Modeling Preferences over bundles of goods

The demand price function $p_i(q)$ is the consumer’s marginal willingness to pay or marginal benefit from the q-th unit,

$$MB = \frac{dB_i}{dq} = p_i(q).$$

This is depicted below. Integrating this expression, the total benefit from purchasing q units is

$$B_i(q) = \int_0^q p_i(x)dx$$
**Example:** If the demand price \( p_i(q) = a_i - b_iq \) the total benefit, \( B_i(q) = a_iq - \frac{1}{2}b_iq^2 \)

If the consumer pays \( R \) and receives \( q \) units her net gain or consumer surplus is

\[
u_i(q, R) = B_i(q) - R
\]

**Preferences over quantity and total payments**

In the discussion to follow it will prove helpful to depict the preferences of the consumer over quantities and payments. Given the offer \((q, R)\) of \( q \) units for \( R \) dollars, the net gain to the consumer or “consumer surplus” is \( u_i(q, R) = B_i(q) - R \). We can then draw the set of \((q, R)\) combinations that yield any particular level of consumer surplus, \( \bar{u}_i \).

Consider the indifference curve

\[
u_i(q, R) = B_i(q) - R = \bar{u}_i
\]

Rearranging we obtain,

\[
R = B_i(q) - \bar{u}_i
\]

Two of these indifference curves are drawn below.
The indifference curve through any point \((\bar{q}, \bar{R})\) is \(u(q, r) = B(q) - R = u(\bar{q}, \bar{R})\). Rearranging,

\[
R = B(q) - u(\bar{q}, \bar{R})
\]

Hence the slope of the indifference curve at \((\bar{q}, \bar{R})\) is

\[
\frac{dR}{dq} = B'_i(\bar{q}) = p_i(\bar{q})
\]

Thus the slope is the demand price.

Note that the consumer chooses a quantity \(\bar{q}\) so that \(p_i(q) = \bar{p}\). She therefore pays \(\bar{R} = \bar{q}p(\bar{q})\).

Below we map out the choices \((q, qp(q))\) for different prices.

To maximize profit find where the gap between the green revenue curve \(qp(q)\) and the cost curve \(C = cq\) is greatest.

Choose \(q^M\) where the slope of the green curve \(MR = \frac{d}{dq} qp(q)\) is equal to \(MC\).
Two part pricing

In Figure A-4 the consumer is offered a fixed price \( p > c \) and chooses the point C. Suppose that the firm switches from simple “linear” pricing to a two-part payment scheme. The consumer pays a fixed “access” fee \( F \) and in addition a “user” fee of \( t \) per unit. Consider any \( F > 0 \) and \( t_i \), where \( c < t_i < p \). The total payment is then \( R = F_i + t_i q \).

Figure A-4: Two-part pricing

Can you see why the profit must be higher?

**General Principle:** For any simple pricing scheme, there is a two-part scheme with a lower markup, which makes the consumer better off and increases the profit of the firm.
Optimal two part pricing

Consider Figure A-6. What is the scheme which maximizes the profit of the firm?

![Figure A-6: Extracting all the surplus](image)

To extract all the surplus, the firm offers a user fee equal to the marginal cost $c$ and an access fee of $F^*$ which leaves the consumer almost indifferent between making the purchase and purchasing nothing.
Price Discrimination

Two-part pricing

Suppose we divide customers into two “types.” A type 1 customer is a “low” demander and type 2 is a “high” demander.

If offered \( q \) units for a total payment of \( R \), a type \( i \) consumer has a gain of

\[
 u_i(q, R) = B_i(q) - R = \int_0^q p_i(x)dx - R
\]

Consider the indifference curve \( U_i(q, R) = U_i(q_0, R_0) \) through an arbitrary point \((q_0, R_0)\). From the previous section we know that the slope of the indifference curve is the demand price \( p_i(q_0) \).

Thus a high demander has everywhere steeper indifference curves.
Start with a 2-part scheme chosen to extract all surplus from low demanders

\((F_i, t_i) = (F^*_i, c)\)

How can the firm do better? Consider the 2-part scheme \(R = F^*_i + cq\) depicted in Figure B-2 below. The choices of low and high demanders are shown as the points \(E_{1i}\) and \(E_{2i}\) in the figure.

Figure B-1: Extracting all the surplus from low demanders
But we can do better. Add the second plan \((F_2, c)\) chosen so that high demanders are almost indifferent between \(E_{21}\) and \(E_{22}\).

**Principle A:** A seller profits by offering lower usage fees and higher access fees to higher demanders.

Suppose that the two types choose different plans. Suppose low demanders have a payoff \(D\). Then consider two new plans in which the access fees for each are raised by \(D/2\). The payoff to both consumer types falls by \(D/2\) thus the choices remain. Thus the cost of production is unchanged while revenue rises. It follows that the original plan does NOT maximize profit.

**Principle B:** In the profit-maximizing scheme, low demanders are indifferent between purchasing and staying out of the market.

Figure B-2: Extracting all the surplus with 2 types
This same argument can be applied to any other type of buyer. If a high demander strictly prefers the plan \((F_2, t_2)\), the monopolist can increase \(F_2\) and thus his profit without affecting the high demander’s optimal choice. We thus have the following generalization of Principle B.

**Principle B’**: In the profit-maximizing scheme, a type \(i\) buyer is indifferent between the plan \((F_i, t_i)\) designed for him and the plan designed for type \(i - 1\).

From the graphical analysis we have also seen the following.

**Principle C:**

The highest demanders will be offered a plan in which the use fee is equal to marginal cost

Solving analytically for the optimal scheme is not easy. However, we can apply our principles to develop a numerical approach.

First consider a type \(i\) demander who chooses the plan \((F_j, t_j)\) designed for type \(j\). His utility gain if he purchases \(q\) units is

\[
u_i(q, R) = B_i(q) - F_j - t_j q.
\]

Differentiating by \(q\), his optimal purchase, \(q_{ij}\) satisfies the first order condition

\[
B_i'(q_{ij}) = p_i(q_{ij}) = t_j.
\]  \hspace{1cm} (B.1)

Substituting into the utility function, type \(i\)’s consumer surplus is

\[
u_{ij} = B_i(q_{ij}) - t_j q_{ij} - F_j.
\]  \hspace{1cm} (B.2)

Consider the two-type case. Suppose we pick two initial usage rates. By Principle A, \(t_1 \geq t_2\).

Using the first order condition we can solve for the quantities \(q_{ij}\) and hence the \(N_{ij}\) terms in the expression for utility. By principle B, a low demander must be just indifferent between choosing the plan designed for him and dropping out. Thus

\[
u_{11} = N_{11} - F_1 = 0.
\]
By Principle $B'$, a high demander must be just indifferent between selecting his own plan and the plan designed for low demanders. Thus

$$u_{22} = N_{22} - F_2 = N_{21} - F_1 = u_{21}.$$ 

Rearranging these two expressions we can solve for the fixed fees as follows.

$$F_1 = N_{11}$$
$$F_2 = N_{22} - N_{21} + F_1$$

In this way we can compute the profit maximizing access fees for any usage fees. We can then use Solver to solve for the profit maximizing use fees.
Non-linear Pricing

Two-part pricing is a special case of non-linear pricing. More generally, the firm can choose to sell goods in packages. Package $i$, $(q_i, R_i)$ has $q_i$ units in it and sells for a markup over cost of $M_i$. Suppose that low demanders choose the package $E_i = (q_i, R_i)$.

This is illustrated in Figure B-3. With the low demanders choosing $E_1$ either the high demanders choose the same offer, or they must choose some $E_2$, which they prefer, but which is worse for low demanders. Since it must be worse for low demanders, the other offer cannot lie in the shaded region. Therefore to be better for high demanders it must be the case that $q_2 \geq q_1$.

**Principle A:** Any pair of offers which separates two types must have the property that high demanders purchase more.

![Figure B-3: Separating the two types of consumer](image)

Suppose that the firm makes two offers that separate. That is

$$u_1(q_1, R_1) = B_1(q_1) - R_1 \geq B_2(q_2) - R_2 = u_1(q_2, R_2)$$

and

$$u_2(q_2, R_2) = B_2(q_2) - R_2 \geq B_1(q_1) - R_1 = U_2(q_1, R_1)$$
then increasing the total payments to \( R_1 + \Delta \) and \( R_2 + \Delta \) leaves these inequalities unchanged. It follows that if a low demander has a strictly positive gain, the firm can increase the surcharge to all customers by the same amount and increase profit.

**Principle B:**

The firm maximizes profit by extracting all consumer surplus from low demanders.

Similarly, if a high demander is strictly better off choosing \( E_2 \) rather than \( E_1 \), no consumer’s choice is affected by a sufficiently small increase in the total payment \( R_2 \). This leads to the following principle.

**Principle B’:**

Suppose \( p_1(q) < p_2(q) < \ldots < p_n(q) \). Then the firm maximizes profits by making type \( i \) (almost) indifferent between the plan designed for him, \((q_i, M_i)\), and the plan designed for type \( i-1 \).