A Model of Firm Formation and Skills Acquisition

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Abstract: We present a model of production that begins with an exogenously specified set of technologies, accessible to each potential firm. The choice of technologies and organizational structures occurring in equilibrium are endogenous. Labor skills are differentiated, and acquired endogenously by workers—possibly by bearing private costs, and possibly by attending school. Each technology can be used by a group of agents having the appropriate skills. We allow for the possibility that workers care about various aspects of their jobs (including working conditions and production plans), so that compensating differentials may exist. In a continuum model, we show that pricing goods and jobs guarantees the existence of price-taking equilibrium and the equivalence of price-taking equilibrium outcomes with core outcomes.

Keywords: clubs, production, skills, schools, groups
1 Introduction

The usual general equilibrium model of production (see Mas-Colell, Whinston and Green (1995) for instance) takes as given an exogenous set of firms, described in terms of an exogenously given production function or production possibility set. When labor is an input to production, the productivity of workers is also given exogenously. In this paper we build a model in which technological possibilities are given exogenously, but firms and the skills of workers are determined endogenously at equilibrium.

We view a technology as a means of converting inputs to outputs, and we view a firm as a group of agents employing a technology. Employing a given technology requires a particular mix of skills, so some groups of agents will not be able to use some technologies. Thus there will be competition for agents with useful skills. Because skills give access to technologies, agents have incentives to acquire skills, but the acquisition of skills may be costly and unpleasant, and there may be competition for the means by which skills are acquired. This simultaneous problem of skill acquisition and firm formation is the heart of this paper.

Given our notion of group membership, we define a price-taking notion of equilibrium and establish its fundamental properties. In particular, we show that equilibrium exists, that equilibrium states are Pareto optimal, and that our equilibrium notion passes a familiar test of competition: coincidence of the core with the set of equilibrium states. As the reader may surmise, we work in a model with a continuum of agents. Because group memberships are indivisible choices, equilibrium will generally not exist in an economy with a finite number of agents. And of course, economies with a finite number of agents will typically not be competitive.

Our model rests on the notion of a productive group — more properly, group type. A group type is defined by the set of roles that must be filled, the skills of the agents required to fill these roles, infrastructure and organization, and the production plan that the skills, infrastructure, and organization enable. The model here parallels rather closely the clubs model of our earlier work (1999a,b); remarkably few alterations need be made to accommodate the necessary extensions:
In our earlier work, the description of a club included the vector of inputs necessary for formation of the club; here, the description of a group includes the vector of inputs necessary for formation of the club and the vector of outputs the group produces. Formally, our earlier description of a club included a vector $y \in -\mathbb{R}_{+}^{N}$; the present description of a group includes a vector $y \in \mathbb{R}^{N}$, negative components representing inputs and positive components representing outputs.\(^1\)

In our earlier work, we treated all club members (having a particular external characteristic) symmetrically; here we allow for different members to fill different roles. This seems especially important in the present context, in which we address both production and the acquisition of skills: the building contractor oversees many subcontractors — skilled carpenters, masons, plumbers, electricians — all of whom have learned their skills in different places and perform different tasks. Formally, allowing for roles requires only a trivial expansion in the description of a group and a corresponding change in the description of membership in a group.

In our earlier work, we treated the characteristics of agents as given exogenously; here we treat (at least some of) these characteristics as chosen endogenously. Incorporating this endogenous choice of characteristics is subtle but not difficult (at least formally). Skills can be acquired by private study, by attending school, by apprenticing oneself to another who has already mastered the skill (and other forms of learning-by-doing) — and by combinations of these routes. The different means of acquiring skills may each involve psychic costs and benefits (the pain or pleasure of training), material costs and benefits (fees and subsidies for schools and apprenticeships), and personal costs (expenses for books and uniforms). We model the psychic costs by allowing group memberships to enter individual utility functions, fees and subsidies as entry prices into groups (determined endogenously at equilibrium), and personal costs by an exogenously given cost function.

An important feature of our earlier work is absolutely crucial here: we allow individuals to belong to several groups simultaneously. Thus an individual

\(^{1}\)Our earlier work used the opposite sign convention.
might join a particular construction firm as an apprentice contractor in order to learn the skills necessary to be the general contractor in another firm.

This might seem a little strange because it is natural to think of the acquisition of skills as preceding the application of those skills, rather than simultaneous with them. Joining a construction firm as an apprentice contractor is valuable (in part) because it teaches the skills of a general contractor — but the apprentice today will only be a general contractor in the future. Because our model is atemporal, like most of general equilibrium theory, we do not make this distinction in a formal way, but it is easily accommodated by the usual general equilibrium device of viewing dates as part of the description of commodities, groups, and individual characteristics. Thus the individual who apprentices today and becomes a general contractor in the future joins today’s firm as an apprentice, acquires the skills of a future general contractor, and joins the future firm as a general contractor. Uncertainty can be accommodated in the same way. (Assuming in both cases that markets are complete.)

As in our earlier work, we consider price systems that encompass both prices for private goods and prices for group membership. The latter prices might variously be interpreted as cost sharing, as entry fees, or as wages. An important feature of these prices for group membership is that all aspects of group membership are priced. Although this might seem odd at first sight, it is entirely natural. We expect the foreman to command higher wages than the line worker — so group membership prices should depend on roles. We expect that a firm which offers poor working conditions or unpleasant (kinds of) co-workers or noxious production methods will be forced by the marketplace to pay higher wages than competing firms — so group membership prices should depend on the infrastructure of the group and on the membership of the group and on the production plan of the firm. Group membership prices thus capture many familiar compensating differentials. Note, however, that we do not allow prices to depend on individual names, so our equilibrium notion is not like Lindahl equilibrium.

Because we allow group membership prices to depend on all aspects of group membership, the price space is very large. In general, this is a necessity — equilibrium will not exist with any smaller price space. However, it is of considerable interest to know circumstances in which prices are inde-
pendent of various aspects of group membership. In particular, we identify circumstances in which worker’s wages depend only on their skills, and not on (the attributes of) the firms in which they work.

Various kinds of groups — social clubs, schools, firms — come naturally to mind. Our model makes no formal distinction between these kinds of groups — intentionally, because we think such distinction is misleading, and the dividing lines between social clubs, schools, and firms are blurry at best. Agents belong to groups for a variety of reasons — for pleasure or for training or for profit (and perhaps for other reasons), and it might be tempting to classify groups as social clubs or as schools or as firms, according to the purpose for which agents belong. But different individuals might belong to a given group for different purposes: students attend school for training, but a faculty position is (at least to some extent) a job. And a single individual may belong to a group for a variety of reasons: some of us like our jobs and view them as training for future careers and derive income from them. Alternatively, one might classify social clubs as groups which require no inputs and produce no output, schools as groups which require inputs but produce no output, and firms as groups which produce positive output. But trade and vocational schools frequently produce a marketed output (although that might be a by-product of the training which is their most important activity), and even the Thursday night poker club requires inputs: chips, beer, a kitchen table. Finally, one might identify as a firm any group that makes a profit — defined to be the value of outputs less the cost of inputs. But both the value of output and the cost of inputs depend on equilibrium prices, so this view would identify a quilting group as a social club when the market price of quilts is low (so that the group makes a loss) but as a firm when the market price of quilts is high (so that the group makes a profit). (And what of the Beardstown ladies who formed a stock-picking club and then wrote a book about their experiences?)

Our model of production has much in common with Keiding (1973), who treats exogenous technologies and differentiated labor in a continuum model. In Keiding’s framework, private goods are sold in a competitive market, but firms are formed cooperatively. The central result is that core states can be decentralized by competitive prices. However, in Keiding’s model labor skills are exogenously given and immutable, and agents have access to a single technology, rather than to several technologies, so there is no question of the
acquisition of skills and no problem of matching agents consistently in groups. Drèze (1989) also has a concept of labor management in equilibrium, where any group of agents has access to an exogenously given technology. Labor is divisible and differentiated, but again there is no matching problem. Agents do not have preferences for working in specific firms, apart from its affect on income.

Our model of production is also in the spirit of coalition production models (see Ichiishi (1993), for instance) in that production is accomplished by particular groups of agents. However, in the usual coalition production models, the production possibilities are given exogenously for each set of agents, whereas in our model it is only the technologies which are given exogenously; the acquisition of skills and the formation of productive groups is endogenous.

Our formalization of a group type has much in common with the competitive theory of organizations proposed by Louis Makowski (1978) in an unpublished paper.

Following this Introduction, Section 2 sets forth the basics of our model, and the notions of feasibility and equilibrium. Section 3 presents our main results: existence and optimality of equilibrium and equivalence of equilibrium states and core states. Section 4 describes simple circumstances in which group membership prices do not depend on all aspects of group memberships. Finally, Section 5 presents a number of illustrative examples.
2 General Equilibrium with Productive Groups

We extend our (1999a) club model to allow for productive groups, endogenous choice of characteristics, and roles. To avoid confusion, we use “groups” and “group types” rather than “clubs” and “club types.”

2.1 Private goods

There are $N \geq 1$ perfectly divisible private goods.\(^2\)

2.2 External characteristics

Let $\Omega$ be a finite set of external characteristics of agents. In keeping with our previous discussion, these external characteristics might be skills (a college education) or groups of skills (a college education and a teaching certificate). More generally, these external characteristics might be anything which matters to other agents or matters in production. We do insist that these characteristics be public information — hence our terminology.

We normally think of some characteristics as acquired (education, training) and some as innate (height, eye color), but our model treats all characteristics as acquired. However, this is merely a matter of modeling convenience, since the description of an agent will include the collection of characteristics that agent might acquire. Thus, the fact that a short brown-eyed agent cannot become a tall blue-eyed agent is modeled by restricting the set of characteristics such an agent can acquire. See Subsection 2.4 for further discussion.

2.3 Groups, group types, and memberships

Groups are described in reference to an exogenous set of group types. To define group types, let $\Gamma$ be an abstract, finite set of activities. A group type is a 4-tuple $(R, \pi, \gamma, y)$ where

\(^2\)There would be no special difficulties in allowing for some indivisible private goods.
• $R$ is a non-empty finite set of roles

• $\pi : R \times \Omega \to \mathbb{Z}_+ = \{0, 1, \ldots\}$ is a profile, which specifies how many agents having each characteristic are required in each role; we assume $\sum \pi(r, \omega) \geq 1$ so that each group type has at least one member\(^3\)

• $\gamma \in \Gamma$ is an activity

• $y \in \mathbb{R}^N$ is a production vector; as usual, we interpret the negative part $y^-$ as inputs and the positive part $y^+$ as outputs

(For instance, a construction group type might require 1 supervisor with a college education, 4 carpenters with trade school degrees, 2 apprentices currently in trade school, take as inputs the raw materials of a house and produce as output 1 finished house.) We take as given a finite set of possible group types $\mathcal{G} = \{(R, \pi, \gamma, y)\}$. For $g \in \mathcal{G}$, we sometimes find it convenient to write

$$g = (R_g, \pi_g, \gamma_g, y_g)$$

A membership is an opening in a particular group type for an agent having a particular external characteristic, so specifies a characteristic, and a role, and a group type consistent with that characteristic and role:

$$m = (r, \omega, (R, \pi, \gamma, y))$$

such that

$$\begin{aligned}
(R, \pi, \gamma, y) &\in \mathcal{G} \\
\pi(r, \omega) &\geq 1
\end{aligned}$$

Write $\mathcal{M}$ for the (finite) set of memberships. For $m \in \mathcal{M}$ we sometimes find it convenient to write

$$m = (r_m, \omega_m, (R_m, \pi_m, \gamma_m, y_m))$$

Each agent may choose many group memberships or none. A membership list is a function $\ell : \mathcal{M} \to \mathbb{Z}_+ = \{0, 1, \ldots\}$; $\ell(m)$ specifies the number of memberships of type $m$ that are chosen.

\(^3\)In our earlier work on clubs, we required that club types have at least two members, but because we allow for production it is natural here to allow for group types which have a single member.
2.4 Agents

The set of agents is a finite nonatomic measure space \( (A, \mathcal{F}, \lambda) \); i.e., \( A \) is a set, \( \mathcal{F} \) is a \( \sigma \)-algebra of subsets of \( A \) and \( \lambda \) is a non-atomic measure on \( \mathcal{F} \) with \( \lambda(A) < \infty \).

A complete description of an agent \( a \in A \) consists of a consumption set, an endowment of private goods, a utility function and a cost function which gives the personal cost of acquiring various characteristics:

- Agent \( a \)'s consumption set \( X_a \) specifies the feasible bundles of private goods and feasible lists of memberships that the agent may choose. For simplicity, we assume that the only restriction on private good consumption is non-negativity, so for each agent \( a \) there is a set of lists \( \text{Lists}(a) \) such that \( X_a = \mathbb{R}_+^N \times \text{Lists}(a) \). Agents may choose no lists, so \( 0 \in \text{Lists}(a) \). We assume that the number of memberships that each agent can choose is bounded by \( M > 0 \); in particular \( \text{Lists}(a) \) is a finite set.

The natural requirement that an agent must possess a single external characteristic (which may consist of a number of attributes) imposes on feasible lists the restriction that all memberships in that list specify the same characteristic. Formally, this requirement on a list \( \ell \) is that there is an external characteristic \( \omega \) such that if \( \ell(r, \omega, g) > 0 \) then \( \omega = \omega_\ell \). We say that such a list is coherent. If \( \ell \) is a coherent list, write \( \omega(\ell) \) for the common external characteristic; we frequently say that \( \omega(\ell) \) is associated to \( \ell \).

- Agent \( a \)'s endowment is \( (e_a, 0) \in X_a \). Note that agents are endowed with private goods but not with group memberships.

- Agent \( a \)'s utility function \( u_a : X_a \to \mathbb{R} \) is defined over private goods consumptions and lists of group membership. We assume throughout that, for each \( \ell \in \text{Lists}(a) \), \( u_a(\cdot, \ell) \) is continuous and strictly monotone; i.e., utility is strictly increasing in private goods consumption. We make no assumption about the dependence of utility on group memberships.

- Agent \( a \)'s personal cost function is a mapping \( c_a : \text{Lists}(a) \to \mathbb{R}^N \). If \( \ell \in \text{Lists}(a) \) and \( \omega(\ell) \) is the associated external characteristic, then
we interpret \( c_a(\ell) \) as the cost to agent \( a \) (expressed as a bundle of private goods) of acquiring the characteristic \( \omega(\ell) \) given that agent \( a \) chooses the various group memberships specified by \( \ell \). We allow for the possibility that different lists entail different costs. (Some schools require uniforms, some do not.) We allow personal costs to be negative (even though the interpretation of negative costs seems strained).

Choosing \( 0 \in \text{Lists}(a) \) means entering into no groups, and we assume the personal cost of such a decision is zero: \( c_a(0) = 0 \).

Because endowments belong to \( \mathbb{R}_+^N \times \{0\} \), agents are not endowed with external characteristics. However, the specification of a particular consumption set might impose constraints on the external characteristics an agent might acquire; endowed characteristics can be coded into these constraints. More formally, for \( a \in A \), write

\[
\Omega_a = \{ \omega \in \Omega : \exists \ell \in \text{Lists}(a), \ell \text{ is coherent and } \omega = \omega(\ell) \}
\]

This is the set of characteristics that the agent \( a \) might acquire, and this set codes the innate characteristics.

For instance, suppose the relevant characteristics in a particular example are height (Short, Tall) and education (Educated, Not Educated); we would define the set of external characteristics as

\[
\Omega = \{SE, SN, TE, TN\}
\]

If we (naturally) view agents as inherently short or tall but all capable of choosing to be educated, then for short agents \( a \in A \) we will have \( \Omega_a = \{SE, SN\} \), while for tall agents \( a \in A \) we will have \( \Omega_a = \{TE, TN\} \). Of course different short agents might also face different costs of acquiring an education — and these costs are coded into the functions \( c_a \).

### 2.5 Economies

An economy \( \mathcal{E} \) is a mapping \( a \mapsto (X_a, e_a, u_a, c_a) \) for which:

- the consumption set correspondence \( a \mapsto X_a \) is a measurable correspondence
• the endowment mapping \( a \mapsto e_a \) is an integrable function

• the utility mapping \( (a, x, \ell) \mapsto u_a(x, \ell) \) is a jointly measurable function of its arguments

• the cost function \( (a, \ell) \mapsto c_a(\ell) \) is a jointly measurable function of its arguments

We assume that the aggregate endowment \( \bar e = \int_A e_a \, d\lambda(a) \) is not zero, so that at least some private goods are represented in the aggregate. (Of course, other private goods may be produced.)

2.6 States

A state of an economy is a measurable mapping

\[
(x, \mu) : A \to \mathbb{R}^N \times \mathbb{R}^M
\]

A state specifies choices of private goods and lists of group memberships for each agent, ignoring individual and social feasibility. (For agents \( a \) for whom \( \mu_a \neq 0 \), so that the chosen list is coherent, the state implicitly specifies the external characteristic chosen.) Individual feasibility means that each agent chooses in his consumption set: \( (x_a, \mu_a) \in X_a \). Social feasibility entails market clearing for private goods and consistent matching of agents.

Consistent matching of agents will be expressed in terms of an aggregate membership vector \( \mu \in \mathbb{R}^M \), representing the total number of memberships of each type. We say that an aggregate membership vector \( \mu \in \mathbb{R}^M \) is consistent if for every group type \( (R, \pi, \gamma, y) \in \mathcal{G} \), there is a real number \( \alpha(R, \pi, \gamma, y) \) such that

\[
\mu(r, \omega, (R, \pi, \gamma, y)) = \alpha(R, \pi, \gamma, y) \pi(r, \omega)
\]

for each \( r \in R, \omega \in \Omega \). Equivalently, for \( r, r' \in R, \omega, \omega' \in \Omega \) we have

\[
\frac{\mu(r, \omega, (R, \pi, \gamma, y))}{\pi(r, \omega)} = \frac{\mu(r', \omega', (R, \pi, \gamma, y))}{\pi(r', \omega')}
\]
whenever at least one of the denominators is not zero. Write \( \text{Cons} \subseteq \mathcal{M} \) for the subspace of consistent aggregate membership vectors.

If \( B \subseteq A \) is a measurable set of agents, a measurable function \( \mu : B \to \mathbb{R}^\mathcal{M} \) is consistent for \( B \) if \( \mu(a) \in \text{Lists}(a) \) for each \( a \) and the corresponding aggregate membership vector \( \bar{\mu} = \int_B \mu_a d\lambda(a) \in \mathbb{R}^\mathcal{M} \) is consistent. Note that if \( \mu \) is consistent for \( B \) then the corresponding aggregate membership vector \( \bar{\mu} \) is non-negative: \( \bar{\mu} \in \text{Cons}_+ \).

The state \((x, \mu)\) is feasible for the measurable subset \( B \subseteq A \) if it satisfies the following requirements:

(i) **Individual feasibility** \((x_a, \mu_a) \in X_a \) for each \( a \in B \)

(ii) **Material balance**

\[
\int_B x_a d\lambda(a) + \int_B c_a(\mu_a) d\lambda(a) \leq \int_B e_a d\lambda(a) + \int_B \sum_{m \in \mathcal{M}} \mu_a(r_m, \omega_m, (R_m, \pi_m, \gamma_m, y_m)) \frac{y_m}{|\pi_m|} d\lambda(a)
\]

(That is, private consumption plus private expenditures on the costs of acquiring characteristics do not exceed endowments plus net production.)

(iii) **Consistency** \( \int_B \mu_a d\lambda(a) \in \text{Cons} \)

The state \((x, \mu)\) is feasible if it is feasible for the set \( A \) itself.

If \((x, \mu)\) is a state of the economy and \( g \in \mathcal{G} \) is a group type, then

\[ A_g = \{ a \in A : \mu_a(r, \omega, g) \geq 1 \text{ for some } r, \omega \} \]

is the set of agents who choose membership in the group type \( g \). If \( A_g \) is of measure 0 then "no" groups of type \( g \) form. If \( A_g \) is of positive measure, then "many" (indeed, infinitely many) groups of this type form. Because members of a group care only about the external characteristics of other members, and not about their identities, it is not necessary to identify the agents belonging to each individual club.
2.7 Equilibrium and quasi-equilibrium

Both private goods and group memberships are priced, so prices \((p, q)\) lie in \(\mathbb{R}_+^N \times \mathbb{R}^M\); \(p\) is the vector of prices for private goods and \(q\) is the vector of prices for group memberships. Because utility functions are assumed monotone in private goods, prices of private goods will be non-negative, but prices of group memberships may be positive, negative or zero. We emphasize that membership prices depend on the group, the role, and the external characteristic.

An equilibrium consists of a feasible state \((x, \mu)\) and prices \((p, q)\) \(\in \mathbb{R}_+^N \times \mathbb{R}^M\), \(p \neq 0\) such that

(1) **Budget feasibility for agents** For almost all \(a \in A\),

\[
(p, q) \cdot ((x_a + c_a(\mu_a)), \mu_a) \leq p \cdot e_a
\]

(2) **Optimization by agents** For almost all \(a \in A\):

\[
(x'_a, \mu'_a) \in X_a \text{ and } u_a(x'_a, \mu'_a) > u_a(x_a, \mu_a) \Rightarrow (p, q) \cdot ((x'_a + c_a(\mu'_a)), \mu'_a) > p \cdot e_a
\]

(3) **Budget balance for group types** For each group type \((R, \pi, \gamma, y)\) \(\in \mathcal{G}\):

\[
\sum_{\omega \in \Omega, r \in R} \pi(r, \omega)q(r, \omega, (R, \pi, \gamma, y)) + p \cdot y = 0
\]

Thus, at an equilibrium individuals optimize subject to their budget constraint and the sum of membership prices in a given group type is exactly equal to the net cost of production.

A quasi-equilibrium satisfies (1), (3) and (2') rather than (2):

(2') **Quasi-optimization** For almost all \(a \in A\):

\[
(x'_a, \mu'_a) \in X_a \text{ and } u_a(x'_a, \mu'_a) > u_a(x_a, \mu_a) \Rightarrow (p, q) \cdot ((x'_a + c_a(\mu'_a)), \mu'_a) \geq p \cdot e_a
\]
That is, no consumption bundle that is feasible and strictly preferred can cost strictly less than agent $a$'s wealth. An equilibrium is necessarily a quasi-equilibrium.

Note again that both private goods and memberships are priced.

Two contrasts with the usual general equilibrium model of production are worth making. In the usual model, equilibrium requires that firms maximize profits, defined as revenue from output less the cost of inputs, and that these profits be distributed to the owners of the firm. In the usual model, however, firm choices do not enter into the utility functions of agents, so profit maximization by firms is a natural requirement. In our framework, firm choices do enter the utility functions of agents, so profit maximization by firms (in that sense) would be quite an unnatural requirement. Indeed, in a real sense, our firms (groups) have no profits to maximize; budget balance for group types is merely an accounting identity.

The reader might think that budget balance for group types that are not chosen in equilibrium should be an inequality rather than an identity, but that is wrong — because all group memberships are priced. To understand the point, consider an economy with two types of firms. The first type of firm can turn 1 unit of good 1 into 1 unit of good 2; the second type of firm can turn 2 units of good 1 into 1 unit of good 2. If only private goods were priced and equilibrium prices were such that budget balance holds for the first type of firm, it certainly could not hold for the second type of firm. However, because both private goods and group memberships are priced, equilibrium prices will be such that the total costs of memberships in the second type of firm will exactly cover the operating loss. If individuals do not care about the firms to which they belong, the second type of firm will not form in equilibrium — but budget balance will hold in any case. For comparison, see Section 4.

Finally, we note that the clubs framework of our earlier papers is (equivalent to) the special case of the present framework in which the vectors $y$ are all non-positive (so there is no production), and for all $a \in A$ the same

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And even if firm choices do not directly enter the utility function of agents directly, they may enter indirectly through the acquisition of external characteristics. Indeed, agents may well wish to belong to a firm for which cost of inputs exceeds revenue from output if by so doing they acquire characteristics that are required elsewhere.
external characteristic \( \omega_a \in \Omega \) is associated to every coherent list (so each agent can choose only a single external characteristic), and the personal cost function is identically zero (so we may regard agents as endowed with an external characteristic).
3 Main Results

As usual, the feasible state \((x, \mu)\) is weakly Pareto optimal if there is no feasible state \((x', \mu')\) such that \(u_a(x'_a, \mu_a) > u_a(x_a, \mu_a)\) for almost every \(a \in A\); \((x, \mu)\) is strongly Pareto optimal if there is no feasible state \((x', \mu')\) such that \(u_a(x'_a, \mu_a) \geq u_a(x_a, \mu_a)\) for almost every \(a \in A\), with strict inequality for a set of agents of positive measure. The feasible state \((x, \mu)\) is in the weak core if there is no subset \(B \subset A\) of positive measure and state \((x', \mu')\) that is feasible for \(B\) such that \(u_b(x'_b, \mu'_b) > u_b(x_b, \mu_b)\) for almost every \(b \in B\); \((x, \mu)\) is in the strong core if there is no subset \(B \subset A\) of positive measure and state \((x', \mu')\) that is feasible for \(B\) such that \(u_b(x'_b, \mu'_b) \geq u_b(x_b, \mu_b)\) for almost every \(b \in B\), with strictly inequality for a set of agents of positive measure.

Say that endowments are desirable if for every agent \(a\) and every list \(\ell \in \text{Lists}(a)\) we have \(u_a(e_a, 0) > u_a(0, \ell)\).

The First Welfare Theorem follows in the usual straightforward fashion; see Theorem 4.1 in our (1999a) paper.

**Theorem 3.1** Every equilibrium state belongs to the weak core and in particular is weakly Pareto optimal. If endowments are desirable then every equilibrium state belongs to the strong core and in particular is strongly Pareto optimal.

The Second Welfare Theorem does not hold in the clubs framework, hence not in the present framework; see our (1999a) paper.

Because we work in a continuum framework, equilibrium in our model passes a familiar test of perfect competition: coincidence of the core with the set of equilibria.

**Theorem 3.2** If endowments are desirable then every core state can be supported as a quasi-equilibrium.

Finally, quasi-equilibria exist.

**Theorem 3.3** If \(E\) be an economy in which endowments are desirable and uniformly bounded above, then a quasi-equilibrium exists.
Theorems 3.2 and 3.3 are proved as in our (1999a) paper, with only one substantial difference. Given an agent \( a \in A \) and a list \( \ell \in \text{Lists}(a) \), our earlier paper defines \( \tau_a(\ell) \) to be the resource cost to agent \( a \) for choosing this list, assuming inputs to clubs are imputed equally to all members; thus

\[
\tau_a(\ell) = \sum_{(\omega,\pi,\gamma) \in \mathcal{M}} \frac{1}{|\mathcal{M}|} \text{inp}(\pi, \gamma)
\]

In the present framework, the corresponding resource cost \( \tilde{\tau}_a(\ell) \) must take into account the personal cost of acquiring the characteristic \( \omega(\ell) \) and the costs of group inputs and group outputs; thus

\[
\tilde{\tau}_a(\ell) = c_a(\ell) - \sum_{(\tau,\omega,(R,\pi,\gamma,y)) \in \mathcal{M}} \frac{1}{|\mathcal{M}|} y
\]

(Note that the signs on the summations are different because in our earlier work inputs are positive, while in the present framework we have used the opposite sign convention, so inputs are negative and outputs are positive.)

4 Compensating Differentials

As we have noted, our model of production is quite different from the standard general equilibrium model of production. It is useful to provide an explicit contrast.

The standard model specifies a set of possible production plans for each firm. The choice of a particular production plan is made endogenously at equilibrium, the chosen production plan must maximize the difference between the value of outputs and the cost of inputs, this difference is distributed to the owners of the firm according to exogenously given firm shares\(^5\), and this difference may be identified unambiguously as profit. Labor may enter as an input to production and is priced as wages. Not operating means choosing the production plan 0; firms that do not operate make 0 profit.

\(^5\)In more elaborate models, shares in firms might be traded, so the ownership of the firm might be determined endogenously at equilibrium.
Our model specifies a single production plan for each firm (group), so there is no choice of production plan to be made; that the production plan maximizes the difference between the value of outputs and the cost of inputs is tautologous. (See Example 5.1 to see how we would model the choice of production plan.) Budget balance entails that this difference is distributed among the members of the firm, but the particular distribution is endogenously determined. Moreover, this difference cannot be unambiguously identified as profit. It would be natural, for instance, to model the corner candy store as a firm/group with Mom and Pop as the members/owners, in which case they would share the difference between the value of outputs and the cost of input of the store — but it is not clear what portion of this distribution should be identified as profit and what portion should be identified as wages. Labor may enter explicitly as an input to production, in which case it is priced explicitly as ordinary wages, but it may also enter implicitly as part of the contributions of the members of the firm/group. Again, the corner candy store exemplifies the difficulty of identifying membership payments as wages. Not operating means that the memberships in the firm are not chosen at equilibrium. For such firms, however, the value of outputs might exceed the value of inputs, whence at least one membership price must be negative. Agents might not choose to belong to such a firm because membership is unpleasant, or requires costly personal investment, or precludes the choice of a better alternative.

We have chosen our particular specification of a firm/group because we want to allow for the possibility that production plans (and membership profiles) enter agents’ utility functions. If that is the case then we expect to see compensating differentials, hence different prices for memberships in firms/groups that use different production plans (or have different membership profiles). If production plans and membership profiles do not enter agents’ utility functions, we might expect compensating differentials to be absent, so that prices for memberships should be independent of production plans (or membership profiles). This is not quite right, because membership in a firm/group also confers skills and requires personal investment, and different membership choices may lead to costs of personal investment. The absence of compensating differentials requires that neither production plans nor membership profiles enter into agents’ utility functions (more generally, that various plans and profiles enter in the same way) and that personal in-
vestment costs are absent (more generally, that the personal costs of various different membership choices be the same). The precise formulation is a bit fussy.

Fix an agent $a$ and memberships $m_1 = (\omega_1, r_1, g_1), m_2 = (\omega_2, r_2, g_2)$. We say $m_1, m_2$ are substitutes for agent $a$ if for every two lists $\ell_1, \ell_2 \in \text{Lists}$ with the properties that $\ell_1(m_1) + \ell_1(m_2) = \ell_2(m_1) + \ell_2(m_2)$ and $\ell_1(m) = \ell_2(m)$ for every $m \neq m_1, m_2$ we have the following:

- $\ell_1 \in \text{Lists}(a)$ if and only if $\ell_2 \in \text{Lists}(a)$
- if $\ell_1, \ell_2 \in \text{Lists}(a)$ then $c_a(\ell_1) = c_a(\ell_2)$
- if $\ell_1, \ell_2 \in \text{Lists}(a)$ then $u_a(x, \ell_1) = u_a(x, \ell_2)$ for all $x \in \mathbb{R}_+^N$

**Proposition 4.1** Let $(p, q), (x, \mu)$ be a quasi-equilibrium for the economy $E$. If the memberships $m_1, m_2$ are substitutes for all agents $a \in A$ and $m_1$ is chosen by a set of agents of positive measure, then $q(m_1) \leq q(m_2)$. If each of $m_1, m_2$ is chosen by a set of agents of positive measure then $q(m_1) = q(m_2)$.

**Proof** If $q(m_1) > q(m_2)$ and $m_1$ is chosen by a set $A_1$ of agents of positive measure, then agents in $A_1$ could choose $m_2$ instead of $m_1$, obtain the same utility as at $(x, \mu)$, and spend strictly less than at $(x, \mu)$. By spending half the saving on consumption goods, they could obtain greater utility than at $(x, \mu)$ and still spend strictly less than at $(x, \mu)$. Since this contradicts the quasi-equilibrium nature of $(p, q), (x, \mu)$, we conclude that $q(m_1) \leq q(m_2)$. The second conclusion follows by reversing the roles of $m_1, m_2$. □

5 Examples

5.1 The Marshallian firm

We begin with an application of our approach to a rather conventional representation of a firm. Although this example ignores roles, infrastructure, and the acquiring of external characteristics, a number of subtleties remain.
Assume $\Omega = \{\omega_0\}$ with the external characteristic acquirable at zero cost. There are two private goods. Firms are of three types:

- **One-person firms**
  
  \[ g_1 = (R_1, \pi_1, \gamma_1, y_1) \quad \text{with} \quad \pi_1(\omega_0) = 1 \quad \text{and} \quad y_1 = (-1, \alpha) \]

- **Two-person firms**
  
  \[ g_2 = (R_2, \pi_2, \gamma_2, y_2) \quad \text{with} \quad \pi_2(\omega_0) = 2 \quad \text{and} \quad y_2 = (-2, 8\alpha) \]

- **Three-person firms**
  
  \[ g_3 = (R_3, \pi_3, \gamma_3, y_3) \quad \text{with} \quad \pi_3(\omega_0) = 3 \quad \text{and} \quad y_3 = (-3, 9\alpha) \]

where $\gamma_1 = \gamma_2 = \gamma_3 = 1$ and $\alpha > 0$. Let $R_1 = R_2 = R_3 = \{r_0\}$. We assume that an agent can join at most one firm. Let $\ell_j$ denote the list consisting of a membership in one firm of type $g_j$ (j=1,2,3).

All agents are identical with endowment $e_a = (k, 0)$, $k > 0$, and utility function

\[
U_a(x, \ell) = \begin{cases} 
2\sqrt{\frac{x_1 x_2}{x_1 x_2}} & \text{if agent } a \text{ does not join a firm;} \\
\gamma_j \sqrt{\frac{x_1 x_2}{x_1 x_2}} & \text{if agent } a \text{ joins a firm of type } g_j.
\end{cases}
\]

Normalize prices such that $p_1 = 1$.

The budget-balance condition implies

\[
q(r_0, \omega_0, g_1) - 1 + \alpha p_2 = 0 \\
2q(r_0, \omega_0, g_2) - 2 + 8\alpha p_2 = 0 \\
3q(r_0, \omega_0, g_3) - 3 + 9\alpha p_2 = 0
\]

and hence membership prices are given by

\[
-q(r_0, \omega_0, g_1) = \alpha p_2 - 1 \\
-q(r_0, \omega_0, g_2) = 4\alpha p_2 - 1 \\
-q(r_0, \omega_0, g_3) = 3\alpha p_2 - 1
\]
In equilibrium agent $a$’s wealth inclusive of wages earned from membership in a firm equals $k$ for non-workers and

$$p \cdot e_a - q(r_0, \omega_0, g_2) = \begin{cases} 
  k + \alpha p_2 - 1 & \text{for agents in a firm of type } g_1; \\
  k + 4\alpha p_2 - 1 & \text{for agents in a firm of type } g_2; \\
  k + 3\alpha p_2 - 1 & \text{for agents in a firm of type } g_3.
\end{cases}$$

Indirect utility conditional on choice of a list is given by

$$V(p \mid 0) = \frac{k}{\sqrt{p_2}}$$

$$V(p \mid \ell_1) = \frac{k + \alpha p_2 - 1}{2\sqrt{p_2}}$$

$$V(p \mid \ell_2) = \frac{k + 4\alpha p_2 - 1}{2\sqrt{p_2}}$$

$$V(p \mid \ell_3) = \frac{k + 3\alpha p_2 - 1}{2\sqrt{p_2}}$$

It follows immediately that only firms of type $g_2$ will form in equilibrium.

If some agents do not work and some are employed by 2-person firms, then $V(p \mid 0) = V(p \mid \ell_2)$, which yields the equilibrium price

$$p_2 = \frac{k + 1}{4\alpha}$$

Equilibrium membership prices are given by

$$-q(r_0, \omega_0, g_1) = \frac{k - 3}{4}, \quad -q(r_0, \omega_0, g_2) = k, \quad -q(r_0, \omega_0, g_3) = \frac{3k - 1}{4}$$

Non-workers and workers receive the equilibrium allocation

$$x_a = \left(\frac{k}{2}, \frac{2\alpha k}{k + 1}\right) \quad \text{and} \quad x_a = \left(\frac{k}{4}, \frac{4\alpha k}{k + 1}\right)$$

respectively, yielding each the same utility. Letting $\lambda$ denote the proportion of agents who are workers, clearing the market for private good 1 requires

$$\lambda \cdot k + (1 - \lambda) \frac{k}{2} + \lambda \cdot 1 = k$$
which implies

$$\lambda = \frac{k}{k + 2}$$

The proportion of agents who are employed lies in the open interval $(0, 1)$, and it approaches 1 as $k$ increases. It is easy to check that the market for private good 2 clears as well:

$$\lambda \cdot \frac{4k\alpha}{k + 1} + (1 - \lambda) \cdot \frac{2k\alpha}{k + 1} = \lambda \cdot 4\alpha$$

Since membership prices are defined for firms that do not form as well as those that do, the total cost function is well-defined:

$$C(p, y_j^+) = -\pi_j(\omega_0) q(r_0, \omega_0, g_j) + p \cdot y_j^- \quad j = 1, 2, 3$$

Using this notation, the budget balance condition can be written

$$-C(p, y_j^+) + p_2 y_j^+ = 0 \quad j = 1, 2, 3$$

which immediately implies that average cost does not vary with output:

$$AC(p, y_j^+) = p_2 = \frac{k + 1}{4\alpha} \quad j = 1, 2, 3$$

Nevertheless, 2-worker firms are the only ones that form in equilibrium. We can clarify what is going on by expanding the definition of the activity vector to include labor input:

$$(-\pi_j(\omega_0), y_{j1}, y_{j2}) = \begin{cases} 
(-1, -1, \alpha) & \text{for 1-worker firms;} \\
(-2, -2, 8\alpha) & \text{for 2-worker firms;} \\
(-3, -3, 9\alpha) & \text{for 2-worker firms}
\end{cases}$$

Thus, the technology exhibits initial increasing returns followed by decreasing returns, a setting which leads to u-shaped average cost curves for Alfred Marshall. In our framework, price equals average cost for all firms, those that form and those that are not economically viable. Nevertheless, firm size is determinate, just as in Marshall.
5.2 Resisting the factory system

Our second example explores some of the implications of differences in infrastructure in the working environment. According to David Landes [1998], when factories first displaced the putting-out system in England, workers hated the change: they much preferred to work at home rather than in a factory. As a consequence, factories were forced to pay a premium for their workers. Nevertheless, factories displaced the putting-out system because the economies associated with using central power sources outweighed the disadvantages to the workers of leaving home. (See Landes [1998], 209–210.)

The setting is identical in most respects to Example 5.1, with 1-worker firms identified with working at home and 2- or 3-worker firms with factories. There are two differences:

- The activity vector for firm type 1 is now
  
  \[ y_1 = (-1, 1) \]

  (so, as the productivity parameter \( \alpha \) increases, only factories benefit, not agents working at home)

- The infrastructure parameter for firm type 1 is now
  
  \[ \gamma_1 = \frac{3}{2} \]

  (so working at home, while less pleasurable than not working at all, is better than work in the factory).

Mimicking the methods of Example 5.1, we conclude that membership prices are given by

\[
\begin{align*}
-q(r_0, \omega_0, g_1) &= p_2 - 1 \\
-q(r_0, \omega_0, g_2) &= 4\alpha p_2 - 1 \\
-q(r_0, \omega_0, g_3) &= 3\alpha p_2 - 1
\end{align*}
\]

and indirect utility, conditional on choice of a list of memberships, by

\[
V(p \mid 0) = \frac{k}{\sqrt{p_2}}
\]

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\[ V(p \mid \ell_1) = \frac{3}{4} \left( \frac{k + \alpha p_2 - 1}{\sqrt{p_2}} \right) \]
\[ V(p \mid \ell_2) = \frac{k + 4\alpha p_2 - 1}{2\sqrt{p_2}} \]
\[ V(p \mid \ell_3) = \frac{k + 3\alpha p_2 - 1}{2\sqrt{p_2}} \]

As in Example 5.1, firms of type \( g_3 \) will never form.

If some agents work at home and some do not work, then
\[ V(p \mid \ell_1) = V(p \mid 0) \]
which reduces to
\[ p_2 = \frac{k + 3}{3} \]

If some agents work at home and some work in factories, then
\[ V(p \mid \ell_2) = V(p \mid 0) \]
which reduces to
\[ p_2 = \frac{k + 1}{4\alpha} \]
(just as in Example 5.1). Consequently, it is possible to have some agents working at home and some working in factories if and only if
\[ \alpha = \frac{3}{4} \left( \frac{k + 1}{k + 3} \right) \]

From now on assume \( k = 3 \), so that this critical value for \( \alpha \) equals 1/2.

In equilibrium
\[ p_2 = \begin{cases} 2 & \text{if } \alpha \leq 1/2; \\ 1/\alpha & \text{if } \alpha > 1/2; \end{cases} \]

while
\[ -q(r_0, \omega_0, g_1) = \begin{cases} 1 & \text{if } \alpha \leq 1/2; \\ (1 - \alpha)/\alpha & \text{if } \alpha > 1/2; \end{cases} \]
\[ -q(r_0, \omega_0, g_2) = \begin{cases} 8\alpha - 1 & \text{if } \alpha \leq 1/2; \\ 3 & \text{if } \alpha > 1/2; \end{cases} \]
\[ -q(r_0, \omega_0, g_3) = \begin{cases} 6\alpha - 1 & \text{if } \alpha \leq 1/2; \\ 2 & \text{if } \alpha > 1/2. \end{cases} \]
(Note that membership prices are defined even for groups that do not form in equilibrium.)

If \( \alpha < 1/2 \), then all agents work at home, each receiving the allocation

\[
    x_a = (2, 1)
\]

If \( \alpha > 1/2 \), then 3/5 of the agents work in a factory while the rest do not work. Each factory worker receives the allocation

\[
    x_a = (3, 3\alpha)
\]

and each non-worker the allocation

\[
    x_a = \left( \frac{3}{2}, \frac{3\alpha}{2} \right)
\]

If \( \alpha = 1/2 \), then to any assignment \( \lambda_1 \) of workers to home production and \( \lambda_2 \) of workers to factories satisfying

\[
    3\lambda_1 + 5\lambda_2 = 1 \quad \lambda_1, \lambda_2 \geq 0
\]

corresponds an equilibrium with allocation

\[
    x_a = \begin{cases} 
    (3/2, 3/4) & \text{to each agent choosing not to work;} \\
    (2, 1) & \text{to each agent who works at home; and} \\
    (3, 3/2) & \text{to each agent who works in a factory.}
    \end{cases}
\]

(Factory workers receive a premium relative to agents who work at home.)

### 5.3 Monitoring

In his discussion of the displacement of the putting-out system, Landes puts great stress on the benefits derived from an enhanced ability to monitor workers in a factory setting. In this example we highlight the way in which the natural tension between monitoring costs and economies to scale can lead to a determinate firm size, with the monitor serving in somewhat the role of a residual claimant.\(^6\)

\(^6\)Our treatment does not attempt the subtlety of the discussion of residual claimants in, for example, Hart and Moore [1990], which we leave for another occasion.
All agents are identical with external characteristic $\omega_0$, acquired without cost. An agent may stay at home or work for a firm, either as a worker or a manager. Let $R = \{W, M\}$ denote this set of roles. There are two private goods. Each agent has endowment $e_a = (2, 0)$. Normalize $p_1 = 1$.

Firm types differ primarily with respect to the number of workers employed. A firm of type $g_n$ employs $n$ workers and has one manager,

$$\pi(W, \omega_0) = n \quad \text{and} \quad \pi(M, \omega_0) = 1$$

A firm of type $g_n$ has activity vector

$$y_n = (-6n, n^2)$$

so that firms with more workers enjoy greater economies to scale. Agent utility is given by

$$u_a(x, \ell) = \begin{cases} 
4\sqrt{x_1x_2} & \text{for agents who remains at home;} \\
2\sqrt{x_1x_2} & \text{for workers in a firm of type } g_n; \\
\frac{4}{n^3+1}\sqrt{x_1x_2} & \text{for managers in a firm of type } g_n.
\end{cases}$$

Worker experience disutility from working, but they do not care about the size of the firm; managers do care about the size of the firm, experiencing increasing disutility as the number of workers to be managed increases. It is this difference that gives the role of manager the flavor of a residual claimant.

Let $q_n^W$ and $q_n^M$ denote the membership prices of a worker and manager, respectively, associated with a firm of type $g_n$. Indirect utility conditional on choice of a list is

$$V_0 = \frac{4}{\sqrt{p_2}} \quad \text{for agents who remain at home}$$

$$V_n^W = \frac{2 - q_n^W}{\sqrt{p_2}} \quad \text{for workers in a firm of type } g_n$$

$$V_n^M = \frac{2(2 - q_n^M)}{(n^3 + 1)\sqrt{p_2}} \quad \text{for managers in a firm of type } g_n$$

Budget balance for firms of type $g_n$ requires

$$ng_n^W + g_n^M = 6n - p_2n^2$$

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We look for an equilibrium in which all firms are the same size, and some workers stay at home. Equating \( V_0 = V_n^W = V_n^M \) yields
\[
q_n^W = -2 \quad \text{and} \quad q_n^M = -2n^3
\]
Substituting into the budget balance equation gives
\[
p_2 = \frac{8}{n} + 2n
\]
Since the utility of all agents in equilibrium equals \( 4/\sqrt{p_2} \) and a competitive equilibrium is Pareto optimal, the equilibrium must minimize \( p_2 \). Evaluating the first-order condition for a minimum yields \( n = 2 \), the optimal number of workers in each firm (and it is a minimum). We conclude that
\[
q_n^M = -16
\]
and
\[
p_2 = 8
\]
At these equilibrium prices demand for the first private good
\[
x_{a1} = \begin{cases} 
1 & \text{for agents who remain at home;} \\
2 & \text{for agents who work for a 2-worker firm;} \\
9 & \text{for agents who manage 2-worker firms.}
\end{cases}
\]
Letting \( \lambda_0, \lambda_1, \lambda_2 \) denote the fraction of agents who remain at home, become workers and become managers respectively, clearing the market for private good 1 requires
\[
\lambda_0[1] + \lambda_1[2] + \lambda_2[9] + \lambda_2[12] = 6
\]
Using the fact that \( \lambda_1 = 2\lambda_2 \) (as required for a consistent matching) and the fact that the \( \lambda_j \)'s must sum to one, we conclude that
\[
\lambda_0 = \frac{7}{22} \quad \lambda_1 = \frac{5}{11} \quad \text{and} \quad \lambda_2 = \frac{5}{22}
\]

5.4 Roles

The attentive reader might wonder whether it was really necessary to add roles to our formulation. And, in fact, they are not really needed. The
following example illustrates, nevertheless, that distinguishing between roles and external characteristics is quite natural.

There is a continuum of agents with identical endowment $e_a = 5$ of the (single) private good and identical preferences. There is a single, 2-person group type, engaged in giving shaves. The members of this club occupy one of two roles, barber or customer, denoted

$$R = \{B, C\}$$

The ability to give a shave is an external characteristic, denoted $\omega_1$ and acquired at cost $c_a(\ell) = 2$; all other agents have characteristic $\omega_0$, acquired at no cost, and can shave no one. Only those who have acquired skill $\omega_1$ qualify as barbers, and barbers cannot shave themselves.\(^7\) In addition to the services of a barber, a shave requires input $y = (-1)$. Normalize the price of the private good $p = 1$. Budget balance requires

$$q(m_1) + q(m_2) = 1$$

where $m_1$ denotes membership as the barber and $m_2$ as the customer.

An agent can belong to at most 3 clubs and be shaved at most once. Let

$$\ell_{n_1, n_2} = n_11(m_1) + n_21(m_2)$$

denote a list, where $n_1$ represents the number of shaves the agent gives and $n_2$ represents the number of shaves the agent receives; $1(m_1)$ [resp. $1(m_2)$] is the indicator function on the space of memberships $\mathcal{M}$, equaling 1 if $m = m_1$ [resp. $m = m_2$] and 0 otherwise. We require $n_1$ and $n_2$ to be non-negative integers summing to no more than 3 with $n_2$ equal either to 0 or 1.

Agent $a$ has preferences represented by the utility function

$$u_a(x, \ell) = (4 - n_1)(1 + 2n_2)x$$

The indirect utility function conditional on choice of a list $\ell_{n_1, n_2}$ is given by

$$V_{n_1, n_2} = (4 - n_1)(1 + 2n_2)(5 - n_1q(m_1) - n_2q(m_2) - c_a(\ell))$$

\(^7\)Apologies to Bertrand Russell.
Using the information provided above, we compute

\[
\begin{align*}
V_{00} &= 20 \\
V_{10} &= 9 - 3q(m_1) \\
V_{20} &= 6 - 4q(m_1) \\
V_{30} &= 3 - 3q(m_1) \\
V_{01} &= 48 + 12q(m_1) \\
V_{11} &= 18 \\
V_{21} &= 12 - 6q(m_1)
\end{align*}
\]

where \( V_{21} \), for example, represents the maximum utility attainable by an agent who has trained to be a barber, shaves two customers (possibly including another barber), and has a shave himself.

It is easy to see that five of these lists are dominated, leaving only two lists for agents to consider, \( \ell_{01} \) and \( \ell_{21} \). Since agents are identical and both lists must appear in equilibrium, agents must be indifferent between these two options. Equating

\[ V_{01} = V_{21} \]

and simplifying yields the equilibrium membership prices

\[ q(m_1) = -2 \quad \text{and} \quad q(m_2) = 3 \]

Letting \( \lambda_{01} \) and \( \lambda_{21} \) denote the fraction of agents choosing these two lists and clearing the market for private good 1 requires

\[ \lambda_{01}(5 - 3) + \lambda_{21}(5 + 4 - 3 - 2) + \lambda_{21}(2) + 1 = 5 \]

which reduces to

\[ \lambda_{01} + 3\lambda_{21} = 2 \]

Since \( \lambda_{01} + \lambda_{21} = 1 \), this implies

\[ \lambda_{01} = \lambda_{21} = \frac{1}{2} \]

Half the agents have a shave but do not learn the barber trade; the other half train to be barbers, give two shaves, and are shaved themselves by a fellow barber.

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In this equilibrium, all customers are charged the same amount for a shave, but that need not be so. If a barber prefers shaving another barber, then the customer role splits into two (customer-ω₀ and customer-ω₁) and, as the reader can easily verify, customers who are also barbers receive a discount.

5.5 Clubs and firms

This example illustrates the interaction between memberships in schools and in firms. There are two private goods and two group types, a school and a firm. Agents choose either the null list or one of the following lists:

\[ \ell_1 : \text{membership as a worker, twice}; \]
\[ \ell_2 : \text{membership in a school (as a student) and in a firm (as a manager)} \]
\[ \ell_3 : \text{membership in a school, twice, once as a student and once as a teacher}. \]

We identify each list with a unique characteristic:

\[ \omega(\ell_j) = \omega_j \quad j = 1, 2, 3 \]

The school type has two roles, \( R = \{S, T\} \), student and teacher. Membership as a student is denoted \( m_1 \), as a teacher \( m_2 \). Each school has four students and one teacher:\(^8\)

\[ \pi(S, \omega_2) + \pi(S, \omega_3) = 4 \quad \text{and} \quad \pi(T, \omega_3) = 1 \]

Schools require inputs,

\[ y = (-4, 0) \]

The firm type also has two roles, \( R = \{W, M\} \), worker and manager. Membership as a worker is denoted \( m_3 \), as a manager \( m_4 \). Each firm has four workers and one manager:

\[ \pi(W, \omega_1) = 4 \quad \text{and} \quad \pi(M, \omega_2) = 1 \]

\(^8\)We assume there is no difference between training a manager and training a teacher, allowing schools with different student mixes to be treated as one (cf. Proposition 4.1).
Each firm has activity vector

\[ y = (-1, 81) \]

All consumers are identical with endowment \( e_a = (6, 0) \) and utility function

\[
u_a(x, \ell) = \begin{cases} 
4\sqrt{x_1 x_2} & \text{if } \ell = 0; \\
\sqrt{x_1 x_2} & \text{if } \ell = \ell_1; \\
2\sqrt{x_1 x_2} & \text{if } \ell = \ell_2; \\
3\sqrt{x_1 x_2} & \text{if } \ell = \ell_3.
\end{cases}
\]

\( c_a(\ell) = 0 \) for all lists and all agents. Let \( p_1 = 1 \).

Budget balance requires

\[ 4q(m_1) + q(m_2) = \alpha \]

for schools and

\[ 4q(m_3) + q(m_4) = 1 - \beta p_2 \]

for firms. Let \( V_j \) denote the indirect utility conditional on choice of list \( \ell_j \) (\( j = 0, 1, 2, 3 \)), where \( \ell_0 \) is the null list. Then

\[
\begin{align*}
V_0 &= \frac{12}{\sqrt{p_2}} \\
V_1 &= \frac{6 - 2q(m_3)}{2\sqrt{p_2}} \\
V_2 &= \frac{6 - q(m_1) - q(m_4)}{\sqrt{p_2}} \\
V_3 &= \frac{3(6 - q(m_1) - q(m_2))}{2\sqrt{p_2}}
\end{align*}
\]

Equating \( V_0 = V_1 = V_3 \) and using the budget balance conditions yields the equilibrium private good price

\[ p_2 = 5 \]

and the equilibrium membership prices

\[ q(m_1) = 2 \quad q(m_2) = -4 \quad q(m_3) = -9 \quad q(m_4) = -368 \]
for students, teachers, workers and managers respectively.

Letting $\lambda_j$ denote the fraction of agents choosing list $\ell_j$, clearing the market for private good 1 requires

$$\lambda_0 \left[ \frac{6}{10} \right] + \lambda_1 \left[ \frac{6 + 2 \cdot 9}{10} \right] + \lambda_2 \left[ \frac{6 - 2 + 368}{10} \right] + \lambda_3 \left[ \frac{6 - 2 + 4}{10} \right] + \lambda_2[1] + \lambda_3[4] = 6$$

Consistent matching requires

$$\lambda_1 = 2\lambda_2 \quad \text{and} \quad \lambda_2 = 3\lambda_3$$

Consequently, in equilibrium

$$\lambda_0 = \frac{41}{71} \quad \lambda_1 = \frac{18}{71} \quad \lambda_2 = \frac{9}{71} \quad \lambda_3 = \frac{3}{71}$$

Stated differently, for those who agents who choose a career,

$$\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{6}{10} \quad \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{3}{10} \quad \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{1}{10}$$

For every 20 agents who choose not to stay at home, 12 are workers working for 6 different firms (recall that each worker works for two firms); 6 are managers, one to a firm; 2 are teachers, teaching the 6 would-be managers and the 2 would-be teachers. As this interpretation suggests, an intertemporal setting would improve the story. Since that is a well-trodden path, we have not done so here.

### 5.6 Discrimination in the workplace

Our final example considers the impact of prejudice among workers on firm formation and membership prices. There is a single private good. Each consumer has endowment $e_a = 1$. There are two external characteristics, (c)omputer programmers and (e)conomists: $\Omega = \{c, e\}$. Roles are redundant ($R = \{r_0\}$) and infrastructure is irrelevant. There are three firm types:

- Firms with a computer scientist and an economist:

  $$g_1 = (R, \pi_1, \gamma_0, 8)$$

  with $\pi_1(r_0, c) = \pi_1(r_0, e) = 1$. Memberships are denoted $m_c^1$ and $m_e^1$. 

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• Firms with two computer scientists:

\[ g_2 = (R, \pi_2, \gamma_0, 8) \]

with \( \pi_2(r_0, c) = 2 \) and \( \pi_2(r_0, e) = 0 \). A membership is denoted \( m_c^2 \).

• Firms with two economists:

\[ g_3 = (R, \pi_3, \gamma_0, 10) \]

with \( \pi_3(r_0, c) = 0 \) and \( \pi_3(r_0, e) = 2 \). A membership is denoted \( m_e^3 \).

Agents can belong to at most one firm. Preferences are given by

\[
u_a(x, \ell) = \begin{cases} 
12x & \text{if the agent joins no firm;} \\
4x & \text{if the agent chooses } m_c^1; \\
9x & \text{if the agent chooses } m_e^1; \\
6x & \text{if the agent chooses } m_c^2; \\
3x & \text{if the agent chooses } m_e^2.
\end{cases} \]

Thus, economists like to work with computer scientists, but not conversely. Agents are alike in all respects save one, the difficulty of becoming a computer scientist. Let \( A = A_1 \cup A_2 \) with \( \lambda(A_1) = 2/3 \) and \( \lambda(A_2) = 1/3 \). For an agent choosing to be an economist, \( c_a(\ell) = 1 \). But for an agent choosing to be a computer scientist,

\[
c_a(\ell) = \begin{cases} 
1 & \text{if } a \in A_1; \\
3 & \text{if } a \in A_2.
\end{cases} \]

Let \( p = 1 \) as numeraire. Budget balance implies

\[ q(m_c^1) + q(m_e^1) = -8 \quad q(m_c^2) = -4 \quad \text{and} \quad q(m_e^3) = -5 \]

Letting \( V^j_\omega \) denote indirect utility conditional on choice of membership \( m_\omega \), we obtain

\[
\begin{align*}
V_0 &= 12 \\
V_c^1 &= \begin{cases} -4q(m_c^1) & \text{if } a \in A_1 \\
-8 - 4q(m_c^1) & \text{if } a \in A_2 \end{cases} \\
V_e^1 &= -9q(m_e^1) \\
V_c^2 &= \begin{cases} 24 & \text{if } a \in A_1 \\
12 & \text{if } a \in A_2 \end{cases} \\
V^3_3 &= 15
\end{align*}
\]
There is a unique equilibrium. All agents \( a \in A_1 \) become computer programmers, all agents \( A \in A_2 \) become economists, and everyone works for a firm. Although firms of type \( g_3 \) have the highest labor productivity, none form. All economists (the agents in \( A_2 \)) join firms of type \( g_1 \), along with half the computer programmers; the remaining computer programmers join firms of type \( g_2 \). Equating \( V^1_c = V^2_c \) yields

\[
q(m^1_c) = -6 \quad \text{and} \quad q(m^1_e) = -2
\]

As already noted,

\[
q(m^2_c) = -4 \quad \text{and} \quad q(m^3_e) = -5
\]

In equilibrium,

\[
V^1_c = V^2_c = 24 \quad \text{and} \quad V^1_e = 18
\]

Economists receive an allocation \( x_a = 2 \) of the private good while computer programmers receive

\[
x_a = \begin{cases} 
  6 & \text{if they work for a firm of type } g_1; \\
  4 & \text{if they work for a firm of type } g_2.
\end{cases}
\]

It is easily confirmed that the market for the private good clears.
References


