FISCAL HEDGING AND THE YIELD CURVE

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ABSTRACT

We analyze optimal fiscal and monetary policy in an economy with distortionary labor income
taxes, nominal rigidities and nominal debt of various maturities. Optimal policy prescribes the
exclusive use of long term debt. Such debt mitigates the distortions associated with hedging
fiscal shocks by allowing the government to allocate them efficiently across states and periods.

I. Introduction

Governments have traditionally financed deficits by selling nominal bonds of varied maturities.
A long standing policy question concerns the optimal management of such liabilities. Various
contributors have posited a role for short term nominal debt. Campbell (1995) argues that a
cost-minimizing government should respond to a steeply sloped nominal yield curve by shortening
the maturity structure since high yield spreads tend to predict high expected bond returns in the
future. Barro (1997) emphasizes tax smoothing considerations. He asserts that governments can
reduce their risk exposure and better smooth taxes by shortening the maturity structure when the
inflation process becomes more volatile and persistent. Barro characterizes the reduction in the
average maturity of US Federal bonds between 1946 and 1976 as an optimal response to changes in
the inflation process. Both lines of argument treat the processes for inflation and nominal interest
rates exogenously.

In this paper, we explore optimal maturity management in a fully specified general equilibrium
model. We find that optimal policy prescribes the exclusive use of the longest term nominal debt

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available and a management of nominal interest rates that departs from the Friedman rule. A switch from a favorable to an unfavorable fiscal environment, triggered by an adverse shock\(^1\) is followed by increases in current and future short term nominal interest rates, with increases in the latter concentrated in future adverse shock states. Thus, realized sequences of adverse shocks are associated with an accumulation of the short term nominal interest rate that continues until all of the debt outstanding at the time of the initial shock has matured. When a spell of adverse fiscal shocks begins, the yield curve takes a corresponding humped shape, with the hump occurring at the longest traded debt maturity. It reverts to a lower level and a flatter shape when this spell ends or when the debt outstanding at the beginning of the spell has matured. These policies imply that long term nominal debt is riskier than short term debt. However, the volatility of long term debt returns is deliberate and managed so as to hedge the fiscal risk the government faces. The risk premium on this debt resembles an insurance premium paid by the government; it does not provide a motive for shortening the maturity structure.

In our model, the government finances its activities by raising taxes, trading debt and manipulating the return on outstanding debt. Since the focus of our paper is on the management of the nominal maturity structure, we consider an economy in which only non-contingent nominal debt is traded. This assumption implies that the government can hedge fiscal shocks only indirectly through contemporaneous inflations or variations to the nominal term structure. Following Siu (2004), we introduce two nominal rigidities that enrich the government’s policy problem.\(^2\) First, we assume that some firms set their prices before the realization of the current state. This rigidity implies that contemporaneous innovations to inflation are associated with costly misallocations of production across sticky and flexible price firms. The government must trade such distortions off against the hedging benefits that inflation innovations provide. Second, we assume that households face a cash-in-advance constraint applied to some goods (cash goods), but not others (credit goods). Variations in the nominal term structure imply positive short term nominal interest rates after some histories and, hence, misallocations of consumption across cash and credit goods. The government must trade these consumption distortions off against the hedging benefits obtained from variations in the term structure.

Long term nominal debt is useful because it mitigates the effects of nominal rigidities. As noted, reductions in the nominal value of the government’s liabilities in the aftermath of an adverse fiscal

\(^1\)Here, adverse fiscal shocks comprise positive shocks to government spending and negative shocks to productivity.

\(^2\)Absent these rigidities all hedging could be achieved through contemporaneous adjustments in inflation and nominal debt would effectively function as a real contingent claim. See, for example, Chari et al (1991).
shock involve positive short term interest rates and corresponding distortions to the cash-credit
good margin over the term of these liabilities. Long term debt allows the government to allocate
these distortions more efficiently across dates and states. It permits their postponement and their
concentration in future adverse shock states when they can contribute to the hedging of these later
shocks. Such postponement and concentration effects underpin a gradual upward response of short
term nominal interest rates during spells of adverse fiscal shocks.

The literature on optimal fiscal and monetary policy has made various assumptions about the
asset structure confronting the government. Our paper is closest to Siu (2004). We follow him
in restricting the government to the use of nominal debt and incorporating frictions that render
state-contingent inflations distortionary. In contrast to Siu, we allow the government to trade
nominal debt of more than one period maturity. Thus, we are able to consider the optimal maturity
structure. Additionally, in our model the government can influence the price of outstanding nominal
bonds via current and future nominal interest rate policy. This opens up a second channel for
hedging fiscal shocks that is absent in Siu’s earlier contribution.

The plan for the paper is as follows. We describe the environment and characterize competitive
allocations in Sections II and III. Section IV gives the Ramsey problem for our economy and
contrasts it with those obtained under alternative asset market structures. Section V provides a
general recursive formulation, while Section VI uses a simple analytical example to provide intuition.
In Section VII we use the recursive formulation to obtain the optimal policy in a calibrated economy.

II. A model with sticky prices

The economy is inhabited by infinitely-lived households, firms and a government. Let $s_t \in S = \{s_t\}_{i=1}^N$ denote a period $t$ shock and $s^t \in S^{t+1}$ a $t$-period history of shocks. We assume that $s_0$
is distributed according to $\pi^0$ and that subsequently shocks evolve according to a Markov process
with transition $\pi$. The implied probability distribution over shock histories $s^t$ is denoted $\pi^t$.

A. Households

Preferences Households have preferences over stochastic sequences of cash goods $\{c_{1t}\}_{t=0}^\infty$, credit
goods $\{c_{2t}\}_{t=0}^\infty$ and labor $\{l_t\}_{t=0}^\infty$ of the form:

$$E\left[\sum_{t=0}^\infty \beta^t U(c_{1t}, c_{2t}, l_t)\right], \quad (1)$$
where \( U : \mathbb{R}_+^2 \times [0, T] \to \mathbb{R} \) is twice continuously differentiable on the interior of its domain, strictly concave, strictly increasing in its first two arguments and decreasing in its third argument. We assume that \( U \) satisfies the Inada conditions for \( j = 1, 2 \) and each \((c_i, l), i \neq j\), \( \lim_{c_j \to 0} \frac{\partial U}{\partial c_j}(c_1, c_2, l) = \infty \) and for each \((c_1, c_2), \lim_{l \to T} \frac{\partial U}{\partial l}(c_1, c_2, l) = -\infty \). Finally, we assume that \( U \) is homothetic in \((c_1, c_2)\) and weakly separable in \( l \). Let \( U_{jt}, j = 1, 2 \), \( l \) denote the derivatives of \( U \) with respect to each of its arguments at date \( t \) and let \( U_{jkt}, j, k = 1, 2 \), \( l \) denote its second derivatives at \( t \).

Trading Each household enters period \( t \) with a portfolio of money \( M_t \geq 0 \) and nominal (zero coupon) bonds \( \{B^k_t\}_{k=1}^K \in \mathbb{R}_+^K \), where the superscript \( k \) denotes the maturity of the bond and \( K \) is the maximal maturity traded. The shock \( s_t \) is then realized. Asset market trading occurs in two rounds during the course of the day. The first liquidity trading round occurs in the morning immediately after the shock realization. In this, households are able to liquidate their bond holdings in light of their post-shock cash needs. On the other side of the market, the government (one may think of it as the Fed) responds to these needs by trading bonds for money. The household’s budget constraint in this trading round is

\[
A_t(s^{t-1}) + \sum_{k=1}^{K-1} Q^k_t(s^t) B^{k+1}_t(s^{t-1}) \geq \tilde{M}_t(s^t) + \sum_{k=1}^K Q^k_t(s^t) \tilde{B}^k_t(s^t),
\]

where \( Q^k_t \) is the nominal price of the \( k \)-th maturity bond, \( A_t = B^1_t + M_t \) and \( \tilde{M}_t \) and \( \{ \tilde{B}^k_t \}_{k=1}^K \) denote the portfolio of money and bonds purchased by households. Households then shop for cash and credit goods, exert effort in production and receive after-tax wage and dividend income. Since money is required for cash goods consumption, households face the cash-in-advance constraint:

\[
P_t(s^t)c_{1t}(s^t) \leq \tilde{M}_t(s^t).
\]

In the afternoon, asset markets reopen allowing households to settle credit balances accrued whilst shopping and invest income. This second trading round also allows the government (one may now think of it as the Treasury) to finance its budget deficit and purchase a portfolio that hedges itself against future shocks. Define: \( \tilde{A}_t(s^t) \equiv \tilde{B}^1_t(s^t) + \{ \tilde{M}_t(s^t) - P_t(s^t)c_{1t}(s^t) \} - P_t(s^t)c_{2t}(s^t) + (1 - \tau_t(s^t))I_t(s^t) \). Here \( P_t \) is the period \( t \) price level and \( \tau_t \) is the income tax rate and \( I_t \) is the household’s nominal income. The latter is given by \( I_t = W_t l_t + \int_0^1 \Pi_i d\bar{d}_i \), where \( W_t \) is the nominal
wage and \( \Pi_{i,t} \) is the nominal profit of intermediate goods firm \( i \) at date \( t \). The household’s budget constraint in the second *hedging* trading round is:

\[
\tilde{A}_t(s^t) + \sum_{k=2}^{K} \tilde{Q}_k^t(s^t) \tilde{B}_k^t(s^t) \geq A_{t+1}(s^t) + \sum_{k=2}^{K} \tilde{Q}_k^t(s^t) B_{t+1}^k(s^t-1),
\]

(4)

where \( \tilde{Q}_k^t \) denotes the price of a \( k \)-maturity bond in this round.

Following Chari and Kehoe (1993), we assume that household participation in bond markets is anonymous, so that bonds issued by households are unenforceable and no one is willing to buy them.\(^4\) Formally, we assume, for all \( t, s^t \) and \( k \),

\[
B_k^t(s^t-1) \geq 0, \quad \tilde{B}_k^t(s^t) \geq 0.
\]

(5)

This constraint precludes lending by the government to households in the bond market. Consequently, we will refer to it as a *no lending constraint*. Both the repayment of government loans and the payment of taxes are transfers to the government. Ramsey models typically assume that the first is lump sum, while the second is not. This distinction is arbitrary. In practice, costs associated with enforcing repayments and monitoring household effort and productivity are likely to render loan repayments contingent on observed income or consumption. Hence, they will distort household decisions just as taxes do. We do not explicitly model such costs, rather we simply rule government loans out.\(^5\)

Households maximize (1) subject to the constraints for all \( i, t, c_{it} \geq 0, l_t \in [0, T] \) and (2)- (5).

### B. Final goods firms

Final goods firms produce output \( Y_t \) from intermediate goods \( Y_{it} \) using the technology: \( Y_t = \int_{0}^{1} Y_{it}^{1/\mu} di \), \( \mu > 1 \). Intermediate goods are produced by sticky price firms who set their price \( P_{st} \) before \( s_t \) is realized, and flexible price ones who set their price \( P_{ft} \) after \( s_t \) is learned. Letting \( \rho \) denote the fraction of sticky price firms and assuming symmetry across each type of intermediate good firm, the total output of final goods firms is given by:

\[
Y_t = [(1-\rho)Y_{ft}^{1/\mu} + \rho Y_{st}^{1/\mu}]^{\mu},
\]

where \( Y_{ft} \) and \( Y_{st} \) are, respectively, the amount of flexible and sticky price intermediate good used. Final

\(^3\)We assume that the latter is paid as a dividend to the household. To economize on space and without loss of generality, we omit a detailed description of the stock market and assume instead that households own a diversified and non-tradeable portfolio of shares.

\(^4\)On the other hand, we do allow households to borrow from local stores to finance credit good consumption.

\(^5\)Weaker restrictions on government lending of the form \( B_k^t(s^t-1) \geq -B \) would lead to qualitatively similar results.
goods firms are competitive and choose quantities of intermediate goods to maximize their profits:

\[
\sup_{Y_{ft}(s^t),Y_{st}(s^t)} P_t(s^t) \left[ (1 - \rho)Y_{ft}(s^t)^{1\over \alpha} + \rho Y_{st}(s^t)^{1\over \beta} \right]^{1\over \alpha} - (1 - \rho)P_{ft}(s^t)Y_{ft}(s^t) - \rho P_{st}(s^{t-1})Y_{st}(s^t) .
\]

(6)

### C. Intermediate goods

Intermediate goods are produced with labor according to the technology: \( Y_{lt} = \theta_l L_{lt}^\alpha \), where \( \theta_l(s^t) = \theta(s_t) \), \( \theta : S \rightarrow \mathbb{R}_+ \), is a productivity shock. Substituting this and the demand curves stemming from (6) into its objective, a flexible price intermediate goods firm chooses its price \( P_{ft}(s^t) \) to solve:

\[
\sup_{P_{ft}(s^t)} P_{ft}(s^t) \left( P_{ft}(s^t) / P_{t}(s^t) \right)^{\alpha - 1} Y_t(s^t) - W_t(s^t) \left\{ \left( P_{ft}(s^t) / P_{t}(s^t) \right)^{\beta - 1} Y_t(s^t) / \theta_t(s_t) \right\} ^{1\over \beta} .
\]

In contrast, a sticky price firm chooses its price \( P_{st}(s^{t-1}) \) before \( s_t \) is determined, so as to solve:

\[
\sup_{P_{st}(s^{t-1})} E_{s^{t-1}} \left[ (1 - \tau_t)U_{2t} / P_t \left( P_{st}(P_{st}/P_{t})^{\alpha - 1} Y_t - W_t \left\{ (P_{st}/P_{t})^{\beta - 1} Y_t / \theta_t \right\} ^{1\over \beta} \right) \right] .
\]

### D. Government

The government faces a stochastic process for government spending \( \{G_t\}_{t=0}^\infty \) of the form \( G_t(s^t) = G(s_t) \), where \( G : S \rightarrow \mathbb{R}_+ \). The government finances its spending by levying taxes on labor and trading non-contingent nominal bonds. Its budget constraint from the first liquidity trading round is \( A_{gt}(s^{t-1}) + \sum_{k=1}^K Q_{kt}(s^t)B_{kt+1}^{k+1}(s^{t-1}) \leq \tilde{M}_t(s^t) + \sum_{k=1}^K \tilde{Q}_{kt}(s^t)\tilde{B}_{kt}^k(s^t) \), where we use a \( g \) subscript to distinguish elements of the government’s portfolio and \( A_{gt} = B_{gt}^0 + M_t \). Its budget constraint in the second hedging trading round is:

\[
\tilde{A}_{gt}(s^t) + \sum_{k=2}^K \tilde{Q}_{kt}(s^t)\tilde{B}_{kt}^k(s^t) \leq A_{gt+1}(s^t) + \sum_{k=2}^K \tilde{Q}_{kt}(s^t)\tilde{B}_{kt+1}^k(s^t) ,
\]

where \( \tilde{A}_{gt}(s^t) = \tilde{M}_t(s^t) + \tilde{B}_{gt}^1(s^t) - \tau_t(s^t)L_t(s^t) + P_t(s^t)G(s_t) \).

### E. Competitive equilibria and allocations

Define an allocation to be a sequence \( e^\infty = \{c_{1t}, c_{2t}, L_{ft}, L_{st}\}_{t=0}^\infty \) and a continuation allocation by \( e^\infty(s^{t-1}) = \{c_{1t+r}(s^{t-1}, \cdot), c_{2t+r}(s^{t-1}, \cdot), L_{ft}(s^{t-1}, \cdot), L_{st}(s^{t-1}, \cdot)\}_{r=0}^\infty \).

**Definition 1.** \( \{c_{1t}, c_{2t}, l_t, L_{ft}, L_{st}, \tau_t, W_t, P_{st+1}, P_{ft}, P_t, \{Q^k_{kt}\}_{k=1}^K, \{Q^k_{kt}\}_{k=1}^K, \{B^k_{kt}\}_{k=1}^K, \{B^k_{kt}\}_{k=1}^K, M_t, \{B^k_{kt}\}_{k=1}^K, \{B^k_{kt}\}_{k=1}^K, \tilde{M}_t\}_{t=0}^\infty \) is a competitive equilibrium at \( \{P_{s0}, M_0, \{B^k_{0}\}_{k=1}^K \} \) if each \( c_{it} \geq 0, l_t \in [0, T] \) and

1. \( \{c_{1t}, c_{2t}, l_t, \{B^k_{kt}\}_{k=1}^K, M_t, \{\tilde{B}^k_{kt}\}_{k=1}^K, \tilde{M}_t\}_{t=0}^\infty \) solves the household’s problem given \( \{P_{s0}, M_0, \{B^k_{0}\}_{k=1}^K \} \)
2. the sequence of input amounts \( \{L_{ft}^{\alpha}\}_{t=0}^{\infty} \) and \( \{L_{st}^{\alpha}\}_{t=0}^{\infty} \) solve the final goods firm’s problem; the price sequences \( \{P_{ft}\}_{t=0}^{\infty} \) and \( \{P_{s,t+1}\}_{t=0}^{\infty} \) solve the intermediate firms’ problems;

3. the government’s budget constraints hold at each date;

4. the labor, bonds and goods markets clears: \( \forall t, s^t, \; l_t = (1-\rho)L_{ft} + \rho L_{st}, \; B^k_t = \tilde{B}^k_t = \tilde{\tilde{B}}^k_t, \; c_{1t} + c_{2t} + G_t = \theta_t[(1-\rho)L_{ft}^{\alpha/\mu} + \rho L_{st}^{\alpha/\mu}] ; \)

5. the no lending constraints hold: \( \forall t, s^t, k, \; B^k_{g,s,t}(s^{t-1}) \geq 0, \; \tilde{\tilde{B}}^k_{g,s,t}(s^t) \geq 0. \)

e^\infty is a competitive allocation if it is part of a competitive equilibrium.

III. Characterizing competitive allocations

We take a primal approach to the government’s problem. To this end in Proposition 2, we provide a set of conditions that characterize competitive allocations. Before stating the proposition, we discuss those conditions that are new and refer the reader to Siu (2004) for further details of the other more standard conditions.

A. Implementability and Measurability constraints

Implementability and measurability constraints are central elements of any dynamic Ramsey taxation model. We describe and interpret these conditions under our asset market structure and then contrast them with the corresponding constraints from earlier work.

**Primary Surplus Value** First, we define the primary surplus value: \( \xi_t(s^t) \equiv E_{s^t}\left[ \sum_{j=0}^{\infty} \beta^{t+j} \Lambda_{t+j}(s^{t+j}) \right] \), where \( \Lambda_{t+j} = U_{1t+j}c_{1t+j} + U_{2t+j}c_{2t+j} + U_{lt+j}G_{t+j} \) and \( \Psi_{t+j} \equiv \theta_t [(1-\rho)L_{ft} + \rho L_{st}]^{\mu} \). \( \xi_t(s^t) \) gives the present discounted value of future primary surpluses accruing to the government after \( s^t \).

To see this, note that the definition of \( \Lambda_t \), the household’s first order conditions, the expression for profits from an intermediate goods firm and the resource constraint imply: \( \Lambda_t = U_{2t} \left\{ i_t^{\frac{J_t}{T_t}} + \left[ \tau_t s^t - G_t \right] \right\} \), where \( i_t = \frac{1}{\frac{J_t}{T_t}} - 1 \) is the one period nominal interest rate.

**Unanticipated Inflation** Define: \( N_t(s^t) \equiv [(1-\rho)L_{ft}^{\alpha}(s^t) + \rho L_{st}^{\alpha}(s^t)]^{\mu}/L_{st}^{\alpha}(s^t) \). Later we show that in a competitive equilibrium, \( \frac{P_{st}(s^{t-1})}{P_{st}(s^t)} = N_t(s^t) \). \( N_t(s^t) \) can be interpreted as an “unanticipated inflation” term, that departs from one when shocks to the environment prompt flexible price firms to set their prices differently from those set by sticky price firms.
**Bond Pricing** Define the sequence \( \{D_{t+1}^{k}\}_{k=1}^{K} \) by \( D_{t+1}^{1} = 1 \) and for \( k > 1 \),

\[
D_{t+1}^{k}(s^{t}) = \sum_{s^{t+k-1}} \left( \prod_{j=1}^{k-1} \frac{U_{2t+j}(s^{t+j})}{U_{1t+j}(s^{t+j})} \prod_{j=1}^{k-1} \frac{N_{t+j}(s^{t+j})U_{1t+j}(s^{t+j})}{E_{s^{t+j-1}}[N_{t+j}U_{1t+j}]} \right) \pi^{-k-1}(s^{t+k-1}|s^{t}).
\]

Below we establish that in a competitive equilibrium, the liquidity trading round bond price \( Q_{t}^{k}(s^{t}) \) equals \( U_{t}(s^{t})/E_{t}(s^{t})D_{t+1}^{k}(s^{t}) \). Hence, the price of longer term bonds is given by the expected product of cash-credit good wedges under the “distorted” probability measure:

\[
\tilde{\pi}^{-k-1}(s^{t+k-1}|s^{t}) = \prod_{j=1}^{k-1} \left( \frac{N_{t+j}(s^{t+j})U_{1t+j}(s^{t+j})}{E_{s^{t+j-1}}[N_{t+j}U_{1t+j}]} \right) \pi^{-k-1}(s^{t+k-1}|s^{t}). \tag{7}
\]

This measure weights states with higher than average values of \( \prod_{j=1}^{k-1} N_{t+j}(s^{t+j})U_{1t+j}(s^{t+j}) \), and in which unanticipated inflations are modest and cash goods scare, more heavily.

**Portfolio Weights** Finally, we define the portfolio weights \( a_{t}(s^{t-1}) = A_{t}(s^{t-1})/P_{s,t}(s^{t-1}) \) and for each \( k, b_{t}^{k}(s^{t-1}) = B_{t}^{k}(s^{t-1})/P_{s,t}(s^{t-1}) \).

Using this notation, the implementability/measurability constraints are:

\[
\frac{\xi_{t}(s^{t})}{U_{1t}(s^{t})} = N_{t}(s^{t}) \left\{ a_{t}(s^{t-1}) + \frac{U_{2t}(s^{t})}{U_{1t}(s^{t})} \sum_{k=1}^{K-1} b_{t}^{k+1}(s^{t-1})D_{t+1}^{k}(s^{t}) \right\}. \tag{8}
\]

The portfolio weights at date 0 are taken to be predetermined and we will refer to the date 0 version of (8) as the implementability constraint. At dates \( t > 0 \), the portfolio weights will be chosen as part of the competitive equilibrium. However, since the \( \{a_{t}, \{b_{t}^{k}\}_{k=2}^{K}\} \) terms are measurable with respect to \( s^{t-1} \) they place cross-state restrictions on the process for \( \xi_{t} \). We will refer to these conditions as measurability constraints. The left hand side of (8) can again be interpreted as a government primary surplus value. The right hand side of this equation can be interpreted as the value of the government’s liabilities. To see the latter, notice that:

\[
\frac{A_{t}(s^{t-1})}{P_{t}(s^{t})} + \sum_{k=1}^{K-1} \frac{B_{t}^{k+1}(s^{t-1})}{P_{t}(s^{t})}Q_{t}^{k}(s^{t}) = \left\{ a_{t}(s^{t-1}) + \sum_{k=1}^{K-1} b_{t}^{k+1}(s^{t-1}) Q_{t}^{k}(s^{t}) \right\}. \]

unanticipated inflation

\[
\frac{P_{s,t}(s^{t-1})}{P_{t}(s^{t})} \]

term structure.
Using \( N_t(s^t) = \frac{P_{s,t}(s^t-1)}{P_{s^t}(s^t)} \) and \( Q^k_t(s^t) = \frac{U_{1t}(s^t)}{U_{2t}(s^t)} D^k_{t+1}(s^t) \) in this equation gives the right hand side of (8). In any competitive equilibrium, this liability value must equal the value of the government’s primary surpluses and (8) must hold.

Although the portfolio weights \( \{ a_t, \{ b^{k+1}_{k-1}\}_{k=1} \} \) are predetermined at \( t \), the values of the different liabilities in the government’s portfolio are not. These can be altered by changes to either the price level (i.e. to \( N_t(s^t) \)) or the nominal term structure (i.e. to \( \frac{U_{2t}(s^t)}{U_{1t}(s^t)} D^k_{t+1}(s^t) \)). Both types of change alter the value of the government’s liabilities in (8) and allow the government to hedge shocks.

The \( N_t(s^t) \) and \( \frac{U_{2t}(s^t)}{U_{1t}(s^t)} D^k_{t+1}(s^t) \) terms also capture the costly distortions associated with unanticipated price level and term structure changes. If events at \( t \) induce flexible price firms to alter their prices relative to their previously expected level, then \( N_t(s^t) \) departs from 1 and an inefficient allocation of production across firms will occur. If the price of the \( k \)-th maturity outstanding bonds falls, then \( \{ \frac{U_{2t}(s^t)}{U_{1t}(s^t)} D^k_{t+1}(s^t) \}_{k=1}^K \) also departs from 1 and the short run nominal interest rate must exceed zero either now or in some future state. This results in a misallocation of consumption across cash and credit goods as households seek to economize on their use of cash.

**B. Comparison with Existing Models**

The key difference between our model and others becomes apparent in the measurability constraints. It is worth contrasting our version of these conditions with those in the less restrictive environment of Lucas and Stokey (1983) and the more restrictive ones of Siu (2004) and Aiyagari et al (2002).

In the model of Lucas and Stokey, the government can trade real state contingent debt, and so the analogue of (8) is:

\[
\frac{\xi_t(s^t)}{U_{1t}(s^t)} = a_t(s^t). \tag{9}
\]

The portfolio weight \( a_t \) is \( s^t \)-measurable, so that unlike (8), (9) does not represent a collection of cross state restrictions. Except at date 0, when \( a_0(s^0) \) is fixed, the constraints in (9) are redundant.

On the other hand, in Siu (2004), nominal debt of only one period is traded and (8) reduces to:

\[
\frac{\xi_t(s^t)}{U_{1t}(s^t)} = N_t(s^t)a_t(s^{t-1}). \tag{10}
\]

Thus, \( \frac{\xi_t(s^t)}{U_{1t}(s^t)} \) can be varied across states \( s_t \) only by changing \( N_t(s^t) \) through inflation. Since there is no long term debt, there is clearly no opportunity to devalue this debt through increases in future nominal interest rates.
Finally, in Aiyagari et al. (2002), only real debt of one period is traded, which implies:

\[ \xi_t(s^t)/U_{1t}(s^t) = a_t(s^{t-1}). \]  

(11)

In this case, there is not even the opportunity to devalue debt through unexpected inflation.

C. No Lending and Nominal Wealth-in-Advance Constraints

The no lending constraints ensure that the household’s bond holdings are non-negative. For maturities \( k > 1 \), we have \( b^k_t \geq 0 \), while for one period nominal liabilities \( a_t \geq 0 \). Finally, we have a sequence of nominal wealth-in-advance constraints. Households must use some fraction of their nominal wealth in the liquidity trading round to obtain the money necessary for cash good consumption. Consequently, their total nominal wealth restricts their cash good consumption. Using the measurability constraints (8), this restriction can be expressed as:

\[ \xi_t(s^t) \geq U_{1t}(s^t)c_{1t}(s^t). \]  

(12)

The following proposition formally characterizes competitive allocations.

**Proposition 2.** \( e^\infty = \{c_{1t}, c_{2t}, L_{ft}, L_{st}\}_{t=0}^\infty \) is a competitive allocation at \( \{P_{s,0}, M_0, \{B_k^0\}_{k=1}^K\} \) if there exists a sequence of portfolio weights \( \{a_t, \{b^k_t\}_{k=1}^{K-1}\}_{t=0}^\infty \) with \( a_0 = \frac{M_0 + B_0^1}{P_{s,0}} \) and \( b^{k+1}_0 = \frac{B^{k+1}_0}{P_{s,0}} \), such that the portfolio weight sequence and \( e^\infty \) satisfy \( \forall i, t, s^t, \ c_{it}(s^t) > 0, \ (1-\rho)L_{ft}(s^t) + \rho L_{st}(s^t) \in (0,T), \ (8), \ no \ lending: \ \forall k, t, s^{t-1}, \ b^k_t(s^{t-1}) \geq 0 \) and \( a_t(s^{t-1}) \geq 0, \ (12) \) and

1. (No arbitrage) for all \( t, s^t \),

\[ U_{1t}(s^t)/U_{2t}(s^t) \geq 1; \]  

(13)

2. (Resource) for all \( t, s^t \),

\[ G(s_t) + c_{1t}(s^t) + c_{2t}(s^t) = \theta(s_t)[(1-\rho)L_{ft}^\pi(s^t) + \rho L_{st}^\pi(s^t)]^\mu; \]  

(14)

3. (Sticky price optimality) for all \( t > 0, s^{t-1} \),

\[ \sum_{s'^t | s^{t-1}} \pi(s'^t | s^{t-1})U_{it}(s^t)[L_{ft}(s^t)^{1-\alpha}L_{st}(s^t)^{\alpha} - L_{st}(s^t)] = 0. \]  

(15)
If \( e^\infty = \{c_{1t}, c_{2t}, L_{f1}, L_{st}\}_{t=0}^\infty \) is a competitive allocation at \( \{P_0, M_0, \{B_0^k\}_k\} \) with each \( c_{it} > 0 \) and \( (1 - \rho)L_{f1} + \rho L_{st} \in (0, T) \), then \( e^\infty \) satisfies (8), no lending, (12)-(15) for some sequence of portfolio weights \( \{a_t, \{b_k^{k+1}\}_{k=1}^{K-1}\}_{t=0}^\infty \), with \( a_0 = \frac{M_0 + B_0^1}{P_0} \) and \( b_0^{k+1} = \frac{B_0^{k+1}}{P_0} \).

PROOF: See Appendix A. ■

IV. Ramsey problems for incomplete and complete markets economies

The Ramsey problem with non-contingent nominal debt

Given Proposition 2, the optimal policy problem in an economy with initial triple \( \{P_0, M_0, \{B_0^k\}_k\} \) can be formulated as:

\[
\text{Problem 1: } \sup_{\{c_{1t}, c_{2t}, L_{f1}, L_{st}, a_t, \{b_k^k\}_{k=1}^{K-1}\}_{t=0}^\infty} E \left[ \sum_{t=0}^\infty \beta^t U(c_{1t}, c_{2t}, (1 - \rho)L_{f1} + \rho L_{st}) \right] \tag{RP}
\]

subject to \( a_0 = \frac{M_0 + B_1^1}{P_0} \) and \( b_0^k = \frac{B_0^k}{P_0} \), for all \( t, c_{1t}, c_{2t} \geq 0 \) and \( (1 - \rho)L_{f1} + \rho L_{st} \in [0, T] \), (8), no lending and (12)-(15).

The Ramsey problem with complete markets

Before analyzing (RP) in detail, we briefly turn to the benchmark complete markets economy. In this the government faces no restrictions of lending and can trade contingent claims. The corresponding Ramsey problem is essentially that considered by Siu (2004) and others. We merely state two key properties of its solution.

Proposition 3. Under our assumed preferences, from period 1 onwards: 1) the Friedman rule holds and \( U_{1t} = U_{2t} \), 2) flexible price firms always set their prices to the same value as sticky price firms and \( N_t = 1 \).

Thus, when markets are complete, optimal policy implies nominal yields equal to zero at all maturities and an absence of inflation surprises, i.e. \( P_t/P_{t-1} \) is \( s^{t-1} \)-measurable, \( t \geq 1 \).

Comparing policy in incomplete and complete markets economies

It is convenient to rewrite the measurability constraints (8) in matrix form. Given an allocation, let \( \{\Xi_t\}_{t=1}^\infty \) denote the corresponding sequence of primary surplus vectors, where \( \Xi_t(s^{t-1}) \) is an \( N \times 1 \) vector with \( n \)-th element \( \xi_t(s^{t-1}, \hat{s}_n)/U_{1t}(s^{t-1}, \hat{s}_n) \). Let \( \{\Psi_t\}_{t=1}^\infty \) denote the sequence of matrices obtained by stacking the liability values from the right hand side of (8). Thus, \( \Psi_t(s^{t-1}) \) is an \( N \times K \) matrix with \( (n, 1) \)-th element \( \psi_{1t}(s^{t-1}, \hat{s}_n) = N_t(s^{t-1}, \hat{s}_n) \) and \( (n, k) \)-th element, \( k > 1, \psi_{kt}(s^{t-1}, \hat{s}_n) = \).
\[ N_t(s^{t-1}, \hat{s}_n) U_{2t}(s^{t-1}, \hat{s}_n) D_t^{k+1}(s^{t-1}, \hat{s}_n). \] The measurability constraints can then be rewritten as, for all \( t, s^{t-1}, \):

\[ \Xi_t(s^{t-1}) \in \text{Span} (\Psi_t(s^{t-1})). \quad (16) \]

It follows from Proposition 3 that when markets are complete, the optimal continuation allocation equates the marginal utilities of cash and credit goods and the prices of the sticky and flexible-type firms. In that case, \( \Psi_t(s^{t-1}) \), \( t \geq 1 \), is reduced to the unit matrix. By (16), implementation of this allocation in the nominal debt economy would require that \( \xi_t(s^t)/U_{1t}(s^t) \) be \( s^{t-1} \)-measurable.

If this requirement does not hold, and in general it will not, then the optimal complete markets allocation cannot be implemented with non-contingent nominal debt. The logic is straightforward: this allocation typically requires that fiscal shocks be hedged and liability values varied, yet it also precludes state-contingent variations in interest rates and inflation. The latter are precisely the means by which hedging is attained in an economy with non-contingent nominal debt. To put it differently, at the optimal complete markets allocation all nominal assets (regardless of maturity) have the same risk free return and no fiscal hedging is possible.

Although, the optimal complete markets allocation does not usually satisfy the measurability constraints, there are arbitrarily small perturbations of it that do. Small, state-specific substitutions of cash for credit good consumption at the optimal complete markets allocation can be used to perturb the matrices \( \Psi_t(s^{t-1}) \) so that (16) holds. However, while allocations very close to the optimal complete markets one can be implemented in non-contingent nominal debt economies, their implementation usually requires very large negative (and positive) asset positions at some maturities. Such positions are needed to obtain sufficient hedging off of the small variations in interest rates and inflation implied by the perturbation. Clearly, restrictions on short selling in general, and our no lending constraints in particular, prevent the government from obtaining large negative asset positions. In doing so they usually preclude allocations in a neighborhood of the optimal complete markets one.

**Comparing policy in economies with real and nominal incompleteness** Buera and Nicolini (2004) and Angeletos (2002) consider economies in which the government can trade real non-contingent claims of various maturities. Angeletos shows that generically the optimal complete markets allocation can be implemented with non-contingent real debt if the number of maturities traded exceeds the number of states. However, the calibrated examples of Buera and Nicolini suggest that the government may need to take large debt positions to achieve this implementation.
Buera and Nicolini regard this as a problem. In contrast, with non-contingent nominal debt, the optimal complete markets allocation cannot usually be implemented since, as noted, it implies that all nominal assets offer the same riskless return. Moreover, since the implementation of allocations close to the optimal complete markets typically requires an extreme asset market position, the problem identified by Buera and Nicolini is more severe in an economy with nominal debt.

V. A recursive formulation

We now look for a recursive formulation of problem (RP). This formulation must ensure that continuation choices attain the primary surplus and liability values implied by past implementability/measurability constraints. More formally, the measurability constraint may be rewritten as:

\[ [U_1(s^t)a_{t-1}(s^{t-1}) + U_2(s^t) \sum_{k=1}^{K-1} b_{k+1}^{t-1}(s^{t-1})D_{k+1}^t(s^t)]N_t(s^t) = \Lambda_t(s^t) + \beta \phi_{t+1}(s^t), \quad (17) \]

where \( \phi_{t+1}(s^t) = E_{s^t} [\xi_{t+1}(s^{t+1})] \). Additionally, (12) can be recast in terms of \( \phi_{t+1} \) as:

\[ \Lambda_t(s^t) + \beta \phi_{t+1}(s^t) \geq U_1(s^t)c_{1t}(s^t). \quad (18) \]

The tuple \( \{ \phi_{t+1}, \{D_k^t\}_{k=1}^{K-1} \} \) may be interpreted as a list of implicit “promises” made by the government at \( t \) concerning the value of its primary surplus stream and of specific bonds within its portfolio. Satisfaction of (17) requires that future choices implement these promises. Our earlier definitions imply that the \( \phi_t \) and \( D_k^t \) variables evolve recursively according to:

\[ \phi_t(s^{t-1}) = E_{s^{t-1}} [\Lambda_t(s^t) + \beta \phi_{t+1}(s^t)], \quad (19) \]

\[ D_1^t := 1 \text{ and for } k = 2, \ldots, K - 1, \]

\[ D_k^t(s^{t-1}) = E_{s^{t-1}} \left[ \frac{U_2(s^t)}{U_1(s^t)} D_{k+1}^{t-1}(s^t) \frac{N(s^t)U_1(s^t)}{E_{s^{t-1}}[N(s^t)U_1(s^t)]} \right]. \quad (20) \]

Our recursive approach to (RP) treats the variables \( x_t = \{s_{t-1}, \phi_t, \{D_k^t\}_{k=1}^{K-1} \} \) as state variables that summarize relevant aspects of the past history of the economy and ensure that past constraints are satisfied. As with most Ramsey problems, the initial period of (RP) differs from subsequent ones. In the initial period, the government faces a fixed vector of portfolio weights \( \{a_0, \{b_k^{t+1}\}_{k=1}^{K-1} \} \)
rather than a fixed vector of state variables $x_0$. In later periods, this is reversed: the government can be modeled as entering period $t \geq 1$ with a state vector $x_t$ and choosing portfolio weights $\{a_t, \{b^{k+1}_t\}_{k=1}^{K-1}\}$ along with a current allocation $\{c_{1t}, c_{2t}, L_{ft}, L_{st}\}$ and a continuation state vector $x_{t+1}$. Thus, the continuation of the Ramsey problem is recursive in the state variables $\{x_t\}$. In the remainder of this section, we formally state the associated dynamic programming problem. The policy functions that solve this problem can be used to generate an optimal continuation allocation along with corresponding optimal policies.

Let $X$ denote the set of tuples $\{s, \phi, \{D^k\}_{k=2}^{K-1}\}$ that are attained by some continuation competitive allocation in its initial period. We collect recursive versions of the constraints that define a competitive allocation into a correspondence $\Gamma$. Given an inherited tuple of state variables, these constraints ensure that a current consumption-labor allocation and tuple of future state variables are consistent with the requirements of a competitive allocation.

**Definition 4.** Let $\Gamma(s, \phi, \{D^k\}_{k=2}^{K-1})$ equal all tuples $\{a, \{b^k\}_{k=2}^{K-1}, c_1, c_2, L_f, L_s, \zeta', \phi', \{D^{k'}\}_{k=2}^{K-1}\}$ that satisfy for each $s'$, $c_i(s') \geq 0$, $i = 1, 2$, $(1-\rho)L_f(s') + \rho L_s(s') \in [0, L]$, (13)-(15), $a \geq 0$, $b^k \geq 0$, (17)-(20) and for each $s'$ $(s', \phi'(s'), \{D^{k'}(s')\}_{k=2}^{K-1}) \in X$.

The correspondence $\Gamma$ provides the constraint set for our dynamic programming problem:

$$V(s, \phi, \{D^k\}_{k=2}^{K-1}) = \sup_{\Gamma(s, \phi, \{D^k\}_{k=2}^{K-1})} E_s[U(c_1, c_2, (1-\rho)L_f + \rho L_s) + \beta V(s', \phi', \{D^{k'}\}_{k=2}^{K-1}, s')]. \quad (21)$$

Problem (21) can be solved numerically, its policy functions may then be used to compute the optimal continuation Ramsey allocation along with the supporting optimal fiscal and monetary policies at each initial state vector $\{s_0, \phi_1, \{D^k_1\}_{k=2}^{K-1}\}$. We pursue this approach in Section VII. Before doing so, we provide an example that permits an analytical solution and that builds intuition.

**VI. An analytical example: the postponement effect**

In our model, fiscal hedging entails the distortion of private production and consumption decisions. The key advantage of long term debt is that it permits the government to allocate these distortions more efficiently across time and states. The example in this section isolates one aspect of this: long term debt allows the government to postpone the distortionary nominal interest rate rises used to hedge fiscal shocks.

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6LSY (2006) formally demonstrate the recursivity of the continuation Ramsey problem in these state variables.
Setup  We consider a continuation problem beginning in period 1. We suppose that the government entered the preceding period 0 with only one period debt so that there are (effectively) no liability value promises $D_k^1$ to keep. The only promise the government must honor is one concerning its primary surplus value $\phi_1 > 0$.

There is one shock drawn in period 1, from the set $S = \{\bar{s}, \bar{\pi}\}$ according to probability distribution $\pi$. Thereafter, no further shocks occur. Since the hedging results we derive below apply fairly generally to stochastic variations in the government’s financing needs that stem from shocks to the resource constraint, we leave the precise nature of the shock unspecified. It may alter government spending, productivity or both. The essential requirement will be that the shock induces some variation in the shadow value of the primary surplus stream across states. Since all uncertainty is resolved in period 1, the sticky price optimality, measurability and no lending constraints in later periods will not bind and can be dropped. Additionally, we assume that the household’s preferences are given by $U(c_1, c_2, l) = (1 - \gamma) \log c_1 + \gamma \log c_2 + v(l)$, for some smooth, decreasing, concave function $v$ with $\lim_{l \to L} v'(l) = -\infty$. These preferences ensure that the consumption allocation does not affect primary surplus values, significantly simplifying the analysis.

Government Problem  The government’s continuation problem reduces to:

$$\sup_{\{a_1, b_1^k\}_{k=2}^{K-1}, \{c_1, c_2, L_{ft}, L_{st}\}} \sum_{s \in S} \left[ \sum_{t=1}^{\infty} \beta^{t-1} U(c_{1t}(s), c_{2t}(s), (1 - \rho)L_{ft}(s) + \rho L_{st}(s)) \right] \pi(s)$$

subject to the boundary conditions for all $i$ and $t$, $c_{it} \geq 0$ and $(1 - \rho)L_{ft} + \rho L_{st} \in [0, L]$, the no arbitrage (13), resource (14), first period sticky price optimality (15) and no lending conditions $a_1, b_1^k \geq 0$, the nominal wealth-in-advance constraints (12), the implementability condition,

$$\phi_1 = \sum_{s \in S} \xi_1(s) \pi(s),$$

and the first period measurability constraint,

$$\forall s \in S, \quad \left[ \sum_{k=1}^{K-1} \prod_{j=1}^{k} \frac{U_{2j}(s)}{U_{1j}(s)} b_1^{k+1} + a_1 |U_{11}(s) N_1(s) = \xi_1(s) \right].$$

The key first order condition  Let $-\omega_0$ denote the Lagrange multiplier on the implementability condition (23) and $-\omega_1(s) \pi(s)$ the multiplier on the $s$-th measurability constraint (24). We assume
that $\omega_0 > 0$; it is the ex ante shadow value of the primary surplus stream prior to the realization of the shock. Similarly, $\omega_1(s)$ is the additional shadow value of this stream contingent on the realization of $s$. It may be interpreted as a measure of the government’s additional desire for funds after the shock. Under our assumed log preferences, the first order conditions for $c_{it}, i = 1, 2, t = 2, \ldots, K - 1$ can be combined to give:

$$
\beta^{t-1}[U_{1t} - U_{2t}] = -\beta^{t-1}\eta_t[U_{11t} + U_{22t}] - \omega_1U_{11}N_1 \sum_{k=t}^{K-1} \left[b^{k+1}_{1} \prod_{j=2}^{k} \frac{U_{2j}}{U_{1j}} \right] \left[\frac{U_{11t}}{U_{1t}} + \frac{U_{22t}}{U_{2t}} \right], \quad (25)
$$

where $\beta^{t-1}\eta_t(s)\pi(s)$ is the multiplier on the $(s, t)$-th no arbitrage constraint. (25) conveys the key logic underpinning the optimal postponement of nominal interest rate rises and the use of long term debt. It describes the costs/benefits of a small substitution of cash for credit goods at $t$. The term on the left hand side captures the utility cost of the substitution, those terms on the right hand side capture the shadow benefits from relaxing the no arbitrage condition and from hedging. Notice that the hedging benefit (the last term) is undiscounted. If the government uses only the longest maturity debt $K$, then the hedging benefit appears symmetrically in each date $t$ first order condition, $t = 2, \ldots, K - 1$. In contrast, the utility cost $U_{1t} - U_{2t}$ is discounted more heavily as $t$ rises. Intuitively, nominal debt prices are given by undiscounted products of cash-credit marginal rates of substitution (MRS’s) over the term of the debt. Each MRS enters the debt price formula symmetrically so that a reduction in the credit-cash MRS at date $t$, $K > t > 1$, is as effective at perturbing the price of maturity $K$ debt and hedging shocks as a reduction at $r$, $1 < r < t$. Since the cost of the later date $t$ reduction is discounted more heavily, the government should postpone cash-credit MRS reductions and the associated interest rate rises. Since long term debt affords the government the greatest flexibility in using nominal interest rates to hedge shocks and, in particular, permits the greatest postponement, it should be used.

More formally, we have the following results whose proofs are supplied in the appendix.

**Lemma 5.** (Use of long term debt) When the shocks are such that $\omega_1(s) > 0 > \omega_1(s')$, some pair $s, s' \in S$, then it is strictly optimal for the government to use only the longest term debt. When $\omega_1(s) > 0$, the nominal interest rate is greater than zero in periods $t = 2, \ldots, K - 1$.

**Lemma 6.** (Postponement effect) In the state $s$ such that $\omega_1(s) > 0$, $Q_{t+1}^1(s) < Q_t^1(s), k = 1, \ldots, K - 1$. For $t > K - 1$, $Q_t^1(s) = 1$. In the state $s$ such that $\omega_1(s) < 0$, $Q_t^1(s) = 1$ for all $t$.  

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Implications for the yield curve and term premia  Lemma 6 has immediate implications for the yield curve. Since all uncertainty is resolved at date 1, the expectations hypothesis holds from that date onwards and \( Q_1^k = \prod_{j=1}^k Q_1^{1+j} \). It follows that if \( \omega_1(s) \leq 0 \), then the date 1 yield curve remains at zero. On the other hand, if \( \omega_1(s) > 0 \), the yield curve rises and steepens over the horizon \( k = 1, \ldots, K - 1 \). Yields at maturities greater than \( K - 1 \) asymptote towards zero as the maturity increases. Thus, the yield curve is hump shaped, with the hump occurring at maturity \( K - 1 \). As time passes and the debt outstanding at the time of the shock matures, the hump occurs at a progressively shorter maturity, before disappearing at date \( K \).

Lemma 6 also implies that the proportional variation in the period 1 debt price \( \frac{Q_1^{k-1}(s)}{E_0[Q_1^{k-1}]} \) across states is increasing in the debt’s maturity \( k \), up until \( k = K - 1 \). This contributes to a date 0 term premium that is increasing in the maturity of the debt until \( k = K \).\(^7\) Despite the relative cost of \( K \)-maturity debt, the government uses it because it is able to postpone the distortions associated with fiscal hedging.

Remark  This (single shock) example highlights the role of long term debt in allowing for an efficient allocation of distortions across time. As our subsequent calibrated example shows, long term debt also facilitates the efficient allocation of distortions across states. When sequences of shocks occur, long term debt allows future nominal interest rate rises to be concentrated in future shock states where they can contribute to the hedging of future as well as current shocks.

VII. A Calibrated Example

A. Numerical method and parameter values

Numerical method  Our approach is to solve the dynamic programming problem (21) and then back out the implied optimal policies. The state space \( X \) for these problems is endogenous and of dimension \( K \). In our calculations we restrict the state space to be a \( K \)-dimensional rectangular set \( \tilde{X} \). We check that we can numerically solve the dynamic programming problems at each point in \( \tilde{X} \) and that enlarging \( \tilde{X} \) does not significantly alter the numerical results we report. The dynamic programming problem is solved by a value iteration. The main computational difficulty concerns the dimension of the state space which is increasing in the maximal debt maturity \( K \). To enable us to solve problems with a maturity structure of reasonable length, we use Smolyak’s algorithm

\(^7\)If prices are completely sticky, then the period 0 expected real holding return is given by: \( \frac{1}{HR_0^k} = \beta E_0[U_{11}/U_{10}] + \beta \text{Cov}_0(U_{11}/U_{10}, Q_1^{k-1}/E_0[Q_1^{k-1}]) \). Hence, the covariance term is negative and, following the discussion in the text, decreasing in \( k \). Thus, \( HR_0^k \) rises in \( k \).
to approximate the government’s value function on a sparse grid fitted to $\hat{X}$. For further details on Smolyak’s algorithm see Krueger and Kubler (2004).

**Calibration** To permit comparability of our results to those in Siu (2004) and Chari et al (1991), we first compute a baseline case with parameter values that are close to theirs. In this baseline case, we assume preferences of the form:

$$U(c_1, c_2, l) = \log \left[ (1 - \gamma)c_1^\phi + \gamma c_2^\phi \right] + \psi \log(T - l).$$  

(26)

We set the preference parameters $\gamma$, $\phi$ and $\beta$ to 0.58, 0.79 and 0.96. We choose $\psi$ so that approximately 30% of an agent’s time is spent working. The values of $\gamma$ and $\phi$ are similar to those used by Siu (2004) and Chari et al (1991). The value of $\phi$ used implies an elasticity of substitution between cash and credit goods of approximately 4.8 and thus a high degree of substitutability between these goods. We also compute a version of the model with preferences that are log in both cash and credit goods. This version has an elasticity of substitution equal to one. We follow Siu (2004) and set the production parameters $\alpha$, $\mu$ and $\rho$ to 1.0, 1.05 and 0.08 respectively. Government spending takes on two values $G$ and $\bar{G}$. The government spending process has a mean of around 20% of GDP in a complete markets model with a debt to GDP ratio of 60%. We set the standard deviation of this process to be 6.7% and its autocorrelation coefficient to 0.95. These values are close to those estimated from the data and conform to the values used in Siu. We also consider a version of the model with a larger standard deviation for shocks of 14%. The log of the productivity process is modeled as a two state Markov chain with mean zero, autocorrelation coefficient 0.95 and standard deviation. Productivity and government spending shocks are assumed to be independent.

We allow the maximal maturity $K$ to vary between 1 and 7. We conjecture that all of the effects we identify would be quantitatively reinforced if $K$ were raised above 7.

**B. Results**

All numerical calculations confirm the result from the simpler analytical example: the government uses only the longest maturity debt available. In each hedging trading round, it funds its deficit and refinances its portfolio with debt of maturity $K$. In the remainder of this section, we focus on both the qualitative and quantitative implications of optimal policy for nominal interest rates, inflation and debt holding returns. We illustrate these implications with short run impulse responses to
shocks and with sample moments from long simulations.

B.1. Impulse responses

In each of the impulse responses presented in this section, the government is assumed to have an initial debt value to output ratio of about 40%. For clarity, we incorporate only government spending shocks. The government draws low spending shocks until period 4. A spell of high spending shocks then ensues for between 1 and 10 periods. When this spell ends, low spending shocks are drawn thereafter.

Holding returns Figure 1 shows the evolution of the real holding return on the government’s portfolio for economies with baseline preferences. The solid line is drawn for the case $K = 7$, the dashed line for $K = 3$. In both cases, the holding return falls in period 5 contemporaneously with the high spending shock. However, when $K = 7$, realized holding returns fall by nearly 2.0 percentage points, whereas, when $K = 3$ they fall by only about 0.3 percentage points. Over the next few periods, holding returns rise. In period 15, government spending falls back to the lower level and both holding returns rise sharply. This increase is about eight times larger in the $K = 7$ case. The quantitative difference between the two cases provides a first indication that the government is better able to hedge fiscal shocks by altering the real value of its debt, and, hence, the real holding return, when the maturity of that debt is larger.

Figure 1. Debt holding returns

Nominal capital gains and inflation Since the government uses only the longest term debt, the real holding return on its portfolio at $t$ can be decomposed as: $HR_t = q^K_t - \pi_t$, where
$q^K_t = \log Q^K_{t-1} - \log Q^K_t$ gives the rate of nominal capital gains on the $K$-th maturity bond and $\pi_t$ is the inflation rate. Figure 2 below illustrates the impulse responses of $q^K_t$ and $\pi_t$ for the cases $K = 3$ (dashed line) and $K = 7$ (solid line). Qualitatively, the $q^K_t$ responses are similar. In both cases, the $K$-maturity bond price decreases coincidentally with the onset of high government spending shocks in period 5. In this way, the government reduces the real holding return on its portfolio by delivering a nominal capital loss to bond holders. Conversely, when government spending falls back in period 15, there is an increase in the nominal bond price and a nominal capital gain for investors. Despite these qualitative similarities, the cumulative fall in debt prices in the aftermath of the shock is obviously much greater (2.4% versus 0.4%) when $K = 7$. Additionally, the spell of low debt prices is more persistent and the subsequent rise in debt prices in period 15 much greater in this case.

**Figure 2.** Nominal capital gains and inflation

![Nominal capital gains and inflation](image)

The nominal capital gains and losses in periods 5 and 15 are reinforced by contemporaneous changes in inflation. In both the $K = 7$ and $K = 3$ cases, there are small spikes in inflation, positive in period 5 and negative in period 15. The first spike devalues debt, the second revalues it, at the expense of some distortion to the labor allocation. Moreover, changes in inflationary expectations play a role in implementing the nominal capital losses and gains described in the previous paragraph. When the high spending shock hits in period 5, optimal policy engineers an increase in inflationary expectations in the $K = 7$ case. This increase reduces the expected real payout obtained from long term debt held in period 5 and, hence, contributes to the lower debt price in that period. Higher inflationary expectations manifest themselves in the run up in inflation that occurs between periods 7 and 10 in this case. These higher inflation rates are expected with
high probability and therefore cause little distortion to the labor allocation.\footnote{The Euler equations for money and short term bonds imply a risk augmented Fisher equation that links expected inflation to short term nominal interest rates. The higher inflation rates over periods 7 to 10 are, thus, linked to the build up in nominal interest rates that we discuss in the next subsection.}

**Short run nominal interest rates** Recall that the $k$-th maturity debt price in period $t$ is given by: $Q^k_t(s^t) = \tilde{E}_t[s^t][\prod_{j=0}^{k-1} U_{2t+j}/U_{1t+j}]$ where the expectation $\tilde{E}_t$ is constructed using the distorted probability measure $\tilde{\pi}^k(s^{t+k}|s^t)$. It follows that the nominal capital losses delivered to investors during and after period 5 must be associated with an increase in the conditional expectation of future short run nominal interest rates (under the distorted probability measure).

**Figure 3.** One period nominal interest rates

Figure 3 shows the impulse response of one period nominal interest rates for the $K = 7$ and $K = 3$ cases. In each case, this interest rate gradually rises to a peak after period 5 and the advent of the high government spending shocks. It then falls back (at date $5 + K - 1$) as the debt outstanding at the time of the first high spending shock matures. The gradual increase is consistent with efforts to delay the distortion from positive nominal interest rates identified in the example from Section VII. Quantitatively, the initial rise in nominal interest rates is larger and more protracted when $K = 7$. In this case, rates peak in period 10 at about 0.85%, whereas when $K = 3$, they peak at 0.30% in period 6. We conjecture that as $K$ is increased further, short term nominal interest rates would peak at a higher value and at a progressively later date. We take these results as further indication that the government is better able to hedge using longer term nominal debt, by delaying the associated costly nominal interest rate distortions.
Yield curves Figure 4 plots the evolution of the yield curve for the $K = 7$ case as the economy is hit by a series of high spending shocks. Initially, at $t = 4$, government spending is low and the yield curve (dotted line) is fairly flat and close to zero. With the realization of the first high spending shock at $t = 5$, nominal interest rates rise at all maturities (solid line). Consistent with the pattern of short run nominal interest rates and our earlier simple example, the increase is greatest at the longest outstanding maturity $K - 1 = 6$. Thus, the yield curve is hump shaped, tilting upwards over maturities $k = 1$ to $K - 1$ and downwards from $K - 1$ onwards. Over periods 6 to 10, the hump rises and passes to lower maturities. Once all initially outstanding debt has matured in period 11, the yield curve falls back to a lower level and adopts a flatter shape (solid-circle line).

Sensitivity analysis: Variations in preferences and shock volatility Figure 5 shows the evolution of one period holding returns on the government’s portfolio and nominal interest rates for the baseline economy (Case 1, solid line), an economy with a large shock volatility (a standard deviation of 14%, dashed line) and an economy with a large shock volatility and preferences that are log-log in cash and credit goods (Case 3, solid line with circles). After the high government spending shock occurs, the holding return falls by 2.4% in Case 1, 3.9% in Case 2 and 7% in Case 3. Similarly, the peak in nominal short term nominal interest rates climbs from 0.85% in Case 1, to 1.8% in Case 2, to 3.3% in Case 3. As the volatility of shocks rises, the government hedges to a greater degree, in part by raising nominal interest rates further and distorting the cash-credit good consumption more. When preferences are log-log in cash and credit goods, the elasticity of substitution between these goods is reduced relative to the baseline case. Distortions to the cash-credit good consumption margin are less costly and the government is prepared to distort this
margin even further. Thus, the holding return falls most in Case 3, and nominal interest rates rise most in this case. Clearly, the elasticity of substitution between cash and credit goods is important in determining the degree of fiscal hedging that occurs in an economy with nominal debt.

**Figure 5.** Debt holding returns and nominal interest rates

B.2. Long simulations

In this section we report results from long simulations of various economies. Each simulation is of length $T_{\text{sample}} = 20,000$. Figure 6 shows simulated values for the debt to output ratio (dashed line) and nominal interest rates (solid line) for the first 1,000 periods of the baseline economy. The figure shows that the volatility of nominal interest rates is increasing in the debt level. At high debt levels a given interest rate volatility induces greater absolute variation in the government’s total liability value and provides a more effective hedge against fiscal shocks.

Tables 1 and 2 below summarize the remainder of our results. Table 1 gives a rough measure of the impact of a positive fiscal shock on the value of the government’s debt. In particular, the first row of this table gives the average variation in the real value of government debt across high and low spending states:

$$
\Delta B = \frac{1}{T_{\text{sample}}} \sum_{t=0}^{T_{\text{sample}}} \left[ \frac{Q_t^{K-1}(G)}{P_t(G)} - \frac{Q_t^{K-1}(G)}{P_t(G)} \right] B_t^K,
$$
Figure 6. Debt levels and nominal interest rates

The next two rows break this adjustment down into a component that comes purely from nominal capital losses and a component that comes from a contemporaneous price inflation. The remainder of the table gives these values normalized in the variation in government spending $\Delta G = \bar{G} - G$. Table 1 reports results for the baseline economy with $K = 1, 3$ and $7$. As $K$ rises both the degree of hedging (as measured by $\Delta B$) and the extent to which this hedging is obtained from movements in debt prices rather than contemporaneous inflations increases. When $K = 7$, $\Delta B$ equals about 71% of the variation in government spending $\Delta G$. Over 80% of this variation comes from a movement in the nominal debt price and less than 20% from a contemporaneous inflation. The table also reports results for the economy with more volatile shocks and with log-log preferences. There is more hedging of shocks in both cases, with considerably increased reliance on adjustments in debt price in the latter case.

Table 1: Financing Government Spending

<table>
<thead>
<tr>
<th></th>
<th>K = 1</th>
<th>K = 3</th>
<th>K = 7</th>
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<tbody>
<tr>
<td></td>
<td>Base</td>
<td>Base</td>
<td>Base</td>
</tr>
<tr>
<td>$\Delta$ in real value of debt</td>
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<td>-0.0125</td>
<td>-0.0356</td>
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<tr>
<td>change in inflation</td>
<td>-0.0065</td>
<td>-0.0066</td>
<td>-0.0065</td>
</tr>
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<td>-0.0292</td>
</tr>
<tr>
<td>$\Delta$ in real value of debt (norm.)</td>
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<td>-0.2491</td>
<td>-0.7117</td>
</tr>
<tr>
<td>change in inflation</td>
<td>-0.1293</td>
<td>-0.1328</td>
<td>-0.1294</td>
</tr>
<tr>
<td>change in price of debt</td>
<td>0.0000</td>
<td>-0.1189</td>
<td>-0.5844</td>
</tr>
</tbody>
</table>

Table 2 reports statistics from long simulations of different versions of the model. For reasons of space, we focus below on the contrast between the baseline economies with $K = 1$ and $K = 7$. First, the correlation between government spending shocks and one period nominal interest rates
is increasing in the maximal debt maturity $K$. Relatedly, the correlation between government spending shocks and the nominal price of the government’s debt portfolio is decreasing (towards $-1$) in $K$. This captures the fact that the government raises nominal interest rates further from 0 for longer in the aftermath of an adjustment from low to high spending as $K$ rises. The correlation between interest rates and spending shocks is negative for $K = 1$, but positive for $K > 1$. In the former case, the measurability constraints are of the form: $\xi_t = N_t U_{1t} a_{t-1}$. When a high spending shock occurs, $\xi_t$ falls. The government partly accommodates this by decreasing $U_{1t}$ (relative to $U_{2t}$) and, hence, reducing the current nominal interest rate. For $K > 1$, the measurability constraint at the optimal debt portfolio takes the form: $\xi_t = N_t U_{2t} D^{K-1}_{t+1} b^K_t$. In this case, to accommodate the fall in $\xi_t$, the government depresses $U_{2t}$ (relative to $U_{1t}$) and $D^{K-1}_{t+1}$. Hence, it raises current and future nominal interest rates. The mean, standard deviation and autocorrelation coefficient for nominal interest rates all increase with $K$, while the standard deviation of tax rates and their correlation with government spending shocks decrease slightly with $K$ (holding the shock volatility fixed). The latter indicates greater tax smoothing at higher $K$ values.
Table 2: Statistics from Long Simulations

<table>
<thead>
<tr>
<th>Statistics</th>
<th>$K = 1$</th>
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<th>$K = 7$</th>
<th>$K = 7$</th>
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<tr>
<td></td>
<td>Base</td>
<td>Base</td>
<td>High volatility</td>
<td>High vol.; log preferences</td>
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<tr>
<td>inflation</td>
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<tr>
<td>st. deviation</td>
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<td>0.290</td>
<td>0.334</td>
<td>0.688</td>
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<td>taxes</td>
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<td>mean</td>
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<td>0.995</td>
<td>0.997</td>
<td>0.988</td>
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<td>0.970</td>
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<td>0.832</td>
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<tr>
<td>correlation with G-shock</td>
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<td>-0.192</td>
<td>-0.658</td>
<td>-0.760</td>
</tr>
<tr>
<td>1-period nom. interest rate</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean</td>
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<td>0.084</td>
<td>0.264</td>
<td>0.521</td>
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<tr>
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<td>0.181</td>
<td>0.364</td>
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<tr>
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<td>0.152</td>
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<tr>
<td>correlation with G-shock</td>
<td>-0.396</td>
<td>0.420</td>
<td>0.509</td>
<td>0.573</td>
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VIII. Conclusion

We have explored optimal debt management and taxation when the government is restricted to using non-contingent nominal debt of various maturities and is limited in its ability to lend. Our model prescribes the exclusive use of long term debt. Such debt mitigates the distortions associated with hedging fiscal shocks by allowing the government to postpone them. This policy of postponement has implications for the management of short term interest rates and the evolution of the yield curve in the aftermath of a fiscal shock. Other contributors have argued that the use of long term debt may raise the government’s financing costs or expose it to unnecessary risk. Their arguments have implicitly treated inflation and/or the yield curve as parameters. In our model, which endogenizes all prices, the holding return on long term nominal debt is more volatile than that on short term debt. However, this volatility is deliberate and is used to hedge fiscal shocks. Higher risk premia on long term debt are the analogues of insurance premia paid by the government.
and are not a rationale for shortening the maturity structure.

References


Appendix

**Proof of Proposition 1**

**Necessity** Suppose \( \{c_{1t}, c_{2t}, L_{f,t}, L_{s,t}\}_{t=0}^{\infty} \) is an interior competitive allocation with no government lending at \( \{P_0, A_0, \{B_{0}^{k+1}\}_{k=1}^{K-1}\} \). We show that it satisfies the conditions in the proposition.
There is no loss of generality in assuming that at the equilibrium bond prices households have no desire to borrow. The interiority of the competitive allocation implies that the constraints $\forall i, t, c_{it} \geq 0$ and $(1 - \rho)L_{ft} + \rho L_{st} \in [0, T]$ are non-binding. We assume the existence of optimal Lagrange multipliers on the households’ constraints. Let $\mu_t(s^t)$ denote the multiplier on the household’s period $t$ cash-in-advance constraint. Similarly, let $\lambda_t(s^t)$ and $\tilde{\lambda}_t(s^t)$ denote, respectively, the multipliers on the liquidity and hedging round budget constraints. The transversality condition: \( \lim_{t \to \infty} \beta^t E[\lambda_t(s^t) \{ \tilde{\lambda}_t(s^t) + \sum_{k=0}^{K-1} Q_k^t(s^t) \tilde{B}_{t+1}^k(s^t) \}] = 0 \). The first order conditions for consumption and labor supply are:

\[
c_{1t} : \quad \{ \tilde{\lambda}_t(s^t) + \mu_t(s^t) \} P_t(s^t) = U_{1t}(s^t) \quad (A1)
\]
\[
c_{2t} : \quad \tilde{\lambda}_t(s^t) P_t(s^t) = U_{2t}(s^t) \quad (A2)
\]
\[
l_t : \quad \tilde{\lambda}_t(s^t)(1 - \tau_t(s^t)) \frac{\partial I}{\partial l_t}(s^t) = -U_{lt}(s^t). \quad (A3)
\]

The first order conditions for each of money and bonds are:

\[
\tilde{M}_t : \quad \lambda_t(s^t) = \mu_t(s^t) + \tilde{\lambda}_t(s^t) \quad (A4)
\]
\[
M_{t+1} : \quad \tilde{\lambda}_t(s^t) = \beta E_{st}[\lambda_{t+1}^t] \quad (A5)
\]
\[
\tilde{B}_t : \quad Q_k^t(s^t) \lambda_t(s^t) = \tilde{Q}_k^t(s^t) \tilde{\lambda}_t(s^t) \quad (A6)
\]
\[
B_{t+1}^k : \quad \tilde{Q}_k^t(s^t) \tilde{\lambda}_t(s^t) = \beta E_{st}[Q_{k}^{t+1}\lambda_{t+1}] \quad (A7)
\]

Combining (A1) and (A2), we obtain: \( \frac{U_{ft}}{U_{st}} = \frac{\lambda_{t+1} - \mu_t}{\lambda_t} \geq 1 \). This establishes (13). Adding the household’s and the government’s hedging round budget constraints and using the definition of firm profits gives (14).

The first order condition from the final goods firm implies \( P_{st}(s_t^t) = (Y_{ft}(s_t)/Y_{st}(s_t))^\frac{\alpha - 1}{\alpha} P_t(s^t) = (Y_{ft}(s_t)/Y_{st}(s_t))^\frac{\alpha - 1}{\alpha} P_{ft}(s^t) \). (A8)

The flexible price firm’s first order conditions gives \( P_{ft}(s_t) = \frac{\partial W_t(s_t)}{\partial (s_t)} L_{ft}(s_t)^{1-\alpha} \). Combining this with (A8) we obtain \( P_{st}(s_t^{-1}) = \left( \frac{Y_{ft}(s_t)}{Y_{st}(s_t)} \right) ^{\frac{\alpha - 1}{\alpha}} \frac{\partial W_t(s_t)}{\partial (s_t)} L_{ft}(s_t)^{1-\alpha} \). As in Siu (2004), this last expression, the first order condition from the sticky price firm’s problem and the household’s first order conditions imply (15).

Next take the household’s hedging round budget constraint at $t$, multiply it by $\tilde{\lambda}_t(s^t)$, add $\mu_t(s^t)\tilde{M}_t(s^t)$ and use the household’s first order conditions to obtain:

\[
\tilde{\lambda}_t(s^t) \sum_{k=1}^{K} \tilde{Q}_t^k(s^t) \tilde{B}_{t+1}^k(s^t) + \{ \tilde{\lambda}_t(s^t) + \mu_t(s^t) \} \tilde{M}_t(s^t) = U_{1t}(s^t)c_{1t}(s^t) + U_{2t}(s^t)c_{2t}(s^t) + U_{lt}(s^t)l_t(s^t)/W_t(s^t)
\]

\[
+ \beta E_{st} \left[ \tilde{\lambda}_{t+1}(s_t^{t+1}) \sum_{k=1}^{K} \tilde{Q}_{t+1}^k(s_t^{t+1}) \tilde{B}_{t+1}^k(s_t^{t+1}) + \{ \tilde{\lambda}_{t+1}(s_t^{t+1}) + \mu_{t+1}(s_t^{t+1}) \} \tilde{M}_{t+1}(s_t^{t+1}) \right]. \quad (A11)
\]
Using the expressions for profits from the intermediate goods firms problems, $I_t(s^i)/W_t(s^i) = \Upsilon_t(s^i)$. Iterating on (A11) and using the household’s first order and transversality conditions gives: $U_{ttt}(s^i) [\sum_{k=1}^{\infty} Q^k_t(s^i) \frac{\tilde{P}^k_t(s^i)}{\tilde{P}^{k+1}_t(s^i)} + \frac{\tilde{M}_t(s^i)}{\tilde{P}^1_t(s^i)}] = \xi_t(s^i)$, where $\xi_t(s^i) = E_s \{ \sum_{j=0}^{\infty} \beta^{t+j} \{ U_{ttt+j} c_{ttt+j}(s^{i+j}) + U_{ttt+j+j} Y_{ttt+j}(s^{i+j+t}) \} \}$.

The household’s liquidity round budget constraint at $t$ and the last equation imply:

$$A_t(s^{i-1})/P_t(s^i) + \sum_{k=1}^{\infty} Q^k_t(s^i) B^{k+1}_t(s^{i-1})/P_t(s^i) = \xi_t(s^i)/U_{ttt}(s^i). \quad \text{(A13)}$$

Combining (A8), the definition of $N_t$ and the household’s first order conditions gives $\frac{P_t}{\tilde{P}^{t+1}_t} = \frac{1}{\beta E_s[N_{ttt} U_{ttt}]}$. Using this and the household’s first order conditions again, we obtain:

$$Q^k_t = E_t \left[ \frac{U_{ttk-1}}{U_{tt+1}^t} \prod_{j=0}^{k-2} \left\{ \frac{N_{ttt+j+1} U_{ttt+j}}{E_{ttt+j}[N_{ttt+j} U_{ttt+j+1}]} \right\} \right] = \frac{U_{ttk}}{U_{tt+1}^t} D^{k+1}_t. \quad \text{(A16)}$$

Combining (A8), (A13), (A16) and the definitions $a_t(s^{i-1}) = A_t(s^{i-1})/P_{ttt}(s^{i-1})$ and $b^k_t(s^{i-1}) = B^k_t(s^{i-1})/P_{ttt}(s^{i-1})$, we have the implementability/ measurability constraints (8). The definitions of $b^k_t$ and $a_t$, and the fact that $B^k_t \geq 0$ and $A_t \geq 0$ gives the no lending constraints. Finally, from (8), the non-negativity constraints on debt and the cash-in-advance constraint, we obtain:

$$\frac{\xi_t(s^i)}{U_{tt}^t} = \sum_{k=1}^{\infty} Q^k_t(s^i) \frac{\tilde{P}^k_t(s^i)}{\tilde{P}^{k+1}_t(s^i)} + \frac{\tilde{M}_t(s^i)}{\tilde{P}^1_t(s^i)} \geq c_{ttt}(s^i), \text{ and, hence, } (12).$$

**Sufficiency** We construct a candidate competitive equilibrium from an allocation and a portfolio weight sequence satisfying the conditions in the proposition. First we set prices. At date 0, $P_{s,0}$ is a parameter, while $P_0$ and $P_f^0$ are set to $P_0(s^0) = P_{20}(Y_{s0}(s^0)/Y_0(s^0))^{\frac{\alpha-1}{\alpha}}$ and $P_f^0(s^0) = (Y_{s0}(s^0)/Y_0(s^0))^{\frac{\alpha-1}{\alpha}}$ respectively. For $t > 0$, set the relative sticky price to:

$$P_{st}/P_{t-1} = \beta/U_{tt=1} E_{t-1} (\sum_{j=0}^{\infty} \beta^{j+t} (Y_{st/j} Y_{st+j})^{\frac{\alpha-1}{\alpha}}), \quad \text{(A17)}$$

the gross (final goods) rate of inflation to:

$$P_t(s^i)/P_{t-1}(s^{i-1}) = P_{st}(s^{i-1})/P_{t-1}(s^{i-1}) (Y_{st}(s^i)/Y_{st}(s^i))^{\frac{\alpha-1}{\alpha}}. \quad \text{(A18)}$$

and the flexible price to $P_{ft}(s^i) = P_{st}(s^{i-1}) (Y_{st}(s^i)/Y_{ft}(s^i))^{\frac{\alpha-1}{\alpha}}$. These conditions allow us to recursively recover all goods prices. For $k > 0$ and $t \geq 0$, set the asset prices $Q^k_t$ from the period $t$ liquidity round budget constraint to:

$$Q^k_t(s^i) = \frac{U_{ttk}(s^i)}{U_{ttt}(s^i)} D^{k+1}_t(s^i). \quad \text{(A19)}$$

Also, for $k > 0$ and $t \geq 0$, set the asset prices from the period $t$ hedging round budget constraint to be $\tilde{Q}^k_t(s^i) = D^{k+1}_t(s^i)$. For $t > 0$, we set the portfolios purchased by households in the hedging round as follows.
The level of debt of $k > 1$ maturity is fixed at $B_k^k(s^{-1}) = b_k^k(s^{-1})P_k^*(s^{-1})$. Using the no lending constraint, $B_k^k(s^{-1}) \geq 0$. Also by this constraint, $a_k(s^{-1}) \geq 0$, and we can choose $M_k(s^{-1}) \geq 0$ and $B_k^k(s^{-1}) \geq 0$ so that $M_k(s^{-1}) + B_k^k(s^{-1}) = a_k(s^{-1})P_k^*(s^{-1})$. Let $A_k(s^{-1}) = a_k(s^{-1})P_k^*(s^{-1})$. Next we turn to the portfolios purchased in the liquidity round. For $t \geq 0$, the money supply is set to $\tilde{M}_t(s^t) = P_t(s^t)c_{11}(s^t)$. From the measurability constraints (8), the above definitions of goods prices, asset prices and portfolios and the nominal wealth-in-advance constraints (12), we have:

$$\frac{\xi_t(s^t)}{U_{1t}(s^t)} = \tilde{M}_t(s^t) P_t(s^t) = \beta E_t \left( \frac{\xi_{t+1}}{U_{1t+1}(s^{t+1})} \right).$$

It follows that, at each date $t$, we can choose a non-negative debt portfolio $\{\tilde{B}_t^k(s^t)\}_{k=1}^K \in \mathbb{R}_t^K$ so that the liquidity round budget constraints hold with equality. Hence, the no lending, liquidity round budget and cash-in-advance constraints are satisfied. The government’s debt holdings are set equal to the household’s holdings of bonds.

We verify the household’s first order conditions. Set the real wage to $\frac{W_t}{P_t} = a_0 + \beta_t L_{ft}^{-1} \left( \frac{Y_t}{P_t} \right)^{\gamma_t}$, the income tax rate to $(1 - \tau_t) = -\frac{U_{1t}}{U_{2t}} P_t^* - \lambda_t$, and the Lagrange multipliers to $\lambda_t = U_{1t} \geq 0$, $\lambda_t P_t = U_{2t} \geq 0$ and $\mu_t P_t = U_{3t} - U_{2t} \geq 0$. It is then immediate that $\lambda_t = \mu_t + \lambda_t \tilde{\lambda}_t$, $\lambda_t P_t = U_{1t} \tilde{\lambda}_t$, $\lambda_t P_t = U_{2t}$ and $\lambda_t - \mu_t (1 - \tau_t) \partial l_t/\partial l_t = -U_{1t}$. Also, (A17) and (A18) imply $U_{2t} = \beta E_t \left( \frac{P_t}{P_{t+1}} \right) U_{1t+1}$, so that $\lambda_t = \beta E_s [\lambda_{t+1}].$

Finally, the definitions of $Q_t^k$, $\tilde{Q}_{t+1}^k$ and the multipliers gives $Q_t^k \lambda_t = \tilde{Q}_{t+1}^k \tilde{\lambda}_t$ and $\tilde{Q}_{t+1}^k \tilde{\lambda}_t = \beta E_s [Q_{t+1}^{k-1} \lambda_{t+1}].$

Next we verify the household’s hedging round budget constraints. Combining (8), (A18) and (A19) gives:

$$\xi_t(s^t) = \frac{U_{1t}(s^t)}{P_t(s^t)} \left[ A_t(s^{-1}) + \sum_{k=1}^{K-1} Q_t^k(s^t) B_{t+1}^k(s^{-1}) \right].$$

(Hence, using the liquidity round budget constraint and the definitions of $Q_t^k(s^t)$ and $\tilde{Q}_{t+1}^k(s^t)$ and dividing by $U_{2t}(s^t)$, we have $\frac{\xi_t(s^t)}{U_{2t}(s^t)} = \frac{U_{1t}(s^t)}{U_{2t}(s^t)} \frac{\tilde{M}_t(s^t)}{P_t(s^t)} + \sum_{k=1}^{K-1} \frac{\tilde{Q}_t^k(s^t)}{P_t(s^t)} \frac{\tilde{B}_t^k(s^t)}{P_t(s^t)}$ adding $\frac{U_{2t}(s^t) - U_{1t}(s^t)}{P_t(s^t)} \tilde{M}_t(s^t)$ to each side of this equation, using the definition of $\xi_t(s^t)$ and $\tau_t(s^t)$ and $\tilde{M}_t(s^t)$ to each side, we get $\frac{\tilde{M}_t(s^t)}{P_t(s^t)} \beta E_t \left[ \xi_{t+1} \right] = c_{1t}(s^t) + c_{2t}(s^t) - (1 - \tau_t(s^t))I_t(s^t) + \beta \frac{U_{2t}(s^t)}{U_{1t}(s^t)} E_t \left[ \xi_{t+1} \right]$. Then, using (A20) at $t + 1$, the definitions of $\tilde{Q}_{t+1}^k$, $\tilde{Q}_{t+1}^k$ and the condition $U_{2t} - \beta E_{t+1} \left[ \frac{P_{t+1}}{P_{t+1}} U_{1t+1} \right] = 0$, we obtain:

$$\frac{\tilde{M}_t(s^t)}{P_t(s^t)} + \sum_{k=1}^{K} \tilde{Q}_t^k(s^t) \frac{\tilde{B}_t^k(s^t)}{P_t(s^t)} = c_{1t}(s^t) + c_{2t}(s^t) - (1 - \tau_t(s^t))I_t(s^t) + \frac{A_{t+1}(s^t)}{P_t(s^t)} + \sum_{k=1}^{K-1} \tilde{Q}_t^k(s^t) \frac{B_{t+1}^k(s^t)}{P_t(s^t)} \right] .$$

The hedging round budget constraint at $t$ then follows from (A21) and the definition of $\tilde{A}_t(s^t)$.

By (8) and the interiority of the allocation, $\xi_0$ is finite. Using the definition of $\xi_t$, we have for all $T$,

$$E \left[ \xi_0 \right] = E \left[ \xi_0 + \sum_{t=0}^{T} \beta^t \left[ U_{1t+1} + U_{2t+2} + U_{1t+1} Y_t \right] \right] = E \left[ \sum_{t=0}^{T} \beta^t \left[ U_{1t+1} + U_{2t+2} + U_{1t+1} Y_t \right] \right] + \beta^{T+1} E \left[ \xi_{T+1} \right].$$

Taking limits and using the period $T + 1$ measurability constraint then gives: $\lim_{T \to \infty} \beta^{T+1} E \left[ \xi_{T+1} \right] = \lim_{T \to \infty} \beta^{T+1} E \left[ U_{1T+1} \left( \frac{A_{T+1}}{P_{T+1}} + \sum_{k=1}^{K-1} \frac{B_{T+1}^k}{P_{T+1}} \right) \right] = 0$ which confirms the transversality condition. Hence, the allocation is feasible and optimal for households at the derived prices and tax rates. The household’s budget constraints,
the resource constraint and the definitions of $A_{gt}$ and $B_{gt}$ ensure that the government’s budget constraints are satisfied. It is easy to verify that the derived choices of firms satisfy their first order conditions and are optimal. ■

**Proof of Lemma 5:** For a proof that a solution to the government’s problem exists, see LSY (2006). Let $\{a^*_1, \{b^k_1\}_{k=2}^K\}$ denote an optimal portfolio. Since $\phi_1 > 0$, either $a^*_1 > 0$ or $b^k_1 > 0$ for some $k$. Let $\hat{k}$ denote the smallest $k$ such that for all $k > \hat{k}$, $b^k_1 = 0$. Suppose $\hat{k} < K$. Then, for $t \geq \max\{2, \hat{k}\}$, the first order condition for $c_{it}$ reduces to $0 = U_{it} + \eta_t [U_{1t} - U_{2it}] - \chi_t$. If $U_{1t} > U_{2t}$, then $\eta_t = 0$ and this first order condition implies that each $U_{it} = \chi_t$. We deduce that in fact $U_{1t} = U_{2t}$. It then follows from the measurability constraint that the optimal allocation can be implemented with a portfolio in which either $b^k_1 = b^\hat{k}_1$ and $\hat{k} = 0$ or, if $\hat{k} = 1$, $b^\hat{k}_1 = a^*_1$ and $a_1 = 0$. All other portfolio weights remain the same.

Wlog assume that $b^\hat{k}_1 > 0$. The combined first order condition for $c_{1t}$ and $c_{2t}$, $t \in \{2, \cdots, K-1\}$, (25) implies that $U_{1t} - U_{2t} > 0$ if and only if $\omega_1 > 0$, and $U_{1t} - U_{2t} = 0$ otherwise. The first order condition for $b^\hat{k+1}_1$ is:

$$0 = -\sum_{s \in S} \omega_1(s) N_1(s) U_{21}(s) \prod_{j=2}^K \frac{U_{2j}(s)}{U_{1j}(s)} \pi(s) + \kappa^{\hat{k}+1},$$

where $\kappa^{\hat{k}+1} \geq 0$ is the Lagrange multiplier on the corresponding no lending constraint. Since, $b^\hat{k+1}_1 > 0$, it follows from (27) that either A) $\omega_1(s) = 0$ for each $s$ or B) $\omega_1(s) > 0 > \omega_1(s')$ for some pair $s, s'$. We show in LSY (2006) that only Case B holds. Suppose $b^\hat{k+1}_1 > 0$ for $k < \hat{k}$. Then, $-\sum_{s \in S} \omega_1(s) N_1(s) U_{21}(s) \prod_{j=2}^{k-1} \frac{U_{2j}(s)}{U_{1j}(s)} \prod_{j=k}^{K-1} \frac{U_{2j}(s)}{U_{1j}(s)} \pi(s) = 0$. It follows that: $-\sum_{s \in S} \omega_1(s) N_1(s) U_{21}(s) \prod_{j=2}^{K-1} \frac{U_{2j}(s)}{U_{1j}(s)} \prod_{j=k}^{K-1} \frac{U_{2j}(s)}{U_{1j}(s)} \pi(s) > 0$. But this contradicts the first order condition for $b^\hat{k+1}_1 > 0$. Thus, $b^\hat{k+1}_1 = 0$ for $k < \hat{k}$. By a similar argument, using the relevant first order condition, $a^*_1 = 0$ as well. The lemma is proven. ■

**Proof of Lemma 6** It follows from the proof of Lemma 5, that if $\omega_1(s) \leq 0$ or $t \geq K$, then $U_{1t} = U_{2t} = 0$. And $Q^1_t(s) = 1$; if $\omega_1(s) > 0$ and $t = 2, \cdots, K-1$, then $U_{1t} - U_{2t} = 0$ and $Q^1_t(s) < 1$. Also, $a_1, b^k_1 = 0$ at the optimal allocation. Using the first order conditions for $c_{1t}$ and $c_{2t}$, when $\omega_1(s) > 0$ and $t = 2, \cdots, K-1$ and the fact that $U_{1t}/U_{1t} = -U_{1t}/(1-\gamma), U_{2t}/U_{2t} = U_{2t}/(1-\gamma)$, we derive

$$\frac{U_{1t} - 1}{1-\gamma U_{2t}} = \frac{U_{2t} - 1}{1-\gamma U_{2t}} + 1 = \beta^{-(t-1)} \frac{\omega_1}{\beta} b^k_1 \prod_{j=1}^{K-1} \left[ \frac{U_{2j}}{U_{1j}} \right] U_{1t} N_1 = 0. \tag{B2}$$

Since $\beta \in (0,1)$ and $\omega_1 > 0$, we deduce that for $t = 2, \cdots, K-1$, $U_{1t}/U_{2t} > U_{1t}/U_{2t}$. Hence, $1 > Q^1_t > Q^1_{t+1}$, for $t = 2, \cdots, K-2$. If $Q^1_t = 1$, we are finished. If not, manipulation of the household’s first order conditions gives:

$$\frac{U_{1t} - 1}{1-\gamma U_{2t}} + 1 < \frac{\omega_1}{\beta} b^k_1 \prod_{j=1}^{K-1} \left[ \frac{U_{2j}}{U_{1j}} \right] U_{1t} N_1 < \frac{U_{1t} - 1}{1-\gamma U_{2t}} + 1, \quad t = 2, \cdots, K-2. \tag{B3}$$

Thus, $Q^1_{t+1} < Q^1_t$ for $t = 1, \cdots, K-2$. ■