1. Transportation Problem

The Hang Seng Co. has three pea canneries (in Canton, Auckland and Manila). The cans are all shipped to one of 4 distribution centers (in Los Angeles, Toronto, New York and Dallas). The following table indicates the capacity of each cannery, the allocation assigned to each distribution center and the cost per unit $t_{ij}$ of shipping from source $i$ to destination $j$.

<table>
<thead>
<tr>
<th>Source i</th>
<th>destination j</th>
<th>capacity of source i</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1  2  3  4</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>463 650 654 486</td>
<td>75</td>
</tr>
<tr>
<td>2</td>
<td>515 422 441 791</td>
<td>130</td>
</tr>
<tr>
<td>3</td>
<td>995 682 388 685</td>
<td>100</td>
</tr>
<tr>
<td>Allocation to destination j</td>
<td>80 65 70 85</td>
<td></td>
</tr>
</tbody>
</table>

(a) Write down the cost minimization problem. (There are seven constraints.)

(b) Enter the data in Solver and then enter an initial feasible set of shipments.

(c) What would be the effect on revenue of reducing supply by 5 at (i) source 1 (Canton) (ii) Source 2 (iii) source 3.

(d) Do shadow prices help to answer part (d)? Is it surprising that one of the shadow prices is zero? Explain.

(d) Present your results on a sensitivity sheet. Interpret the shadow prices.

(e) What is the imputed profit or loss on shipping from canton to Toronto?

(f) If the cost of shipping from Canton to Toronto falls by (i) 250 (ii) 300 what is the effect on the solution?

(g) If total supply is equal to total demand it is necessarily the case that one of the shadow prices at a source is zero. Why is this?

(h) Is it necessarily the case that, in any transportation problem, all the prices at destinations are strictly positive? Explain.
2. Maximum Flow

The figure below shows the capacity of various channels of a water supply grid. Let \( x_{ij} \) be the flow to from node \( i \) to node \( j \).

(a) Explain why there are 8 constraints of the form \( x_{ij} \leq c_{ij} \) and 8 more of the form \( x_{ij} \geq -c_{ij} \).

(b) You may assume that there is an overflow valve at each node so that the total flowing away from a node is less than or equal to the inflow. Write down the constraint at each node (inflow – outflow \( \geq 0 \)) and the maximand if the objective is to solve for the maximum flow from \( N_1 \) to \( N_6 \).

(c) Solve the problem using SOLVER. You should submit a printout of your set-up page as well as the Sensitivity Report.

(d) Interpret the shadow prices. Explain each of the zero prices and why the others are all the same.

(e) How does the solution change if the capacity of channel \( N_1N_2 \) currently at 20 is increased by 10?

3. Peak load Pricing

The day is divided into 4 periods. Demand in each period is given by the following demand price functions.

\[ p_1 = 160 - 2q_1, \quad p_2 = 120 - 2q_2, \quad p_3 = 100 - 2q_3, \quad p_4 = 90 - q_4 \]

The operating cost per unit in each period is 20. The unit cost of capacity per day is 30.

(a) Solve analytically and graphically for the profit maximizing outputs and prices.

(b) Solve for the surplus maximizing outputs and prices

(c) In the second case, is this marginal cost pricing? Explain.

(d) Confirm your results by using a spread-sheet.
(e) If you are a planner and must keep the prices the same in each period, what outputs would you choose and what would be the price?

4. Peak load pricing with non-linear demands.

Demand price functions in periods 3 and 4 are as follows. (Everything else is the same as in the previous problem.)

\[ p_3 = \left( \frac{16,000}{q_3} \right)^{\frac{1}{2}} = (16,000)^{\frac{1}{2}} q_3^{-\frac{1}{2}}, \]

\[ p_4 = \left( \frac{54,000}{q_4} \right)^{\frac{1}{2}} = (54,000)^{\frac{1}{2}} q_4^{-\frac{1}{2}}, \]

(a) Confirm that if the firm chooses the same outputs as in part (b) of the previous question, the prices will be the same.

(b) Explain why this must be the socially optimal plan.

(c) Total benefit in period 3 is 
\[ \int_0^{q_1} p_3(x)dx = (16,000)^{\frac{1}{2}} \int_0^{q_1} x^{-\frac{1}{2}}dx = (16,000)^{\frac{1}{2}} 2q_3^{\frac{1}{2}}. \]

Obtain a similar expression for the period 4 benefit and then use solver to solve for the socially optimal outputs.

HINT: Using the spread-sheet for the previous question, you need to change the entries for the benefit in the two periods.

(d) Solve also for the profit maximizing outputs and prices.