Lecture 6

Joint Costs

Upstream:

Logging:
Cost per log = $30

Newsprint:
One log yields 1 km of newsprint at a cost of $10
Demand:
$p_1(q_1) = 200 - q_1$

Chipboard:
One log yields 1 sheet of chipboard at a cost of $4
Demand:
$p_2(q_2) = 100 - q_2$

Downstream:

What is the revenue if $q$ logs are cut and shipped each day?

Newsprint: $p_1 = 200 - q$
Chipboard: $p_2 = 100 - q$

Total price per log: $p_1 + p_2 = 300 - 2q$
Total Revenue: $R = 300q - 2q^2$
Marginal revenue: $MR = 300 - 4q$

$MC = 44$

$q$

$MR = 300 - 4q$

$MC = 44$

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PUZZLE

Revenue from chip-board sales:

\[ p_2 = 100 - q = 11 - 64 = 36 \quad \Rightarrow \quad R = 36 \times 64 = 2304 \]

Raise price to 40. Then quantity is 60: Revenue is $2400

View the problem from “upstream”

produce \( q_0 \) logs where \( q_0 \) is small.

Constraints

Newsprint: \( q_1 \leq q_0 \)  \quad \text{Chip-board: } q_2 \leq q_0

Net marginal revenue

Division 1 generates a marginal net revenue of

\[ MNR_1 = MR_1 - MC_1 = (200 - 2q_1) - 10 \]

Division 2 generates the marginal net revenue of

\[ MNR_2 = MR_2 - MC_2 = (100 - 2q_2) - 4 \]

Once the logs are shipped, any contribution to net marginal revenue adds to profit.
\[ MNR_1 = 190 - 2q_1 \]

\[ MNR_2 = 96 - 2q_1 \]
Mathematical approach

\[ R(q) = p_1(q_1)q_1 + p_2(q_2)q_2 = (200 - q_1)q_1 + (100 - q_2)q_2. \]

Total cost is

\[ C(q) = 30q_0 + 10q_1 + 4q_2 \quad \text{where} \quad q = (q_0, q_1, q_2) \geq 0 \]

Once the firm has set its logging capacity, its downstream output is constrained as follows.

\[ 0 \leq q_1 \leq q_0, \quad 0 \leq q_2 \leq q_0. \]

The formal optimization problem can thus be written as follows

\[ \max_q \{ \Pi(q) = R(q) - C(q) \mid 0 \leq q_1 \leq q_0, \ 0 \leq q_2 \leq q_0 \}. \]

We will see later how to solve numerically.
Peak-load Pricing

production and pricing of electricity

$q_0 = \text{capacity in kilowatts per hour, } C_0(q_0) = \text{cost of providing this capacity.}$

The day is divided into sub-periods.

$q_i = \text{amount supplied in sub-period } i, \ C_i(q_i) = \text{operating cost of producing this output.}$

Example:

Capacity cost: $C_0(q_0) = 500 + 30q_0.$

Operating costs: $C_i(q_i) = 10q_i, \ i = 1, 2$

$p_1(q_1) = 150 - q_1, \ p_2(q_2) = 170 - 2q_2$
\[ \delta B_i \approx p_i(q_i)\delta q. \]

Dividing this expression by \( \delta q \) and taking the limit, it follows that

Marginal benefit from one more unit: \( MB_i = B_i'(q_i) = p_i(q) \)

Integrating,

\[ B_i(q_i) = \int_0^{q_i} p_i(x)dx. \]

Note that if \( p_i(q_i) = a_i - b_i q_i \), then

\[ B_i(q_i) = \int_0^{q_i} (a_i - b_i q) dq = a_i q_i - \frac{1}{2} q_i^2 = (a_i - \frac{1}{2} q_i)q_i \]
Interdependent Demand

Review: Independent linear demands

\[ B(q_1, q_2) = (a_1 - \frac{1}{2} b_1 q_1) q_1 + (a_2 - \frac{1}{2} b_2 q_2) q_2 \]

Generalization. We define 2 vectors of parameters as follows.

\[
\begin{bmatrix}
    b_1 \\
    b_2
\end{bmatrix} = \begin{bmatrix}
    b_{11} & b_{12} \\
    b_{21} & b_{22}
\end{bmatrix} \quad b_{12} = b_{21}
\]

Total benefit,

\[ B(q) = (a_1 - \frac{1}{2} (b_{11} q_1 + b_{12} q_2) q_1 + (a_2 - \frac{1}{2} (b_{21} q_1 + b_{22} q_2) q_2 \]

Differentiating by \( q_1 \), the marginal willingness to pay,

\[ p_1(q) = \frac{\partial B}{\partial q_1} = a_1 - b_{11} q_1 - \frac{1}{2} b_{12} q_2 - \frac{1}{2} b_{21} q_2. \]

Since \( b_{12} = b_{21} \)

\[ p_1(q) = \frac{\partial B}{\partial q_1} = a_1 - b_{11} q_1 - b_{12} q_2 \]

Similarly,

\[ p_2(q) = \frac{\partial B}{\partial q_2} = a_2 - b_{21} q_1 - b_{22} q_2 \]

symmetric linear demand price functions.
We will examine the peak-load pricing problem from both the perspective of a profit-maximizing firm and the perspective of a planner attempting to maximize the social surplus.

**Profit maximization**

\[ \Pi(q) = R(q) - C(q) = \sum_{i=1}^{n} p_i(q)q_i - C(q) = \sum_{i=1}^{n} (a_i - b_i \cdot q_i)q_i - C(q) \]

**Social surplus maximization**

The social planner adds consumer surplus to the profit to obtain the total surplus to society.

\[ S(q) = B(q) - p \cdot q + p \cdot q - C(q) = B(q) - C(q) = \sum_{i=1}^{n} (a_i - \frac{1}{2} b_i \cdot q_i)q_i - C(q) \]

Constraints:

Output in any period cannot exceed installed capacity.

\[ 0 \leq q_i \leq q_0, \quad i = 1, \ldots, n \]