[ l_k := \mu N(k/N)[1 - \eta(k/N)], \]

where \( \lambda \) and \( \mu \) are constant parameters that do not concern us here.\(^5\)

The time derivative of this function expresses the net increase in probability that \( k \) agents have chosen 1, which is the difference of the probability influx and outflux given as

\[ \frac{dP(k, t)}{dt} = \text{Influx to } (n_1(t) = k) - \text{Outflux from } (n_1(t) = k), \]

where

\[ \text{Influx} = P(k + 1, t)l(k + 1, t), \]

and

\[ \text{Outflux} = P(k, t)r(k, t), \]

\(^5\)They are called birth and death rate in random walk model in probability textbook. When \( \eta(x) \) is replaced by 1, the model is a standard random walk model. The expression \( \eta(x) \) introduces externalities of choices among agents.
subject to boundary conditions at $k = 0$ and $k = N$, which we do not show here.

We solve this equation for a stationary distribution of $k$ by setting the left-hand side to zero. The stationary distribution, written as $\pi(k)$ is given by

$$
\pi(k) = \text{constant} \prod_{j=1}^{k} \frac{r_{j-1}}{l_{j}}, \quad k \geq 1.
$$

After substituting the above into the expression for the equilibrium distribution above, we derive

$$
\pi(k) = \text{constant} \ C_{N,k} \prod_{j=1}^{k} \frac{r_{j-1}}{l_{j}},
$$

where $C_{N,k}$ is the combinatorial coefficient $N!/k!(N-k)!$. We write this distribution in the exponential function form

$$
\pi(k) = X^{-1}N \exp[-\beta NU(k/N)],
$$

where we write the probability by introducing $U(k/N)$, called potential. By replacing $k/N$ by $x$ which is now treated as a real number between 0 and 1, and replacing the sum by the integral, we see that

$$
U(x) = -2 \int^{x} g(y)dy - \frac{1}{\beta}H(x),
$$

where $H(x) = -x \ln(x) - (1-x) \ln(1-x)$ is the Shannon entropy.

This entropy expression arises from the combinatorial factor of the number of ways of choosing $k$ out of $N$. This combinatorial factor is entirely ignored in the standard economic analysis, but is crucial in large uncertainty choice problems such as this one, since the entropy term is multiplied by $1/\beta$ which is the largest term in the expression for $\pi(x)$.

Locally stable $\phi$ is that which minimizes the potential

$$
0 = U'(\phi) = -2g(\tilde{\phi}) + \frac{1}{\beta} \ln \frac{\tilde{\phi}}{1-\tilde{\phi}}.
$$

When value of $\beta$ is large (case of little uncertainty), this reduces to our earlier expression that showed that $\tilde{\phi}$ is a critical point of $g(\phi)$.

With small values of $\beta$(case of large uncertainty), $\tilde{\phi}$ which minimizes the potential is not a critical point of $g$.

A straightforward variational analysis shows that if $g(x)$ is modified to $g(x) + h$ with some $h > 0$, then this $\tilde{\phi}$ is moved by

$$
\delta \tilde{\phi} = \frac{2\beta h(\tilde{\phi})}{[\tilde{\phi}(1-\tilde{\phi})]^{-1} - 2\beta g'(\phi)} > 0.
$$

However, with values of $\beta$ small, this is approximately equal to

$$
\delta \tilde{\phi} \approx 2\beta h(\tilde{\phi})\tilde{\phi}(1 - \tilde{\phi}) \approx 0.
$$
This expression shows that the effect of increasing the expected advantage of choice 1 over 2 by $h > 0$ is nullified by the presence of small $\beta$. This is why we call this phenomenon uncertainty trap. The economy cannot move out of $\bar{\phi}$ even when $g$ is shifted upwards to favor choice 1.

To summarize, an expansionary effort by shifting $g$ function by $h > 0$ is equal to

$$
\delta\bar{\phi} = -h(\bar{\phi})/g'(\bar{\phi}) > 0,
$$

with large $\beta$, but is nearly zero with small $\beta$.

3 Two Types of Lags in Tree Dynamics: Multiplier and Information lags

3.1 Multiplier Lags

Responses of a macroeconomic price index to shocks to one of its component prices consists of two components. One is the well-known dynamic delays in multipliers or impulse responses which are familiar in economics and econometrics. The other is called information lag in this paper. It refers to delays in the effects of exogenous shocks which originated in one sector spreading to other sectors stochastically. The former is simply illustrated by dynamic responses of a second order ordinary differential equation to a stem input changes. The latter requires solving Chapman-Kolmogorov equations for states that are leaves of trees such as those in Figures 1 and 2. To explain the former, it is convenient to use Laplace transform forms to relate output responses to input changes as

$$
H(s)Y(s) = U(s),
$$

where $s$ is the Laplace transform variable, and $Y(s)$ is the transforms of output. For a simple illustration we assume that dynamics are described by a second order differential equation with zero initial conditions; $y(0) = 0$, and $dy(0)/dt = 0$. For input, we use a step input, at time zero. Its Laplace transform is $U(s) = 1/s$. To be very concrete suppose that $H(s) = (s + a)(s + b)$ with some positive $a$ and $b$. This is the transfer function of a dynamic system described by a second order differential equation with two stable eigenvalues $-a$, and $-b$.

The dynamic response is obtained by taking the inverse Laplace transform of

$$
Y(s) = \frac{1}{s(s + a)(s + b)} = \frac{c}{s} + \frac{A}{s + a} + \frac{B}{s + b},
$$

where $c$, $A$, and $B$, are the constants, $c = 1/ab$, $A = -1/[(b - a)a]$, and $B = 1/[(b - a)b]$, that is,

$$
y(t) = c + Ae^{-at} + Be^{-bt}.
$$

If $a < b$, then after time span of about $4/a$ time units, the transient portion of the output die out and the output settles to a steady state value $y(t) \approx 1/ab$. Hence the dynamic lag is approximately $4/a$. We call this the multiplier lag.
If the input has a long decaying component such as \( u(t) = 1 - e^{-\lambda t} \), with \( \lambda \) a positive constant much smaller than \( a \) and \( b \), then \( y(t) \) is approximately equal to \( (1/ab)(1 - e^{-\lambda t}) \) which takes a long time to reach its steady state value. This is the kind of behavior we encounter in the next two subsections.

3.2 Information lags: Lags in spillover probabilities in trees

We next turn to the second type of lags that exist in trees with several levels of nodes. In this paper we measure distance between clusters using the notion of ultrametrics in the sense of Schikhof (1978).\(^6\)

To explain our idea simply, we compare two economies with four sectors each. They are organized in two different ways. One is organized as one level tree, and the other as two level trees a shown in Fig. 1. and 2, respectively. As shown in these figures, trees are usually drawn upside down with roots at the top level. Several branches come out of a root, each ending with a node. Further branches may sprout from these nodes. Nodes are arranged into one or more levels. Nodes at the the bottom levels of these upended trees are called leaves. We call them clusters (of agents or goods) or sites.\(^7\)

Think of the root of a tree as representing the whole macroeconomy. We think of price indices of macroeconomy as some weighted sums of the prices at the leaves, that is at the bottom sites (clusters) of the trees.

We next show how magnitudes and speeds with which the price changes triggered or originated in one sector of economy spread stochastically through the macroeconomy.

Exogenous disturbances to any one of the agents or goods will be felt first by agents or goods in the same cluster, and then the effects will gradually and stochastically propagate to other leaves, that is, clusters of agents or goods of the trees. Propagations of shocks are treated in this paper stochastically by modeling shock propagation processes as continuous-time Markov chains. Therefore we speak of the expected changes in price indices as the results of such shocks to one cluster.

Without loss of generality we assume that exogenous disturbances are felt at site 1 at time zero. This disturbance is felt at site \( i \) at time \( t \) with probability \( P_i(t) \). The initial condition is \( P_1(0) = 1 \), and \( P_i(0) = 0 \), \( i \neq 1 \).

Evolutions of these probabilities over time is described by the backward Chapman-Kolmogorov equation, also known as master equation. The probability at site \( i \) is changing over time as the difference of the influx and outflux of probabilities. Denote the transition rate between site \( i \) and \( j \) by \( w(i, j) \). The master equation which describes the dynamics of the probabilities is

\[
\frac{dP_i(t)}{dt} = I_i(t) - O_i(t),
\]

\(^6\)This notion is also explained and simple examples are given in Aoki (1996, p.35)/This notion is used in numerical taxonomy, see Jardine and Sibson (1971) for example, together with the method of minimal spanning trees for graphs, Murtagh (1983). In numerical taxonomy literature mere correlations are shown to be inadequate by Feigelman and Ioffe (1991), for example.

\(^7\)Agents in the same leaf are regarded to be more homogeneous compared with those in other leaves.
where the influx to site $i$ is
\[ I_i(t) = \sum_{j \neq i} P_j(t) w(j, i), \]
and
\[ O_i(t) = P_i(t) \sum_{j \neq i} w(i, j). \]

We also assume that $w(i, j) = w(j, i)$ for all $i$ and $j$.

Next, we posit that the transition rates $w(i, j)$ depends only on $d(i, j)$, which is called ultrametric distance. It is equal to the number of levels leaf $i$ and $j$ traverse towards the root until a common node is found for the first time. Ultrametric distance, denoted as $d(i, j)$, is positive unless $i = j$, then it is zero, symmetric $d(i, j) = d(j, i)$, and satisfies
\[ d(i, j) \leq \max_k \{d(i, k), d(j, k)\}. \]

More generally, for any three leaves, the ultrametric distance satisfies the inequalities $d(i, j) \leq \max\{d(i, k), d(j, k)\}$, and all permutations of these three indices. In our example the tree in Fig. 1 has only one level, so that $d(i, j) = 1$, for all $i \neq j$. In Fig. 2, $d(1, 2) = d(3, 4) = 1$, and $d(1, 3) = d(1, 4) = 2 = d(2, 3) = d(2, 4)$.

Some rudimentary notion of social distances between different clusters of agents is found in economic literature. For example, in some model agents are assumed to be located at sites of lattices, and agents in nearest lattice sites are assumed to interact, in a very crude analogy with Ising model in ferromagnetics literature in physics. However, no models exists with more formal notion of distances among different groups of agents such as ultrametrics.

The concept of ultrametrics has been in the literature of mathematics and applied in taxonomy, and physics, especially in spin glass models. For these, see Schikhof (1984), or M\'ezard and Virasoro (1985), among others. In clustering objects, more appropriate notion is needed than correlation, since correlation is not transitive.\(^8\)

Aoki (1996) has several elementary economic applications of the notion of ultrametrics. Aoki and Yoshikawa (2003) has a more complex example of labor market dynamics and Okun’s law, where unemployed workers from different sectors of economy or different human capitals or job experiences form separate clusters and ultrametrics are used to measure distances between clusters. This distance is then used to generate probabilities of unemployed being recalled by a given industry. Taylor’s well-known analysis of adjustment of wages is different from ours. He treats groups of workers with different wage contracts, but these groups are not hierarchically

\(^8\)Feigelman and Ioffe (1991) have an example of three patterns: $A=(1, 1, 1, 1), B=(1,1,1,1)$ and $C=(1,1,1,1)$. Calculating correlations by $\rho = (1/4) \sum x_i y_i$ where $x$s and $y$s are the components of the patterns above, we see that $\rho_{A,B} = \rho_{A,C} = 1/2$ but $\rho_{B,C} = 0$. To avoid this intransitivity of correlations as a measure of similarity of patterns, we use the notion of ultrametrics as a measure of distance between clusters of agents. See Aoki (1996, p.34) for further detail.
arranged. There is no notion of adjustment speeds as functions of some similarity measures among clusters in Taylor (1980).

By making transition rates between nodes or clusters of agents functions of ultrametric distance, we demonstrate in this section that the notion of hierarchically structured clusters of goods or producers leads to models with sluggish adjustment processes.

For the one-level tree of Fig. 1 transition rates are all the same

\[ w(i, j) = q, \]

\( i \neq j, \) where \( q = \exp(-\gamma d(i, j)) = \exp(-\gamma), \) for all \( i \) and \( j \) between 1 and 4, and \( \gamma \) is some positive parameter.

For the two-level tree of Fig. 2 we have\(^9\)

\[ w(1, 2) = w(3, 4) = q, \]

and

\[ w(1, 3) = w(1, 4) = q^2 \]

because \( d(1, 3) = 2, \) hence \( w(1, 3) = \exp(-2\gamma) = q^2. \)

We start with the model of Fig. 1. The master equation for the probability vector \( \mathbf{P}(t) \) consists of probabilities at the four leaves

\[ \frac{d\mathbf{P}(t)}{dt} = W\mathbf{P}(t), \]

with

\[ W = \begin{bmatrix} W_1 & W_2 \\ W_2 & W_1 \end{bmatrix}, \]

where

\[ W_1 = \begin{pmatrix} -3q & q \\ q & -3q \end{pmatrix}, \]

\[ W_2 = qe_2e_2^t, \]

where \( e_2 = [1 \ 1]^t. \)

This matrix \( W \) has eigenvalue 0 with eigenvector \( (1 \ 1 \ 1)^t, \) and triple repeated eigenvalue \(-4q\) with three independent eigenvectors \( (1 \ -1 \ -1)^t, \) \( (1 \ -1 \ 0)^t, \) and \( (0 \ 0 \ 1 \ -1)^t. \)

The probabilities evolve with time according to

\[ P_1(t) = 1/4 + (3/4)e^{-4qt}, \]

and

\[ P_2(t) = P_3(t) = P_4(t) = (1/4) - (3/4)e^{-4qt}. \]

Approximately after time span of \( 1/q, \) the probabilities are all about \( 1/4.\)\(^{10}\) It takes about this time span for the initial shock to propagate to all

\(^{9}\)This model is the same as the one in Aoki (1996, p. 38). Details of analysis differ somewhat.

\(^{10}\)Note that \( e^{-1} = 0.018. \)
the sectors. Hence this is the time lag for the shock initiated at sector 1 to spread probabilistically to all the other sectors. Macroeconomic price index to fully reflect the price shock to one of its sectors.

In the case of tree of Fig. 2, the matrix $W$ is given by

$$W = \begin{bmatrix} W_1 & W_2 \\ W_2 & W_1 \end{bmatrix},$$

where

$$W_1 = \begin{pmatrix} -(q + 2q^2) & q \\ q & -(q + 2q^2) \end{pmatrix},$$

$$W_2 = q^2 e_2 e_2',$$

where $e_2 = [1 \ 1]'$.

This matrix $W$ has eigenvalues 0, with eigenvector $(1 \ 1 \ 1 \ 1)'$, eigenvalue $\lambda_1 = -4q^2$, with eigenvector $(1 \ 1 -1 \ -1)'$, and double repeated eigenvalue $\lambda_2 = -2(q + q^2)$, with eigenvectors $(1 \ -1 \ 0 \ 0)'$, and $(0 \ 0 \ 1 \ -1)'$. Note that the magnitude of $\lambda_1$ is less than that of $\lambda_2$ because $q$ is less than one. The associated with eigenvalue $\lambda_1$ is faster than that associated with eigenvalue $\lambda_2$. It represents the escape rate of probability from site 1 to site 2.

The probabilities evolve with time as

$$P_1(t) = (1/4 + (1/4)e^{-\lambda_1 t} + (1/2)e^{-\lambda_2 t},$$

$$P_2(t) = (1/4 + (1/4)e^{-\lambda_1 t} - (1/2)e^{-\lambda_2 t},$$

$$P_3(t) = P_4(t) = 1/4 - (1/4)e^{-\lambda_1 t}.$$

After time span of $2/q(1 + q)$, the term $e^{-\lambda_2 t}$ is approximately zero. After time span of $1/q^2$ all probabilities are approximately equal to 1/4. Note however, that the time span $1/q^2$ is much longer than that of $1/q$, that is, the tree of Fig. 2 is much more sluggish than that of Fig. 1.

We can show that the larger the number of hierarchies the slower the process of disturbance propagation, and response of macro-price index to shocks to one of the sectors. Ogielski and Stein (1985), among several others, have shown that in the limit of the number of hierarchy going to infinity, the response becomes power-law, not exponential decay.

To compare dynamic behavior of this model with the one-level tree of Fig. 1, we can aggregate the tree by defining a two-dimensional state vector with components $S_1(t) = P_1(t) + P_2(t)$, and $S_2(t) = P_3(t) + P_4(t)$ by defining

$$Q(t) = S P(t),$$

where the aggregation matrix $S$ is given by

$$S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$
The dynamic matrix $V$ for this aggregated vector is given by $V = SWS'(SS')^{-1}$ which has eigenvalues 0 and $-4q^2$.

The vector $Q(t)$ has two components $0.5 + 0.5e^{-\lambda_2 t}$, and $0.5 - 0.5e^{-\lambda_2 t}$.

To summarize, dynamics of Fig. 1 is much simpler. It has eigenvalues 0 and $-4q$. We can similarly aggregate the first two sites and the second two sites separately to produce a two node tree. The eigenvalue are still 0 and $-4q$. In other words, after the lapse of time of the order $1/q$, the system of Fig. 1 has approximately reached its equilibrium state, while that in Fig. 2 has not. That is to say, the two-level dynamics is more sluggish than that of the one level tree of Fig. 1.

This fact remains true when a one-level tree of $K$ sites is compared with a $k$ level tree where $K = 2^k$. We can also group $l$ of $K$ sites into one cluster, and the remaining $K - l$ sites into another. The eigenvalues are still 0 and $-Kq$, while those of a $k$ level tree are 0 and $-(2q)^k$.

### 3.3 Sluggish Macroeconomic Price Index: An Example

We present an example of slow adjustments of some macroeconomic price index composed of prices of several sectors of economy. In a nutshell an exogenous shock to one of the sector is transmitted to that part of price index with delay of multiplier which involve dynamics of the linkage from that input to the price index component. This is the direct effect. This shock will probabilistically spread to other sectors as well. This is delayed spillover or information lag effects. The latter effects arrive to other sectors generally much later than the direct multiplier effects. Hence the sluggish movements of the price index.

To exhibit this explicitly in a simple way suppose that a price index $P_I$ is the weighted average of two sectoral output prices, $Q_A$ and $Q_B$. We outline how Sector 1 price $Q_A$ is affected by an exogenous shock to site 1 price, since effects on $Q_B(t)$ are similarly analyzed.

For concreteness suppose that node $A$ is composed of a two level tree with two more nodes $a$ and $b$ with two branches each. There are thus four more basic prices at sites 1 through 4, such as factor prices, prices of intermediate goods and so on. The two-level hierarchical tree traces out the relations among these prices. As shown in the previous section, the tree generates spill-over probabilities of an exogenous shock to one of the basic prices.

From the master equation, the Laplace transform of $P(t)$ is

$$\hat{P}(s) = \frac{1}{4s}u_0 + \frac{1}{2(s + \lambda_2)}u_2 + \frac{1}{4(s + \lambda_1)}u_1,$$

where $s$ is the Laplace transform variable, $u$s are the 4-dimensional column vectors shown above.

Consequently we can write down the explicit expression for $E[\delta Q_A(s)]$. Assuming that multiplier lags of the transfer function $h_a(s), h_b(s), h_i(s)$, $i = 1, \ldots, 4$ are not as large as $1/q^2$, we can extract the slowest decaying term out of this as

$$E(\delta Q_A(t)) \approx \frac{1}{4} h_a(-\lambda_1)\{h_1(-\lambda_1) + h_2(-\lambda_1)\} - h_b(-\lambda_1)\{h_3(-\lambda_1 + h_4(-\lambda_1)\}e^{-\lambda_1 t} + \cdots,$$
where the slowest term is extracted.

In the case where \( h_a(s) = 1/(s+a) \), \( h_b(s) = 1/(s+b) \), \( h_i(s) = 1/(s+\alpha_i) \), \( i = 1, \ldots, 4 \), then a sufficient condition that this term is present is

**Proposition**  When \( q^2 \) is negligibly small compared with \( a, b, \alpha_i \), \( i = 1, \ldots, 4 \), price \( Q_A \) will exhibit sluggish response to an exogenous price change at site 1 if

\[
\frac{1}{a} \left( \frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right) \neq \frac{1}{b} \left( \frac{1}{\alpha_3} + \frac{1}{\alpha_4} \right).
\]

More generally, structures of interconnections between the basic prices and \( Q_A \) are conveniently expressed in terms of the Laplace transforms as

\[
E[\delta \hat{Q}_1(s)] = \sum_{i=1}^{4} \hat{H}_i(s) E[\delta \hat{q}_i(s)],
\]

with \( \hat{H}_i(s) \) being the Laplace transform of the transfer function from site \( i \) to the price \( Q_A \), and where hatted variables are the Laplace transforms with \( s \) denoting the Laplace transform variable. The symbol \( E \) is the expectation operator, that is the expected values of changes in the basic price \( q_i \) with the probabilities of spillover.

The example of this section illustrates the effects of spillover, i.e., information lags due to hierarchical tree structure, in addition to the usual delays due to dynamics of the transfer functions between the component prices and the price index.

## 4 A New Model of Labor Dynamics

### 4.1 A Quantity Adjustment Model

Consider an economy composed of \( K \) sectors, and sector \( i \) employs \( n_i \) workers, \( i = 1, \ldots, K \). The variable \( n_i \) needs not be the number of employees in literal sense. It should be a variable that represents 'size' of the sector in some sense. For example it may be the number of lines in assembly lines, and so on. In this paper we assume that \( K \) is fixed. In another model with growth, \( K \) can change as sectors enter and exit. See Aoki (2002, Sec. 8).

Sectors are in one of two status; either in normal time or in overtime. That is, each sector has two capacity utilization regimes. In normal time, which is indicated by variable \( v_i = 0 \), \( n_i \) workers produce output

\[
Y_i = c_i n_i,
\]

for \( i = 1, 2, \ldots, K \), where \( c_i \) is the productivity coefficient, and \( n_i \) denotes the number of employees of sector \( i \). In overtime, indicated by variable \( v_i = 1 \), \( n_i \) workers produce output equal to

\[
Y_i = c_i (n_i + 1).
\]

The total output (GDP) is given by the sum of all sectors

\[
Y = \sum_{i=1}^{K} Y_i.
\]
Note that we keep the constant coefficient structure of the production function used in our earlier model. The important difference is the notion of normal or overtime, and a pool of unemployed weighted by ultrametric distance which is discussed later.

Demand for good $i$ is given by $s_i Y$, where $s_i$ is a positive share of the total output $Y$ which falls on sector $i$ goods, with $\sum_i s_i = 1$.

Each sector has the excess demand defined by

$$ f_i = s_i Y - Y_i, $$

for $i = 1, 2, \ldots, K$.

Changes in $Y$ due to changes in any one of the sectors affect the excess demands of all sectors. That is, there exists an externality between aggregate output and demands for goods of sectors. Changes in the patterns of $s$'s also affect these sets of excess demands.

The time evolution of the economy is modeled as a continuous-time Markov chain, as described in Aoki (1996, 2002), for example.

At each point in time, the sectors of economy belong to one of two subgroups; one composed of sectors with positive excess demands for their products, and the other of sectors with negative excess demands.

We denote the sets of sectors with positive and negative excess demands by $I_+ = \{ i : f_i \geq 0 \}$, and $I_- = \{ i : f_i < 0 \}$, respectively. These two groups are used as proxies for groups of profitable and unprofitable sectors, respectively. All profitable sectors wish to expand their production. All unprofitable sectors wish to contract their production.

A novel feature of our model is that only one sector succeeds in adjusting its production up or down by one unit of labor at any given time. The sector that has the shortest holding or sojourn time is the sector that jumps first, that is, only the sector that jumps first succeeds in implementing the desired adjustment. See Lawler (1995) or Aoki (2002, p. 28) for the notion of holding or sojourn time of a continuous-time Markov chain. We call that sector that jumps first as the active sector. Variables of the active sector are denoted with subscript $a$.

**Transition rates**

Dynamics of this continuous-time Markov chain are determined uniquely by the transition rates.

Sectors adjust their outputs by hiring or firing workers in response to the signs of excess demands which are used as proxies for profitability of the sectors. We assume that the economy has enough number of unemployed that sectors incur zero costs of firing or hiring, and do not hoard workers. To increase outputs the active sector calls back (one unit of ) worker from the pool of workers who were earlier laid-off by various sectors. The actual rehired worker is determined by a probabilistic mechanism discussed in the section on ultrametrics.

To implement a simple model dynamics we assume the following. Other arrangements of the detail of the model behavior is of course possible. Each sector has three state vector components: the number of employed, $n_i$, the
number of laid off workers, \( u_i \), and a binary variable \( v_i \), where \( v_i = 1 \) means that sector \( i \) is in overtime status producing \( c_i(n_i + 1) \) output with \( n_i \) employees. Sectors in overtime status all post one vacancy sign during overtime status. When one of the sectors in overtime status becomes active with positive excess demand, then, it actually hires one additional unit of labor and cancels the overtime sign. When a sector in overtime becomes active with negative excess demand, then it cancels the overtime and returns to normal time and vacancy sign is removed. When \( v_i = 0 \), sector \( i \) is in normal time producing \( c_i n_i \) output with \( n_i \) workers. When one of the sectors, sector \( i \) say, in normal time becomes active with positive excess demand, then it posts one vacancy sign and \( v_i \) changes into one. If this sector has negative excess demand when it becomes active, it fires one unit of labor. To summarize: When \( f_a < 0 \), \( n_a \) is reduced by one, and \( u_a \) is increased by one, that is one worker is immediately laid off. We also assume that \( v_a \) is reset to zero. When \( f_a \) is positive, we assume that it takes a while for the sector to hire one worker if it has not been in overtime status, i.e., \( v_a \) is not 1. If sector \( a \) had previously posted vacancy sign, then sector \( a \) now hires one worker, and cancels the vacancy sign, i.e., resets \( v_a \) to zero. If it has not previously posted a vacancy sign, then, it now posts a vacancy sign, i.e., sets \( v_a \) to 1, and increases its production with existing number \( n_a \) of workers by going into over-utilization state.

The transition path may be stated as \( s \rightarrow s' \), where \((n_a, u_a, v_a) = 0 \rightarrow (n_a, u_a, v_a) = 1 \), and \((n_a, u_a, v_a) = 1 \rightarrow (n_a + 1, u_a - 1, v_a) = 0 \). In either case the output of the active sector changes into \( Y_a' = Y_a + c_a \).

**Continuum of equilibria**

The equilibrium states of this model are such that all sectors are in normal time and have zero excess demands, that is,

\[
s_iY_e = c_in_i^e, \quad i = 1, 2, \ldots, K,
\]

where subscript \( e \) of \( Y \), and superscript \( e \) to \( n_i \) denote equilibrium values.

Denoting the total equilibrium employment by \( L_e = \sum_i n_i^e \), we have

\[
\left( \sum_i \frac{s_i}{c_i} \right) Y_e = L_e. \tag{2}
\]

This equation is the relation between the equilibrium level of GDP and that of employment. We see that this model has a continuum of equilibria.

**Transition to the closed set from initial states**

In simulations described below we start the model from over-employed states, that is, there are more \( n_i(0) \) to meet the demand \( s_i Y(0) \) at least for some, and possibly for all \( i \). Consequently such sectors, when they become active, start by firing employees. Eventually the number of employees become small enough to be compatible with the demands for the sectors. In other words, the Markov chain enter from transient states to the closed set of states, that
is, those states from which the model does not escape. These states are the ones in the business cycle. See Feller (1968, XV.8).

We next describe the variations in the outputs and employments in business cycles near one of the equilibria.

4.2 Okun's Law

Okun's law in the economic literature usually refers to changes in gross domestic products (GDP) and unemployment rates measured at two different time instants, such as one year apart. There may therefore be growth or decline in the economies.

To avoid confusing the issues about the relations between GDP and unemployment rates during stationary business cycle fluctuations, that is, without growth of GDP, and those with growth, we run our simulations in stationary states assuming no change in the numbers of sectors, productivity coefficients, or the total numbers of labor force in the model.

The Okun's law refers to a stable empirical relation between unemployment rates and rate of changes in GDP: one percent increase (decrease) in GDP corresponds to $x$ percent decrease (increase) in unemployment, where $x$ is about 4 in the United States. It is well known that if labor market is a homogeneous single market operating under neoclassical setup, then the Okun's law does not hold. This numerical value of $x$ is much larger than what one expects under the the standard neoclassical framework. Take, for example, the Cobb-Douglas production function with no technical progress factor. Then, GDP is given by $Y = K^{1-a}L^a$ with $a$ of about 0.7, where the total population is $N = L + U$ of which $U$ is the number of unemployed.

We have $\Delta U = -\Delta L$, where $\Delta K$ and $\Delta N$ are assumed to be negligible in the short run. The production function implies then that $\Delta Y/Y = \alpha \Delta L/L$ in the short run. That is, one percent decrease in $Y$ corresponds to an increase of $\Delta U/N = -\frac{1}{\alpha}(\Delta Y/Y)(1 - U/N)$, i.e., an increase of a little over 1 percent of unemployment rate. To obtain the number 4, as in the Okun's law, we need some other effects, such as increasing marginal product of labor or some other nonlinear effects. See Yoshikawa (2000). In the simulation studies of our new model we can obtain numbers larger than 4, depending on the configurations of demand shares and productivity coefficients as in (2), even though the linear production functions for all sectors in our model may lead us to expect numbers closer to 1.

We assume that economies fluctuate about its equilibrium state, and refer to the relation

$$\frac{\Delta Y}{Y_e} = -x \frac{\Delta U}{N},$$

as Okun's law, where $Y_e$ is the equilibrium level of GDP, approximated by the central value of the variations in $Y$ in simulation. Similarly, $\Delta U$ is the amplitude of the business cycle oscillation in the unemployed labor force, and $N$ is taken to be $L_e + U_e$, where $U_e$ is approximated by the central value of the oscillations in $U$, and $Y_e$ and $L_e$ are related by the equilibrium relation (2).

The changes $\Delta Y/Y$ and $\Delta U/U$ are read off from the scatter diagrams
in simulation after allowing for sufficient number of time to ensure that the
model is in "stationary " state.

In simulations described below we note that after a sufficient number of
time steps have elapased, the model is in or near the equilibrium distribution.
Then, $Y$ and $U$ are nearly linearly related with a negative slope i.e.,

$$\frac{\Delta Y}{Y} = -\beta \frac{\Delta U}{N \bar{Y}},$$

where $U$ is the total number of unemployed, so that $U/N$ is the fraction
of unemployed of the total number $N$ of workers. The total output $Y$ is
related to $N$ by $Y = \sum_i c_i n_i$. In the equilibrium we have $c_i n_i = s_i Y$ or
$n_i = (s_i/c_i)Y$, hence $Y = \bar{c}N$, with $(\bar{c})^{-1} = \sum_i s_i/c_i$, so that we may express
the Okun's law as $\Delta Y = -\beta \Delta U$, where $\beta = x\bar{c}$. That is, $x$ is estimated
by $\beta/\bar{c}$, where $\beta$ is estimated from the scatter diagram in "$Y - U$" plane.
This relation is not exactly true in business cycles, but may be used as an
approximation. Alternatively we can use the average GDP value together
with the average employed number to approximate $c$ in business cycles.

This indicates that the dynamics are indeed nonlinear. By changing the
initial conditions and observing how much time elapses before the model
enters the closed set, we observe the relation that the larger the shares on
more productive sectors the faster the model enters the stationary closed
sets.

4.3 Ultrametric trees

We endogenize job destructions and creation differently from Pissarides
(2000).

To present a simple model we ignore quits and on-the-job searches, and
assume that only unemployed get jobs.

In this model, sectors are differentiated with different 'distances' between
each other. These 'distances' reflect such factors as geographical differences,
differences in technology, and educational qualifications. Workers in different
sectors are different in job experiences and human capitals, and their
differences affect probabilities of being hired of different sectors by using the
concept of ultrametrics.

The stochastic process of filling vacancies of sector $i$ by unemployed
workers from the pool of sector $j$ depends on the ultrametric distance $d(i, j)$
between the two sectors of the economy.

Transitions of the active sector depends on the sign of the excess de-
mand, $f_a$ as indicated above. When $f_a < 0$, then one unit of labor is fired
immediately, and $n_a$ become $n_a - 1$ as indicated at the end of the previous
section.

Hiring a new unit occurs only with $f_a > 0$, and $v_a = 1$. Here we explain
how the active sector employs one additional unit of labor. We need to
distinguish $u_a$, which denotes the size of sector a's laid-off workers, from the
total pool of unemployed from which sector a randomly hires one unit of
labor. This pool is composed of $u_a$ and separate pools of laid-off workers
from sector $j$ $u_j$, $j \neq i$ suitably weighted by ultrametric distance. We denote
the latter by \( \tilde{u}_a \), that is \( u_a + \tilde{u}_a \) is the total size of the pool of the unemployed units of labor for sector \( a \).

These separate sub-pools are organized as a hierarchical tree with ultrametric distance.

The clusters or sub-pools of unemployed have different probabilities of being picked. The highest probability is for the pool of the workers who are laid-off from that sector. Its size is \( u_a \). This reflects the empirical observation that often firms recall laid-off workers first as they become profitable again. Then pools of laid-off workers from other sectors are arranged in increasing order of the ultrametric distance from the pool of size \( u_a \).

We illustrate this notion and its use in the case of \( K = 3 \) where sector 1 and 2 are at ultrametric distance 1, \( d(1,2) = d(2,1) = 1 \), and \( d(1,3) = d(2,3) = 2 \). Suppose that sector 1 is active. It draws from pools \( u_2 \) and \( u_3 \) after deflating them by \( 1 + d(1,2) \) and \( 1 + d(1,3) \) respectively. Thus a vacancy at sector 1 is filled from \( u_1 \) with probability

\[ \frac{u_1}{U_1}, \]

where

\[ U_1 = u_1 + \tilde{u}_1. \]

Similarly \( u_1 \) is reduced by one from pool of unemployed of sector 2 with probability

\[ \frac{u_2/[1 + d(1,2)]}{U_1}. \]

In this example vacancy in sector 1 will be filled from own laid-off pool with probability 6/11, from \( u_2 \) with probability 3/11, and from \( u_3 \) with probability 2/11. A vacancy in sector 3 will be filled from \( u_1 \) with probability 1/5, from \( u_2 \) also with probability 1/5, and from \( u_3 \) with probability 3/5.

**Simulation studies**

Our model behaves randomly because the jumping sectors are random due to holding times being randomly distributed. This is different from the models in the literature which behave randomly by the technology shocks which are exogenously imposed. Apparently, the model states have many basins of attractions each with near equal "potential energy" levels, much as spin glasses are.

Since the model is nonlinear and possibly possesses multiple equilibria, we use simulations to deduce some of the properties of the models. We pay attention to the phenomena of trade-offs between GDP and unemployment,
and the scatter diagrams of GDP vs. unemployment to gather information on business cycle behaviors.

A large number of simulations have been run.

A small set of graphs which show Okun's law phenomena are included. See the descriptions of the case studies in the appendix. See Aoki and Yoshikawa (2003) for detail and more graphs. Here, we summarize the simulations of the three cases, Case 1, 3, and 5. In the first two cases the same demand share vector \( s = (5, 4, 3, 2, 1, 1, 1, 1) / 18 \) is used. In Case 5, the share vector \( s = (3, 3, 4, 4, 1, 1, 2, 2) / 20 \) is used. In all cases the productivity coefficients are \( c_1 = 1 \), and \( c_8 = .225 \) with equally spaced decrease in between. In Case 1 and 3, about 78 per cent of the demands fall on the top 4 productive sectors. In Case 5 the top 4 sectors account for 70 per cent of demand. The sum \( \sum_i s_i / c_i = 1.3 \) in Case 1. In Case 5, the sum is 1.83. Case 1 and Case 3 uses different initial conditions. They appear to settle in different basins since Case 1 has a larger \( Y_e \) values than Case 3.

GDP values are in decreasing order from cases 1, 3, and 5. The number of unemployed is the largest in Case 3, then Case 5 and Case 1 in that order. The ratio of \( U_e / L_e \) are slightly higher in the Case 5. It is 2 per cent vs. 2.8 per cent.

The value of \( x \) is 3.5 in Case 1, 5.9 in Case 3 and 6.0 in Case 5. The values of \( x \) clearly depend on the basins of attractions, if the different starting points lead the model to different \( Y_e \) values. These figures are larger than Okun's. There are other empirical studies on the coefficients. For example, Hamada and Kurosaka (1984) examined the Japanese economy from 1953 to 1995, with the numbers ranging from 10.5 to 32, depending on the time spans and whether the economy was in high unemployment rate period or in low unemployment period.

In Case 1 and 3, the model reaches the closed set in about 600 time steps, while it takes about 750 steps in Case 5.

As a final remark we record that the simulations vary somewhat with the initial conditions.

We have demonstrated by simulation that higher percentages of demands falling on more productive sectors produce three new results: Average GDPs are higher; the Okun's coefficients \( x \) is larger; and transient responses are faster and in near stationary states, excursions of GDP (hence of \( U \)) are larger as well. In addition the amplitudes of the business cycles are higher as well.

It is remarkable that we obtain a relation like Okun's law with linear constant coefficient production functions. The fact that we obtain larger numbers than Okun may indicate the importance of capacity utilization as well as nonlinear interactions among sectors. Capacity utilization of capital is crudely incorporated in our model by the fact that sectors go into overtime under positive excess demand conditions before they can fill the vacancy.

Unlike the Cobb-Douglas production or linear production function which lead to \( x \) values of less than one, we observe the values well over 1 in our simulation. This indicates the importance of stochastic interactions among sectors introduced through the device of stochastic holding times.

We state the qualitative results of these simulation runs as follows:
1) larger shares of demands on more productive sectors result in larger average values of GDP.

2) Under the same circumstances, the systems with larger GDP reach 'near equilibrium' conditions faster.

3) Amplitudes of business cycles are larger, that is the amplitudes of GDP and unemployment rates are larger, the larger the average GDP.

4) Relations between unemployment and average GDP are described by a relation similar to the Okun's law.

Of these four qualitative conclusions, No.2 seems to be new and most interesting. In the existing literature this dynamic aspects of labor market characteristics has not been observed or commented on.

5 Innovation and Imitation Processes in the Long Run

Importance of innovation has received much attention in the context of Schumpeterian dynamics. See Aghion and Howitt (1992, 1997), Iwai (1997, 2001), and Aoki and Yoshikawa (2002) to cite a few recent contributions. To quote Iwai, for example,

The industry does not approach a neoclassical equilibrium of uniform technology in the long run, but at best a statistical equilibrium of technological disequilibria which maintain a relative dispersion of efficiencies in a statistical balanced form.

In this section we explicitly solve disequilibrium stochastic dynamics of a model with two types of firms, one type with innovations and the other with imitation but no innovation. We focus on the long-run behavior to show that two types of firms can coexist in the long run, and demonstrate clearly the role of innovation in the expected sizes and associated variances and covariance of two sectors.

We solve a disequilibrium backward Chapman-Kolmogorov equation by the method of cumulant generating function to derive the long-run stochastic equilibria. As pointed out by Iwai, this type of results refute the neoclassical notion of long-run equilibria.

The Model

Our model has two sectors; one technically advanced sector and the other less so. By a suitable choice of units we denote the sizes of the two sectors by a vector \((n_1, n_2)\). We may think of them as the total sizes of the two sectors in some suitably chosen standard units. Firms in sector one succeed in creating innovative firms at rate \(f\) which is, for simplicity, exogenously fixed in this model.\(^{11}\)

Firms' stochastic behavior is described by a continuous time Markov chain which is uniquely determined by a set of transition rates. We write the transition rate from state \(a\) to \(b\) by \(w(a, b)\). This means that the probability

\(^{11}\)We have examined other, more symmetric specifications as well.
that the system moves from state \(a\) to \(b\) in some small time interval is given by the time interval times the transition rates up to \(o(\text{time interval size})\).

One possible specification is as follows. The first two describe entry (growth) rates

\[
\begin{align*}
w\{(n_1, n_2), (n_1 + 1, n_2)\} &= c_1 n_1 + f, \\
w\{(n_1, n_2), (n_1, n_2 + 1)\} &= c_2 n_2.
\end{align*}
\]

Here \(c_i\) is the rate of growth of type \(i\) firm size, \(i = 1, 2\).

The next two specify exit rates from the model

\[
\begin{align*}
w\{(n_1, n_2), (n_1 - 1, n_2)\} &= d_1 n_1, \\
w\{(n_1, n_2), (n_1, n_2 - 1)\} &= d_2 n_2.
\end{align*}
\]

Here \(d_i\) is the exit (death) rate of type \(i\) firms from the economy, \(i = 1, 2\).

The last set of two transition rates describes how firms change their types

\[
w\{(n_1, n_2), (n_1 + 1, n_2 - 1)\} = \mu g_1 n_2(n_1 + h),
\]

with \(g_2 = c_2/d_2\), and \(h = f/c_1\), and

\[
w\{(n_1, n_2), (n_1 - 1, n_2 + 1)\} = \mu g_2 n_1 n_2,
\]

with \(g_i = c_i/d_i\), \(i = 1, 2\), and \(\mu = \lambda d_1 d_2\). This parameter \(\lambda\) is the coefficient in the transition rates of type changes by firms in the two sectors. The first of the two shows the rate at which one of type 1 firm becomes technologically obsolete and join the cluster made up of type 2 firms. The second equation specifies how firms of type 2 successfully imitate firms of type 1 and join their cluster. For example.

The stochastic dynamic equation is easy to state. It is a backward Chapman-Kolmogorov equation, also known as the master equation. (We use the latter name as it is short, and implies that everything you need to know about stochastic behavior is implicit in the master equation.)

\[
\frac{\partial P(n_1, n_2; t)}{\partial t} = I(n_1, n_2; t) - O(n_1, n_2; t), \tag{3}
\]

where the first term collects all inflows of probability flux into state \((n_1, n_2)\), and the second term collects all outflows of probability fluxes out of this state. There are six distinct flows. In detail we have

\[
I(n_1, n_2; t) = P(n_1 + 1, n_2; t)d_1(n_1 + 1) + P(n_1, n_2 + 1; t)d_2(n_2 + 1)
+ P(n_1 - 1, n_2; t)c_1(n_1 - 1 + h) + P(n_1, n_2 - 1)c_2(n_2 - 1)
+ P(n_1 + 1, n_2 - 1; t)\mu g_2(n_1 + 1)(n_2 - 1) + P(n_1 - 1, n_2 + 1; t)\mu g_1(n_1 - 1 + h)(n_2 + 1).
\]

The second term in (1) is given by

\[
O(n_1, n_2; t) = P(n_1, n_2; t)\{c_1 + f + c_2 n_2 + d_1 n_1 + d_2 n_2 + \mu g_1 n_2(n_1 + h) + \mu g_2 n_1 n_2\}.
\]

To solve the master equation, we first convert it into the probability generating function

\[
G(z_1, z_2; t) = \sum_{n_1, n_2} P(n_1, n_2; t)z_1^{n_1}z_2^{n_2}.
\]
We obtain a partial differential equation for \( G(z_1, z_2; t) \). It is given in Appendix. This partial differential equation is rather intractable, and for that reason we convert it into the cumulant generating function and solve for the expected values of first and second moments.\(^{12}\)

Cumulant generating functions are related to the probability generating functions by

\[
K(\theta_1, \theta_2; t) = \ln G(e^{-\theta_1}, e^{-\theta_2}),
\]

where we change variables from \( z_1 \), and \( z_2 \) into \( \theta_1 \) and \( \theta_2 \).

It is known that the cumulant generating function has a Taylor series expansion of the form

\[
K(\theta_1, \theta_2; t) = k_1 \theta_1 + k_2 \theta_2 + \frac{1}{2}(\theta_1, \theta_2)\Theta(\theta_1, \theta_2)' + \cdots,
\]

where \( k_1 = E(n_1) \), and \( k_2 = E(n_2) \), that is, they are the expected sizes of the two types, and where \( \Theta \) is a covariance matrix made up of the variances and covariances of the two sizes,

\[
\Theta = \begin{pmatrix}
  k_{1,1} & k_{1,2} \\
  k_{1,2} & k_{2,2}
\end{pmatrix}.
\]

See Aoki (2002, Chapt. 7) for further information on these generating functions, and some simple examples.

From the cumulant generating functions we derive a set of five ordinary differential equations for \( k_1, k_2, k_{1,1}, k_{1,2}, \) and \( k_{2,2} \).

Appendix gives the explicit expressions.

### 5.1 Stationary Means and Variances

The equations for the two means (average sizes of the two sectors) are:

\[
\frac{dk_1}{dt} = f - d_1(1 - g_1)k_1 + \lambda f d_2 k_2 + 2\alpha \lambda A_0,
\]  

(4)

and

\[
\frac{dk_2}{dt} = -d_2(1 - g_2 + \lambda f)k_2 - 2\alpha \lambda A_0,
\]  

(5)

where \( \lambda = \mu/d_1 d_2 \), where \( g_i = c_i/d_i \), \( A_0 = k_{1,2} + k_1 k_2 \), and \( 2\alpha = d_1 d_2 (g_1 - g_2) \). Note that \( A_0 =< n_1 n_2 >_0 \geq 0 \).

Since \( A_0 \) depends on \( k_{1,2} \) we need solve for it as well.

Stationary means are described by setting the left-hand sides of (2) and (3) to zero:

\[
f - d_1(1 - g_1)k_1 + \lambda f d_2 k_2 + 2\alpha \lambda A_0 = 0,
\]  

(6)

\[
-d_2(1 - g_2 + \lambda f)k_2 - 2\alpha \lambda A_0 = 0.
\]  

(7)

By adding (6) and (7) to express an important relation between \( f \), \( k_1 \) and \( k_2 \)

\[
f = d_1(1 - g_1)k_1 + d_2(1 - g_2)k_2.
\]  

(8)

\(^{12}\)In some cases the resulting ordinary differential equations for the moments turn out to be an infinite set of coupled ordinary differential equation. Fortunately, the differential equations for the first and second cumulants are self-contained in this model.
This equation clearly shows that a fraction of innovation flow accounts for the new firms in sector 1, and the rest accounts for the net exit flow of firms from sector 2. We later show that the expected value of the stationary values of the size of sector 1 scales with $\gamma_1 := \lambda d_1$, and that of sector 2 scales with $\gamma_2 := \lambda d_2$.

Recalling the definition $d_i(1 - g_i) = d_i - c_i$, we see that the rate of innovation $f$ equals the sum of the expected exit rates of firms of both sectors, since $(d_i - c_i) k_i$ is the net exit rate of firms of sector $i$, $i = 1, 2$.

In this model it is necessary that both $g_1$ and $g_2$ are not greater than one. There are two cases: $g_1 > g_2$ and $g_2 > g_1$. In the former $\alpha > 0$ and $\alpha < 0$ in the latter. Eqs. (6) and (8) exclude the case $g_1 > g_2$. In the other cases, we could have $g_2 > 1 > g_1$ or $1 > g_2 > g_1$.

To reflect these consideration we introduce two parameters $m$ and $n$\(^{13}\) by

$$g_1 = 1 - n \gamma,$$

and

$$g_2 = 1 - m \gamma,$$

where $\gamma = \lambda f$. The constants are bounded by

$$\frac{1}{\gamma} > n > 0,$$

and

$$n > m > -1.$$

The inequality $m > -1$ is derived from (7).

Solving (4) and (5), we obtain the means of the sizes of the two sectors

$$k_1 = \frac{1 - \theta}{n \gamma_1},$$

with

$$\theta := \frac{m(n - m)}{1 + m} d_1 d_2 \lambda^2 A_0,$$

and

$$k_2 = \frac{\theta}{m \gamma_2}.$$

The stationary variances $k_{1,1}$ and $k_{2,2}$ are derived in Appendix; The covariance $k_{1,2}$ is expressed in terms of $\theta$ through the definition $k_{1,2} = A_0 - k_1 k_2$ as

$$k_{1,2} = \frac{1}{\gamma_1 \gamma_2 n m (n - m)} \theta [(1 + n) m + (n - m) \theta].$$

What remains is to determine $\theta$. The self-consistent equation for $\theta$ is derived in Appendix. Although the equation is a fifth order equation of $\theta$ because of the five unknown quantities, the highest term vanishes so that

$$\theta F(\theta) = 0$$

\(^{13}\) not to be confused with $n_1$ or $n_2$
where

\[ F(\theta) = r_0 + r_1 \theta + r_2 \theta^2 + r_3 \theta^3. \]  

(12)

See Aoki, Nakano, and Yoshida for the explicit expression of \( r_i \).

The root \( \theta = 0 \) is of interest because this value of \( \theta \) yields a stationary state in which sector 2 vanishes, \( k_2 = 0, k_{2,2} = 0 \), and \( k_{12} = 0 \).

We expect the innovative sector is larger. Depending on the parameter choices it could be smaller than the imitative sector.

From \( F(\theta) = 0 \) we obtain three values of \( \theta \). The roots must be such that \( \theta \) is real and the obtained values of \( k_1, k_2, k_{1,1} \) and \( k_{2,2} \) are positive; \( k_{1,2} \) is not necessarily positive. Although the analytic solutions may be obtained for special set of parameters, such solutions are not possible in general.

Mathematica, however, enables us to numerically solve \( F(\theta) = 0 \). In order that those solutions exist in reality, the solutions must be the stable fixed points.

As an example we describe in detail the case where \( m = .01; n = 2, \gamma = \gamma_1, \) and \( \gamma_2 = \gamma_1 + \epsilon \), with a small positive \( \epsilon \). In this case there is only one root for which the dynamics are locally stable. It is given by \( \theta = 0.472 \).

The stability of the stationary states is examined in the following way. The starting equations are (4), (5) and (13), (14) and (15) in Appendix. By setting the left hand sides of those equations we have the stationary values, which are confirmed to numerically coincide with the solutions from \( \theta F(\theta) = 0 \). Then the linearized equations for deviations \( \delta k_1, \delta k_2, \delta k_{11}, \delta k_{22} \) and \( \delta k_{12} \) from the stationary values are derived. The eigenvalues of those equations are numerically calculated with a help of Mathematica. If real parts of all five eigenvalues associated with a stationary point are negative, the stationary point is stable.

The value of \( \theta = .472 \) corresponds to a locally stable solution. This leads to

\[ k_1 = \frac{.264}{\gamma_1}, \quad k_2 = \frac{47.2}{\gamma_2}. \]

From (5) we obtain

\[ A_0 = \frac{\theta}{\gamma_1 \gamma_2} \frac{1 + m}{m(n - m)} = \frac{23.956}{\gamma_1 \gamma_2}. \]

From this we derive

\[ k_{1,2} = A_0 - k_1 k_2 = \frac{11.495}{\gamma_1 \gamma_2}. \]

From (9) through (11) we can obtain approximate order of magnitude values for the second moments \( k_{1,1} \) and \( k_{2,2} \) as follows.

\[ k_{1,1} = \frac{C_{1,1}}{\gamma_1^2}, \]

with

\[ C_{1,1} \approx \frac{1}{n^2} (1 + \theta^2) \approx .31, \]

\[ \gamma_1 \approx .01, \quad \gamma_2 \approx .06, \quad \theta \approx .472. \]
which is close to .309 obtained in the numerical example below, and

\[ k_{2,2} = \frac{C_{2,2}}{\gamma_2}, \]

with

\[ C_{2,2} \approx \frac{1}{m^2} \left[ \theta^2 + \frac{m(1+n)}{n} \theta \right] = 2300. \]

We also have an approximate expression for \( k_{12} \) as \( C_{12}/\gamma_1 \gamma_2 \) with

\[ C_{12} = \frac{\theta^2}{mn} \approx 11.15, \]

which is in good agreement with the value obtained above as 11.495.

### 5.2 Numerical Examples

We focus on a stationary solutions. Since there are five parameters, we have many solutions.

To keep the sizes of the two sectors at reasonable values, we examine cases with the death rates close to the birth rates. Namely, we choose \( g_i \) to be close to unity. Previously we have indicated that \( g_2 \) can be either larger than one or smaller than one, while \( g_1 \) is always less than one. First, we consider the case that the death rate \( d_i \) is slightly larger than the birth rate \( c_i \), so that \( n \gamma, m \gamma \ll 1 \). Although the death rate of sector 2 is considered to be larger than that of sector 1, we assume that both are almost the same. We focus on the following parameters; \( \gamma = \gamma_1 = \gamma_2 = 0.01, n = 2.0, m = 0.01 \). Then we have three types of solutions; (1) \( k_1 = 50, k_{11} = 2500, k_2 = k_{22} = k_{12} = 0 \), (2) \( k_1 = 49.97, k_2 = 4.77, k_{11} = 2501, k_{22} = 46505, k_{12} = 3.71 \) and (3) \( k_1 = 26.4, k_2 = 4719, k_{11} = 3093, k_{22} = 2.37 \times 10^7, k_{12} = 114918 \). The stable solution is only the first type; only sector 1 survives. The second and third types are not stable. If we increase \( \gamma_2 \) slightly to \( \gamma_2 = 0.011 \), a remarkable change occurs in the type 3 solution. The numbers for (1) are the same as the previous case. On the other hand, (2) \( k_1 = 49.977, k_2 = 4.122, k_{1,1} = 2501, k_{2,2} = 40232, k_{1,2} = 3.2 \) and (3) \( k_1 = 23.94, k_2 = 4738, k_{1,1} = 3216.7, k_{2,2} = 2.378 \times 10^7, k_{1,2} = 127042 \). The second solution is not stable, but the third solution turns out to be stable in this case.

We vary a value of \( \gamma_2 \) with other parameters fixed. We found that the stable fixed point exists in a narrow range such that \( 0.02 \leq \gamma_2 \leq 0.0102 \).

What parameters are chosen to increase the number of companies? For that purpose we should decrease \( \gamma, \gamma_1, \gamma_2 \). When \( n = 2.0 \) and \( m = 0.01 \) are fixed, we employ \( \gamma = \gamma_1 = 0.001, \gamma_2 = 0.0011 \). Then we have the stable third solution \( k_1 = 264, k_2 = 47193 \) with the correlation coefficient \( k_{1,2}/\sqrt{k_{1,1}k_{2,2}} = 0.42 \). In other sets of parameters with \( n \) and \( m \) fixed at the above values, \( \gamma = \gamma_1 \) and \( \gamma_2 \) being slightly larger than \( \gamma \), we have the following scaling relation

\[ k_1 = \frac{0.264}{\gamma_1}, \quad k_2 = \frac{47.2}{\gamma_2}, \quad k_{1,1} = \frac{0.309}{\gamma^2}, \quad k_{2,2} = \frac{2374}{\gamma^2}, \quad k_{1,2} = \frac{11.5}{\gamma^2}. \]
The correlation coefficient is 0.42.

The coefficient of variations are 2.11 and 1.03 for the two sectors respectively. We also note that with $\gamma_2$ nearly the same as $\gamma_1$ only 0.6 percent of the total sizes of the capital resides in sector 1.

Next we examine negative values of $m$. Take $m = -0.01$ while keeping the values of the other parameters the same as before. The numerical calculation gives a negative value of $k_2$. Although we have not done an extensive study, a negative value of $m$, i.e., $g_2$ is larger than one, may not yield stable stationary situations.

\section{Concluding Remarks}

It is not possible to extensively discuss the implications of our proposed approaches in a single paper. We hope, however, that the examples in this paper give the reader better perspectives of our approach, and encourage some readers to re-think their modeling strategies.

\section{References}


8 Appendix

This appendix describes a microeconomic foundation for the analysis of Section 2. In particular, it explains how \( g(x) \) and \( \beta \) are obtained and why \( \beta \) is a measure of uncertainty.\(^{14}\)

Consider a situation where each of \( N \) microeconomic agents faces a binary choice.\(^{15}\)

Denote by \( V_1(x) \) the return over some time span of choice 1, when fraction \( x \) of agents have chosen 1. For definiteness we may think of the discounted present value of benefits or profit streams, assuming that fraction \( x \) remains the same over some planning horizon, and the discount rate is known.

The conditional mean of difference is \( g(x) = E[V_1 - V_2|x] \), and conditional variance of the difference is \( \sigma^2(x) \). Here, for simpler explanation we assume that \( \sigma \) is a constant. In other words, we use the first two moments of the random variable \( V_1 - V_2 \) to describe how agents choose.

Given the two moments we approximate the distribution of \( V_1 - V_2 \) by a normal distribution with the same two moments.

Thus, we have

\[
\eta(x) = \Pr(V_1 - V_2|x) \approx (1/2)[1 + \text{erf}(x)],
\]

where

\[
\text{erf}(u) := \frac{2}{\sqrt{\pi}} \int_0^u e^{-y^2} dy,
\]

where \( u = (1/\sqrt{2\pi})g(x)/\sigma \). See Abramovitz and Stegun (1968) for example. Note that the expression \( g(x)/\sigma = 1/CV \), where \( CV \) is the coefficient of variation.

Next, we follow the recommendation by Ingber (1982) to approximate the error function by a hyperbolic tangent function

\[
\text{erf}(u) \approx \text{tanh}(\kappa u),
\]

\(^{14}\)This appendix is based in part on Chapter 6 of Aoki (2002).

\(^{15}\)More generally, each agent chooses one out of \( K \) alternative decisions. Without loss of insights, we describe a binary choice situation here.
with \( \kappa = 2/\sqrt{\pi} \).

We see that this approximation is very good for small \( u \) by comparing their Taylor series expansion:

\[
\text{erf}(u) = \kappa(u - \frac{u^3}{3} + \frac{u^5}{5} - \cdots),
\]

and

\[
tanh(\kappa u) = \kappa(u - \frac{u^3}{2.36} + \frac{u^5}{4.67} - \cdots).
\]

We define

\[
\beta = \frac{\sqrt{2}}{\pi \sigma}.
\]

Then,

\[
\eta(x) \approx \frac{e^{\beta g(x)}}{X},
\]

with \( X = e^{\beta g(x)} + e^{-\beta g(x)} \).

Note that \( \beta g(x) = (\sqrt{2}/\pi)CV^{-1} \). The larger the value of \( CV \), the smaller the value of \( u \), and more accurate is the approximation of the error function by the \( \tanh \) function.

The probability generating function

With only a scalar random variable \( X \), its probability generating function is defined by \( G(z, t) = E(z^X) = \sum_k z^k P(k, t) \). Its partial differential equation is obtained by noting that that

\[
\sum_k z^k P(k - 1, t) = zG(z, t),
\]

\[
\sum_{k=1}^{\infty} (k + 1)z^k P(k + 1, t) = \frac{\partial G(z, t)}{\partial z},
\]

\[
\sum_{k=1}^{\infty} k z^k P(k, t) = z\frac{\partial G(z, t)}{\partial z},
\]

and

\[
\sum_{k=1}^{\infty} (k - 1)z^k P(k - 1, t) = z^2 \frac{\partial^2 G}{\partial z^2}.
\]

With two state variables \( n_1 \) and \( n_2 \), similar relations. The result is

\[
\frac{\partial G}{\partial t} = [d_1(1 - z_1) + c_1 z_1(z_1 - 1) + \mu_2(z_2 - z_1)] \frac{\partial G}{\partial z_1}
\]

\[
+ [d_2(1 - z_2) + c_2 z_2(z_2 - 1) + \mu_1 h(z_1 - z_2)] \frac{\partial G}{\partial z_2}
\]

\[
+ [\mu_1 z_1(z_1 - z_2) + \mu_2 z_2(z_2 - z_1)] \frac{\partial^2 G}{\partial z_1 \partial z_2} + f(z_1 - 1)G.
\]

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The cumulant generating function

Noting that
\[ \frac{\partial G}{\partial t} = G \frac{\partial K}{\partial t}, \]
\[ \frac{\partial G}{\partial z_i} = -G e^{\theta_i} \frac{\partial K}{\partial \theta_i}, \]

\( i = 1, 2, \) and
\[ \frac{\partial^2 G}{\partial z_1 \partial z_2} = G e^{\theta_1 + \theta_2} H \]

with
\[ H = \frac{\partial K}{\partial \theta_1} \frac{\partial K}{\partial \theta_2} + \frac{\partial^2 K}{\partial \theta_1 \partial \theta_2}, \]

we convert the partial differential equation for \( G \) into that for \( K \)
\[ \frac{\partial K}{\partial t} = \frac{1}{G} \frac{\partial G}{\partial t} = -\sum_{i=1}^{2} [d_i(e^{\theta_i} - 1) + c_i(e^{-\theta_i} - 1)] \frac{\partial K}{\partial \theta_i} + f(e^{-\theta_1} - 1) + \mu [g_1(e^{(\theta_2 - \theta_1)} - 1) + g_2(e^{(\theta_1 - \theta_2)} - 1)] H. \]

We then extract coefficients of \( \theta_i \) and equate them to \( dk_i/dt, i = 1, 2, \) and those of \( \theta_1^2, \theta_2^2 \) with the derivatives \( dk_{1,1}/dt \) and \( dk_{2,2}/dt \), and the coefficient of \( \theta_1 \theta_2 \) with the derivative \( dk_{1,2}/dt \).

In this way we generate a set of five differential equations for \( k_1, k_2, k_{1,1}, k_{2,2}, \) and \( k_{1,2} \).

Calculations of the variances and covariance

The equations for the variance and covariance are derived as follows:

\[ \dot{k}_{1,1} = f - 2d_1(1 - g_1)k_{11} + d_1(1 + g_1)k_1 + ld_2 f(2k_{1,2} + k_2) + 4\alpha l(k_1 k_{1,2} + k_2 k_{1,1}) / 2\beta l A_0, \]  
(13)

\[ \dot{k}_{2,2} = -2d_2(1 - g_2 + \lambda f)k_{2,2} + d_2(1 + g_2 + \lambda f)k_2 - 4\alpha l(k_{1,2} + k_{2,2}) / 2\beta l A_0, \]  
(14)

\[ \dot{k}_{1,2} = -[d_1(1 - g_1) + d_2(1 - g_2 + \lambda f)]k_{1,2} + ld_2 f(k_{2,2} - k_2) - 2\alpha l(k_1 k_{1,2} + k_2 k_{1,1} - k_1 k_{2,2} - k_2 k_{1,2}) / 2\beta l A_0, \]  
(15)

where \( \beta = d_1 d_2 (g_1 + g_2)/2 \). Stationary values of variances \( k_{11} \) and \( k_{22} \) are obtained by setting the left hand sides of (13) and (14) equal to zero.

Self consistent values of \( \theta \)

Substituting (13) for \( k_{1,1} \), (15) for \( k_{2,2} \) and

\[ k_{1,2} = A_0 - k_1 k_2 = \frac{1}{\gamma_1 \gamma_2 \gamma_2 (n - m)} \theta \left[ 1 - \frac{n - m}{n(1 + m)} (1 - \theta) \right] \]
into the equation which is derived by setting the left hand side of (15) equal to zero yields the fifth order equation for $\theta$. Luckily, however, the highest term vanishes, so that the equation becomes quartic;

$$\theta F(\theta) = 0$$  \hspace{1cm} (16)

where

$$F(\theta) = r_0 + r_1 \theta + r_2 \theta^2 + r_3 \theta^3.$$  \hspace{1cm} (17)