1. The CES family of utility functions
A preference map is in the CES family if and only if indifference curves can be written in the following form.

\[ \sum_{j=1}^{n} u_j(x_j) = k \quad \text{where} \quad u_j'(x_j) = \frac{\alpha_j}{x_j^\sigma}, \quad \sigma > 0, \quad \alpha = (\alpha_1, \ldots, \alpha_n) > 0 \]

(a) If \( \sigma = 1 \) show that there is a logarithmic representation of preferences.
(b) If \( \sigma = 1 \) show that preferences can also be represented by a “Cobb-Douglas” utility function.
(c) Integrate to solve for a utility representation of preferences for \( n = 2 \) if
   (i) \( \sigma = 1/2 \) and (ii) \( \sigma = 2 \).
(d) In each case carefully depict the indifference curve through (1,1). Note, in particular, where it hits the axes (if it does) or approaches an asymptote.

2. Cobb-Douglas Preferences
Alex has a utility function \( U(x) = x_1^\alpha x_2^\beta \). The price vector is \( p \) and his income is \( I \).
(a) Solve for his demand functions and maximized utility.
(b) What is the minimum income Alex needs to achieve a utility of \( \bar{U} \)?
Bev has a utility function \( U(x) = x_1^\alpha x_2^{2-\alpha} + x_3^\alpha x_4^{2-\alpha} \). She faces a price vector \( (p_1, \ldots, p_4) \) and has income \( I \).
(c) If she must spend \( y_{12} \) on the first 2 commodities and \( y_{34} \) on the other two commodities, appeal to part (a) to obtain an expression for her maximized utility.
(d) Now suppose that she is free to decide how to allocate her spending so that \( y_{12} + y_{34} = I \). Under what conditions will she only consume the first 2 commodities?

3. Modified Cobb-Douglas Preferences.
\[ U(x) = \alpha_1 \ln(1 + x_1) + \alpha_2 \ln(1 + x_2) \]
(a) Solve for the demand functions for all incomes and prices.
(b) Under what conditions (if any) is only commodity 1 consumed?

4. Elasticity of consumption ratios
A consumer with CES preferences faces a price vector $p$ and has income $I$.

(a) For the CES family solve for the optimal ratio of demand $\frac{x_j}{x_i}$.

(b) How does this vary with income?

(c) What is the elasticity of this ratio with respect to a change in the price of commodity $i$ and commodity $j$?