Auctions

1. Auction off a single item

(a) by open ascending bid ("English") auction
Auctioneer raises asking price until all but one bidder drops out

(b) by Dutch auction (descending asking price)
Auctioneer starts a clock with high price and lets the clock tick down until a buyer cries out “stop”

(c) by sealed first-price (high-bid) auction.
Sealed bids. High bidder pays his bid. If a tie, the winner is assigned at random.

(d) By sealed second-price auction
Sealed bids. This time high bidder pays the second highest bid. Tie rule as above.

What does game theory have to say about the equilibrium outcome?

In the ascending price and sealed second-price auctions, there is a dominant strategy equilibrium.

In the former, stay in up to your valuation.

In the latter, suppose bidder $i$ bids lower than his valuation

If the highest of the other bids is $b^*$, buyer $i$ loses if he bids either $v_i$ or $b_i < v_i$. If the highest of the other bids is $b^{**}$ buyer $i$ wins and bays $b^{**}$. The only time it matters is if the highest of the other bids is $b^{***}$ between $b_i$ and $v_i$. In this case buyer $i$ wishes he had bid higher since he could have profited by $v_i - b^{***}$.
Simple example: Auction off a $10 bill

Sealed high-bid auction with 10 cent raises

Can you see why there can be no pure strategy equilibrium bid less than $9.90?

If other bidders bid $9.90, your best response is to bid the same since you may then win the coin toss.

If other bidders bid $10.00, you can do no better than bid the same. (Of course you could also bid zero. However it is not an equilibrium for all but one to bid zero. Why is this?

With two equilibria, which is more likely? Apply the Pareto criterion. All buyers are better off in the first equilibrium so it seems reasonable that they would all choose to bid $9.90.

Dutch auction with a “10 cent” tick.

See if you can explain why bidding $9.90 must again be the an equilibrium.

2. Auctions with private information

Two bidders. Each bidder knows his own valuation but neither knows the valuation of his opponent. Each believes his opponents valuation is equally likely to be anywhere between 0 and 100.

What is a strategy in a private information game?

A strategy is a plan indicating what to do for every possible information that a player may have. In this case, every possible valuation.
Can theory guide bidders?

Sealed high bid auction

Suppose you thought your opponent was going to bid his full valuation (overly aggressive.)

How aggressive should you be if your valuation is 100?

If you bid

10 with probability 10/100
20 with probability 20/100
60 with probability 60/100

$b$ with probability $b/100$

Expected gain = Pr{opponent bids less than $b$} × profit

\[ u(v,b) = \frac{b}{100}(v - b) = \frac{1}{100}(vb - b^2) \]

Differentiating by $b$,

\[ \frac{\partial u}{\partial b} = \frac{1}{100}(v - 2b) \]

\[ = 0 \text{ at } b = \frac{1}{2}v. \]
Extending this argument, suppose that your opponent adopts a linear strategy:

\[ b_2 = a_2 v_2 . \]

What should you do?

If you bid \( b \) you win with probability

\[
\Pr\{ b < b_1 \} = \Pr\{ a_2 v_2 < b_1 \} = \Pr\{ v_2 < \frac{b_1}{a_2} \} 
\]

But values are evenly distributed. Thus

\[
\Pr\{ v_2 < v \} = \frac{v}{100} . 
\]

Then buyer 1’s win probability is \( \frac{b_1}{100a_2} \)

His expected gain is therefore

\[
u(v_1, b_1) = \frac{b}{100a_2} \times (v - b) = \frac{1}{100a_2} (b v_1 - b^2) .
\]

Arguing as before, buyer 1’s best response is \( b_1 = \frac{1}{2} v_1 \).
Appealing to the symmetry of the game, each therefore has a best response
\[ b_i = \frac{1}{2} v_i. \]

Comparison with the open ascending bid auction.

In the open auction, if you win and your valuation is

\[ 60, \ \text{what do you expect to pay?} \]
\[ 40, \ \text{what do you expect to pay?} \]
\[ v, \ \text{what do you expect to pay?} \]

\[ \frac{v}{2} \]

In the sealed high bid auction, if you win when your valuation is \( v \) you pay \( b = \frac{v}{2} \).

Thus the expected payments are the same in the two auctions. It follows immediately that the expected receipts of the seller (the revenue) is the same as well.

**Revenue equivalence theorem**

If (a) valuations are independent draws from the same distribution and (b) bidders are risk neutral

Then equilibrium bidding results in the same expected winning bid and hence the same expected revenue.

Sealed high-bid Auctions with more than 2 bidders.

Revenue equivalence theorem continues to hold
Reserve prices?

(Revenue equivalence still holds)

"Auction" with 1 bidder.

<table>
<thead>
<tr>
<th>Reserve price $r$</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pr{sale}</td>
<td>1</td>
<td>0.9</td>
<td>0.8</td>
<td>0.7</td>
<td>0.6</td>
<td>0.5</td>
<td>0.4</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>Expected revenue</td>
<td>0</td>
<td>9</td>
<td>16</td>
<td>21</td>
<td>24</td>
<td>25</td>
<td>24</td>
<td>21</td>
<td>16</td>
</tr>
</tbody>
</table>

Another way to answer the question. Consider the sealed second price auction.

Let the possible valuations be $0,1,2,\ldots,\bar{v}$. Let $f_t$ be the probability that the valuation is $t$. Suppose that the reserve price is raised from $t$ to $t+1$. A particular buyer only affects the outcome if he is the only one with a valuation of $t$ or higher. [Why is this?].

Conditional upon this being the case, with probability $f_t$ the item is not sold. The seller thus gets to keep the item rather than sell it at a price of $t$. With probability $F_t = f_{t+1} + f_{t+2} + \ldots + f_{\bar{v}}$ the price is sold for $1$ more. The expected profit is therefore

$$-f_t t + G_t$$

In the uniform case with values on $[0,100]$ $f_t = 1$ and $G_t = 100 - (t+1)$

Then

$$-f_t t + G_t = -t + (100 - t - 1) = 99 - 2t > 0 \text{ if and only if } t > 49 .$$

Thus the profit maximizing reserve price is 50.
Class exercises:

1. What if valuations are uniformly distributed on $[100,200]$?
2. What if the sellers valuation is $s$?

**Risk aversion**

Consider the Dutch auction.

How does it fare in comparison with the sealed high-bid and open ascending bid auctions?

Modelling

![Diagram showing utility gain and monetary gain]
Special case: $U(x) = x^\alpha$, $0 < \alpha < 1$

Consider the uniform case with 2 bidders. Suppose your opponent’s strategy is linear.

$$b_2 = \theta_2 v_2$$

If you bid $b$ you win if $b_2 = \theta_2 v_2 \leq b$ (this is exact only if you win all ties.)

That is if $v_2 < \frac{b}{\theta_2}$

In the uniform case the probability of this event,

$$\Pr\{v_2 \leq \frac{b}{\theta_1}\} = \frac{b}{100\theta_2}$$

Buyer 1’s expected utility is therefore

$$e(b) = \Pr\{v_2 \leq \frac{b}{\theta_1}\} U(v_1 - b) = \frac{b}{100\theta_2} (v_1 - b)^\alpha = \frac{1}{100\theta_2}[b(v_1 - b)^\alpha]$$

It is left as an exercise to show that this is maximized if $b = \frac{1}{1+\alpha} v_1$.

Auctions with more than 1 item

Example:

Bidder 1’s valuations: 80, 60
Bidder 2’s valuations: 50, 20
Bidder 1’s valuations: 30, 25

Consider the open ascending bid auction