Optimal Pricing - solving analytically

A plan \((q, R)\) is a fee \(R\) for \(q\) units

\[ p_i(q) = a_i - b_i q \]

\[ CB_i(q) = \int_0^q p_i(x) dx \]

Hence the gain to type \(i\) if she chooses the plan \((q, R)\) is

\[ U_i(q, R) = CB_i(q) - R = \int_0^q p_i(x) dx - R. \]

An indifference curve is depicted below.
Around an indifference curve \( U_i(q, R) = k \) we have \( \int_0^q p_i(x)dx - R = k \) and hence \( R = \int_0^q p_i(x)dx - k \). Differentiating, the slope of the indifference curve is \( \frac{dR}{dq} = p_i(q) \).

Special case: Linear demands

\[
CB_i(q) = \int_0^q p_i(x)dx = a_iq - \frac{1}{2}b_iq^2.
\]

Hence the gain to type \( i \) if she chooses the plan \((q, R)\) is

\[
U_i(q, R) = a_iq - \frac{1}{2}b_iq^2 - R.
\]

Example: \( p_1(q) = 14 - 2q, \ p_2(q) = 24 - 2q, \ c = 4 \)

Choose any \( q_1 \) and then the total payment which leaves this type indifferent between choosing this plan and taking the smaller plan (buying nothing.)

\[
U_i(q_1, R_1) = 14q_1 - q_1^2 - R_1 = 0.
\]

Hence

\[
R_1 = 14q_1 - q_1^2.
\]

Let \((q_2, R_2)\) be the plan chosen by type 2. From the figure below,
the firm maximizes profit by moving around the indifference curve for type 2 until the slope of the indifference curve equals the slope of the cost line.

Therefore \( p_2(q_2^*) = 24 - 2q_2^* = 4 \) and so \( q_2^* = 10 \).

If type 2 selects plan 1 his utility is

\[
U_2(q_1, R_1) = 24q_1 - q_1^2 - R_1 = 24q_1 - q_1^2 - (14q_1 - q_1^2) = 10q_1.
\]

If he selects plan 2 his utility is

\[
U_2(q_2^*, R_2) = 24q_2^* - (q_2^*)^2 - R_2 = 140 - R_2.
\]

The firm then chooses the payment so that he is (almost) indifferent between this plan and plan 1. Therefore

\[
U_2(q_2^*, R_2) = U_2(q_1, R_1)
\]

and so

\[
140 - R_2 = 10q_1
\]

Collecting results,

\[
R_1 = 14q_1 - q_1^2 \quad \text{and} \quad R_2 = 140 - 10q_1.
\]

Let \( n \) be the number of type 1 buyers and let \( an \) be the number of type 2 buyers. Total profit is

\[
\Pi = n(R_1 - 4q_1) + an(R_2 - 4q_2)
\]

\[
= n[10q_1 - q_1^2 + a(100 - 10q_1)]
\]

It is now an easy matter to solve for the optimal quantity for plan 1.

\[
\frac{\partial \Pi}{\partial q_1} = n[10 - 2q_1 - 10a] = n[10(1 - a) - 2q_1].
\]
If $a \geq 1$ this is always negative hence $q_1^* = 0$. If $a < 1$, $q_1^* = 5(1 - a)$.

Suppose $a = 0.4$. Then $q_1^* = 3$

Since

$$R_1 = 14q_i - q_i^2 \text{ and } R_2 = 140 - 10q_i.$$  

We can solve for the optimal fees for each plan.

Exercise 1: Solve analytically for the profit maximizing plan if $p_1(q) = 20 - 2q$, $p_2(q) = 24 - 2q$, $c = 2$ and there are equal numbers of each type. Check your answer using Solver.

Exercise 2: Solve again if instead $p_2(q) = 20 - q$.

Hint: Carefully check the constraint for type 2.