Auctions

1. Auction off a $10 bill

(a) by open ascending bid ("English") auction
   Auctioneer raises asking price until all but one bidder drops out

(b) by Dutch auction (descending asking price)
   Auctioneer starts a clock with high price and lets the clock tick down
   until a buyer cries out “stop”

(c) by sealed first-price (high-bid) auction.
   Sealed bids. High bidder pays his bid. If a tie, the winner is assigned
   at random.

(d) By sealed second-price auction
   Sealed bids. This time high bidder pays the second highest bid. Tie
   rule as above.

What does game theory have to say about the equilibrium outcome?

In the ascending price and sealed second-price auctions, there is a dominant
strategy equilibrium.

In the former, stay in up to your valuation.

In the latter, suppose bidder \( i \) bids lower than his valuation

\[
\begin{align*}
    &b^* &\quad &b^{**} &\quad &b^\star &\quad &b \\
    &b_i &\quad &v_i &\quad & &
\end{align*}
\]

If the highest of the other bids is \( b^* \), buyer \( i \) loses if he bids either \( v_i \) or
\( b_i < v_i \). If the highest of the other bids is \( b^{**} \) buyer \( i \) wins and bays \( b^{**} \).
The only time it matters is if the highest of the other bids is \( b^{***} \) between \( b_i \)
and \( v_i \). In this case buyer \( i \) wishes he had bid higher since he could have
profited by \( v_i - b^{***} \).
Sealed high-bid auction with 10 cent raises

Can you see why there can be no pure strategy equilibrium bid less than $9.90?

If other bidders bid $9.90, your best response is to bid the same since you may then win the coin toss.

If other bidders bid $10.00, you can do no better than bid the same. (Of course you could also bid zero. However it is not an equilibrium for all but one to bid zero. Why is this?)

With two equilibria, which is more likely? Apply the Pareto criterion. All buyers are better off in the first equilibrium so it seems reasonable that they would all choose to bid $9.90.

Dutch auction with a “10 cent” tick.

See if you can explain why bidding $9.90 must again be the an equilibrium.

2. Auctions with private information

Two bidders. Each bidder knows his own valuation but neither knows the valuation of his opponent. Each believes his opponents valuation is equally likely to be anywhere between 0 and 100.

What is a strategy in a private information game?

A strategy is a plan indicating what to do for every possible information that a player may have. In this case, every possible valuation.
Class Game: Sealed high-bid auction

Values taken from a bank note (0 - 99)

<table>
<thead>
<tr>
<th>BID</th>
<th>VALUE</th>
<th>Game 1</th>
<th>Game 2</th>
<th>Game 3</th>
<th>Game 4</th>
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Can theory guide bidders?

Sealed high bid auction

Suppose you thought your opponent was going to bid his full valuation (overly aggressive.)

How aggressive should you be if your valuation is 100?

If you bid

10 you win with probability $10/100$
20 with probability $20/100$
60 with probability $60/100$

$b$ with probability $b/100$

Expected gain = Pr{opponent bids less than $b$} × profit

$$u(v,b) = \frac{b}{100}(v-b) = \frac{1}{100}(vb-b^2)$$

Differentiating by $b$,

$$\frac{\partial u}{\partial b} = \frac{1}{100}(v-2b)$$

$$= 0 \text{ at } b = \frac{1}{2}v.$$
Extending this argument, suppose that your opponent adopts a linear strategy:

\[ b_2 = a_2 v_2. \]

What should you do?

If you bid \( b \) you win with probability

\[
\Pr\{b < b_1\} = \Pr\{a_2 v_2 < b_1\} = \Pr\{v_2 < \frac{b_1}{a_2}\}
\]

But values are evenly distributed. Thus

\[ \Pr\{v_2 < v\} = \frac{v}{100}. \]

Then buyer 1’s win probability is \( \frac{b_1}{100a_2} \)

His expected gain is therefore

\[
u(v_1, b_1) = \frac{b}{100a_2} \times (v - b) = \frac{1}{100a_2} (bv_1 - b^2).\]

Arguing as before, buyer 1’s best response is \( b_1 = \frac{1}{2} v_1. \)
Appealing to the symmetry of the game, each therefore has a best response $b_i = \frac{1}{2} v_i$.

Comparison with the open ascending bid auction.

In the open auction, if you win and your valuation is

$60$, what do you expect to pay?
$40$, what do you expect to pay?
$v$, what do you expect to pay? $v/2$

In the sealed high bid auction, if you win when your valuation is $v$ you pay $b = v/2$.

Thus the expected payments are the same in the two auctions. It follows immediately that the expected receipts of the weller (the revenue) is the same as well.

**Revenue equivalence theorem**

If (a) valuations are independent draws from the same distribution and (b) bidders are risk neutral
Then equilibrium bidding results in the same expected winning bid and hence the same expected revenue.

**Sealed high-bid Auctions with more than 2 bidders.**

Break into working groups to plot strategy.
Reserve prices?

(Revenue equivalence still holds)

"Auction" with 1 bidder.

<table>
<thead>
<tr>
<th>Reserve price $r$</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
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<td>Expected revenue</td>
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<td>16</td>
<td>21</td>
<td>24</td>
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Another way to answer the question.

Suppose the reserve price is raised from $r$ to $r'$

Expected gain = $(r' - r) Pr\{v > r'\} - r Pr\{r < v < r'\}$

\[
= (r' - r) (1 - \frac{r}{100}) - r\left(\frac{r' - r}{100}\right)
\]

\[
= \left(\frac{r' - r}{100}\right)(100 - r - r')
\]

Thus whenever $r < 50$, there is always some $r'$ such that the expected gain from raising the reserve price is positive

We can use this insight to understand the optimal selling strategy in an open ascending bid auction.
When does bidder $j$ have an impact if the reserve is raised a small amount from $r$ to $r'$?

We make the difference so small that we ignore the probability that two bidders or more have valuations between $r$ and $r'$.

Then bidder $j$ has an impact only if all the other bidders have valuations below $r$. *Conditional upon this*, the expected gain due to bidder $j$ is

$$(r' - r) \Pr\{v_j > r'\} - r \Pr\{r < v_j < r'\}$$

Exactly as with only 1 bidder.

Same argument holds for each bidder.

Thus the reserve price is exactly the same as with 1 bidder.

**Risk aversion**

Consider the Dutch auction.

How does it fare in comparison with the sealed high-bid and open ascending bid auctions?
Correlated valuations

Consider the jar of quarters

Does knowing what everyone else estimated affect your estimate?

Does knowing what everyone else estimated affect your bid?

If the answer to these questions is yes, the bids of your opponents contains valuable information to you.

What do you learn in the two common auctions?

Winner's curse

If you win, it means that all your opponents thought the item was worth less then you. This needs to be factored into your bid.
Bidding for oil: Big gusher: Value is $40 million
Dry well: Value is zero

Prior to seismic testing, bidders believe that each is equally likely.

Each bidder samples and receives a 'score' from the scientists ranging from 1 to 10.
**Possible messages if it is the "big gusher"**

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Possible messages if the well is Dry

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13
Conduct an experiment

Open auction

Envelope A contains the scores for the "Big Gusher" 
Envelope B contains the scores for the dry well.

The bidders do not know whether the field is dry or has a gusher

(In the experiment the bidders do not know whether 
they get the "Gusher" envelope or the "Dry" envelope)

Each bidder draws a score. Based on that information, bid for the right to drill.
<table>
<thead>
<tr>
<th>Score</th>
<th>Gusher WELL prob.</th>
<th>prob. both get score</th>
<th>DRY WELL prob.</th>
<th>prob. both get score</th>
<th>odds of gusher</th>
<th>prob. gusher</th>
<th>expected value</th>
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<td>1</td>
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<td>0.000</td>
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<td>0.000</td>
<td>361.000</td>
<td>0.997</td>
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If my opponent has a higher score I expect him to bid higher than me and win.

If my opponent has a lower score I expect to outbid him, so if he is still in, then I should be bidding as well.

If my opponent has the same score as me and drops out, what is my expected gain?

We need to compute
Pr\{gusher given that both have my score\} =

\[
\frac{Pr\{\text{both have score } s \text{ given gusher}\}}{Pr\{\text{both have score } s \text{ given gusher}\} + Pr\{\text{both have score } s \text{ given dry}\}}
\]

Example: \(s=3\).

\begin{align*}
Pr\{\text{both have score } 3 \text{ given gusher}\} &= (\frac{5}{100})(\frac{5}{100}) = \frac{25}{10,000} \\
Pr\{\text{both have score } 3 \text{ given dry}\} &= (\frac{15}{100})(\frac{15}{100}) = \frac{225}{10,000}
\end{align*}

Then the probability of a gusher is 
\(\frac{25}{25+225} = 0.1\)

Example: \(s=7\).

\begin{align*}
Pr\{\text{both have score } 7 \text{ given gusher}\} &= (\frac{15}{100})(\frac{15}{100}) = \frac{225}{10,000} \\
Pr\{\text{both have score } 7 \text{ given dry}\} &= (\frac{5}{100})(\frac{5}{100}) = \frac{25}{10,000}
\end{align*}

Then the probability of a gusher is 
\(\frac{225}{225+25} = \frac{225}{250} = 0.9\)

Multiply by the value of a gusher to get the expected gain.