1. The CES family of utility functions

First note that if one has the utility function $U(x)$, his indifference curve that corresponds to a certain utility $\tilde{U}$ will be implicitly expressed as $U(x) = \tilde{U}$.

a) Consider a log-utility function $U(x) = \sum_{j=1}^{n} \alpha_j \ln x_j$.

Its indifference curve for a certain utility level $\tilde{U}$ is expressed as $\sum_{j=1}^{n} u_j(x_j) = k$ where $u_j(x_j) = \alpha_j \ln x_j$ and $k = \tilde{U}$.

Thus, $\sum_{j=1}^{n} u_j(x_j) = k$ and $u_j'(x_j) = \frac{\alpha_j}{x_j}$, which agrees our definition of CES family when $\sigma = 1$.

b) Consider a Cobb-Douglas utility function $U(x) = \prod_{j=1}^{n} x_j^{\alpha_j}$.

Its indifference curve for a certain utility level $\tilde{U}$ is $\prod_{j=1}^{n} x_j^{\alpha_j} = \tilde{U}$.

By taking log of both sides, we can transform this into $\sum_{j=1}^{n} \alpha_j \ln x_j = \ln \tilde{U}$.

Hence, we can express the indifference curve as $\sum_{j=1}^{n} u_j(x_j) = k$, where $u_j(x_j) = \alpha_j \ln x_j$ and $k = \ln \tilde{U}$. Such $u_j(x_j)$ again satisfies $u_j'(x_j) = \frac{\alpha_j}{x_j}$.

Therefore, CES preference can also be represented by Cobb-Douglas utility function when $\sigma = 1$.

c) Suppose a consumer has a preference whose indifference curve satisfies the definition of the CES family, i.e. it can be expressed in the form of $\sum_{j=1}^{n} u_j(x_j) = k$.

, $u_j'(x_j) = \frac{\alpha_j}{x_j}$. Now, consider a utility function $U(x) = \sum_{j=1}^{n} u_j(x_j)$. Then obviously its indifference curve can be expressed in the form $\sum_{j=1}^{n} u_j(x_j) = k$ , $u_j'(x_j) = \frac{\alpha_j}{x_j}$ for $k = \tilde{U}$, and thus $U(x)$ is one way of utility representation of this consumer's preference.

Here $n=2$, so $U(x) = u_1(x_1) + u_2(x_2)$ where $u_j'(x_j) = \frac{\alpha_j}{x_j}$, $j = 1, 2$.

Integrating, we obtain $u_j(x_j) = \frac{1}{1-\sigma} \alpha_j x_j^{1-\sigma}$.

Therefore, $U(x) = \frac{1}{1-\sigma} (\alpha_1 x_1^{1-\sigma} + \alpha_2 x_2^{1-\sigma})$.
<Note>

This is not THE UNIQUE representation of this preference. As I mentioned in the first TA section, any utility function obtained by a monotonic transformation of a utility function expresses the same preference.

Hence,

(i) \( \sigma = \frac{1}{2} \implies U(x) = 2(\alpha_1 x_1^{1/2} + \alpha_2 x_2^{1/2}) \)
(ii) \( \sigma = 2 \implies U(x) = -(\alpha_1 x_1^{-1} + \alpha_2 x_2^{-1}) \)

By plugging in \((x_1, x_2) = (1, 1)\) into the utility function obtained in (c), we find that the indifference curve that goes through \((1, 1)\) corresponds to the utility \(2(\alpha_1 + \alpha_2)\), and so is expressed as \(2(\alpha_1 x_1^{1/2} + \alpha_2 x_2^{1/2}) = 2(\alpha_1 + \alpha_2)\), or
\[
\alpha_1 x_1^{1/2} + \alpha_2 x_2^{1/2} = \alpha_1 + \alpha_2 \quad \text{(1)}
\]
(ii) \( \sigma = 2 \)

By the same argument, the indifference that goes through \((1, 1)\) is,
\[
\alpha_1 x_1^{-1} + \alpha_2 x_2^{-1} = \alpha_1 + \alpha_2 \quad \text{...(2)}
\]

They are depicted as above.

Here I picked \(\alpha_1 = \alpha_2 = 1\) in order to draw the graph by PC, but for your solution, it is advisable to draw the graph for general value of \(\alpha_1\) and \(\alpha_2\).

In order to draw the graph with hand, note that in both cases \(U(x)\) is concave (as they are sum of concave functions) and so quasi-concave, so its upper contour set is convex.

Then,
When (i) $\sigma = \frac{1}{2}$, the indifference curve hits the axes at $(0, (\frac{\alpha_1 + \alpha_2}{\alpha_2})^2)$ and $(\frac{\alpha_1}{\alpha_2 + \alpha_2})^2, 0)$.

(To see this, plug in $x_1 = 0$ or $x_2 = 0$ in (1) and solve for the other variable.)

When (ii) $\sigma = 2$, it does not hit the axes, but instead approaches the asymptotes $x_1 = \frac{\alpha_1}{\alpha_1 + \alpha_2}$ and $x_2 = \frac{\alpha_2}{\alpha_1 + \alpha_2}$.

(To see this, note that $x_i x_i^{-1} \rightarrow 0$ as $x_i \rightarrow \infty$ in (2)).

Using these facts and letting it go through $(1,1)$, you should be able to draw the approximate shape of the indifference curve, without going through deep computation.

2. Cobb-Douglas Preferences

(a) Let us first solve using Lagrangian.

Alex solves,

$$\max \limits_{x_1, x_2} U(x) = x_1^\alpha x_2^\beta$$ \hspace{1cm} (A)

s.t. $px \leq I$ (i.e. $p_1 x_1 + p_2 x_2 \leq I$), $x \geq 0$

Setting up the lagrangian,

$$L(x_1, x_2, \lambda) = x_1^\alpha x_2^\beta + \lambda (I - p_1 x_1 - p_2 x_2)$$

Since $U(x)$ is quasi-concave (here let’s assume $\alpha > 0$ & $\beta > 0$, to make commodities "goods". Any Cobb-Douglas function with $\alpha > 0$ & $\beta > 0$ is quasi-concave) and the constraint is linear, FOCs are sufficient for the maximum.

(FOCs)

$$\alpha x_1^{\alpha - 1} x_2^\beta - \lambda p_1 \leq 0, \text{ with equality if } x_1 > 0$$

$\Leftrightarrow \alpha \frac{U(x)}{x_1} - \lambda p_1 \leq 0, \text{ with equality if } x_1 > 0$ \hspace{1cm} (1)

$$\beta x_1^\alpha x_2^{\beta - 1} - \lambda p_2 \leq 0, \text{ with equality if } x_2 > 0$$

$\Leftrightarrow \beta \frac{U(x)}{x_2} - \lambda p_2 \leq 0, \text{ with equality if } x_2 > 0$ \hspace{1cm} (2)

$\lambda - p_1 x_1 - p_2 x_2 \geq 0, \text{ with equality if } \lambda > 0$ \hspace{1cm} (3)

In general, we need to consider different cases according to whether $x_1, x_2$ and $\lambda$ are $> 0$ or $= 0$.

However, we need not do so here. First note that since the utility is monotonic, the budget constraint ((3)) must hold with equality at the optimal (Since he can always increase his utility by spending more, he will use all his income). Next, notice that $U(x) = 0$ if $x_1 = 0$ or $x_2 = 0$, which is clearly not optimal; therefore, both $x_1$ and $x_2$ need to be strictly positive at the maximum.

Thus, (1) – (3) all holds with equality. Hence,
\[
\lambda = \alpha \frac{U(x)}{p_1x_1} = \beta \frac{U(x)}{p_2x_2} \quad \text{(From (1) and (2))}
\]
\[
\iff \quad \frac{\alpha}{p_1x_1} = \frac{\beta}{p_2x_2} = \frac{\alpha + \beta}{I} \quad \text{ (Using 'equal ratio rule' and (3))}
\]
\[
\therefore x_1(p, I) = \frac{\alpha}{\alpha + \beta} \frac{I}{p_1}, \quad x_2(p, I) = \frac{\beta}{\alpha + \beta} \frac{I}{p_2}
\]
and by plugging these into the utility function, we get the maximized utility
\[
V(p, I) = (\frac{\alpha}{\alpha + \beta})^\alpha (\frac{\beta}{\alpha + \beta})^\beta p_1^\alpha p_2^\beta I^{\alpha + \beta}
\]

<Equal Ratio Rule (Text p134)>
If \( \frac{x_1}{y_1} = \frac{x_2}{y_2} = k \) then for any \( \beta, \frac{x_1 + \beta}{y_1 + \beta} = k. \)
Simple as it is, this rule is quite useful in computation.

<Other Methods>

(1) We can exclude corner solutions by observing that \( U(x) = 0 \) unless the consumption of both commodities are strictly positive (or, by arguing the marginal utility of a good is infinite when its consumption is 0).

After confirming that, we can resort to our familiar relation of MRS = price ratio, and get the same result.

(2) Note that \( \arg \max U(x) \) will be the same as \( \arg \max \ln U(x) \). Hence, let us consider the following problem instead of (A).
\[
\max \ln U(x) = \alpha \ln x_1 + \beta \ln x_2
\]
\[
s.t. \; px \leq I \; i.e. \; p_1x_1 + p_2x_2 \leq I \quad \ldots (B)
\]

Then, by 'log utility rule (to be seen in TA section)' and 'equal ratio rule',
\[
\frac{\alpha}{p_1x_1} = \frac{\beta}{p_2x_2} = \frac{\alpha + \beta}{I}
\]
and therefore we obtain the same result.

(b) By plugging in \( U = \tilde{U} \) into the maximized utility in (a) and solving for \( I \), we obtain
\[
I = (\alpha + \beta)(\frac{p_1}{\alpha})^\frac{\alpha}{\alpha + \beta} (\frac{p_2}{\beta})^\frac{\beta}{\alpha + \beta} \tilde{U}
\]
(c) Here \( y_{12} \) and \( y_{34} \) are fixed, so we are trying to solve the following two questions.
\[
\max x_1^2 \quad \text{s.t.} \; p_1x_1 + p_2x_2 = y_{12}, \quad x_1, x_2 \geq 0
\]
\[
\max x_3^2 \quad \text{s.t.} \; p_3x_3 + p_4x_4 = y_{34}, \quad x_3, x_4 \geq 0
\]

We can apply the result in (b) to the above two questions and get the maximized utility from each commodity pairs \( (x_1, x_2) \) and \( (x_3, x_4) \). The total maximized utility for Bev will be their sum. Hence,
\[
V(p, y_{12}, y_{34}) = (\frac{\alpha}{\alpha + \beta})^\alpha (\frac{\beta}{\alpha + \beta})^\beta p_1^\alpha p_2^{-\beta} y_{12}^\alpha + (\frac{\alpha}{\alpha + \beta})^\alpha (\frac{\beta}{\alpha + \beta})^\beta p_3^{-\alpha} p_4^{-\beta} y_{34}^{\alpha + \beta}
\]
\[
= (\frac{\alpha}{\alpha + \beta})^\alpha (\frac{\beta}{\alpha + \beta})^\beta (p_1^{-\alpha} p_2^{-\beta} y_{12} + p_3^{-\alpha} p_4^{-\beta} y_{34}^{\alpha + \beta})
\]
\[
= (\frac{\alpha}{\alpha + \beta})^\alpha (\frac{\beta}{\alpha + \beta})^\beta (p_1^{-2\alpha} p_2^{-2\beta} y_{12}^2 + p_3^{-2\alpha} p_4^{-2\beta} y_{34}^{2\alpha + 2\beta}) 
\]
\[
\therefore \beta = 2 - \alpha
\]

(d) This time, we are solving the following problem.
\[
\max \quad V(p, y_{12}, y_{34}) \quad \text{s.t.} \; y_{12} + y_{34} = I, \; y_{12}, y_{34} \geq 0 \quad \ldots (A)
\]

One thing to note here is that \( V(p, y_{12}, y_{34}) \) is convex in \( y_{12} \) and \( y_{34} \). Therefore, setting up the Lagrangian, taking FOCs and setting them to 0 won’t lead to the maximum.
I think the quickest way to solve this is as follows. Solve for $y_{34}$ from the constraint equation, and substitute it into the formula for $V(p, y_{12}, y_{34})$. Also note that the term $(\frac{3}{2})^\alpha (\frac{2}{a})^{2-\alpha}$ in $V$ does not affect the value of $y_{12}$ and $y_{34}$ that attains the maximum.

Hence, we can think of the following problem instead of (A).

$$\max_{0 \leq y_{12} \leq I} f(y_{12}) = p_1^{-\alpha}p_2^{2-\alpha}y_{12}^2 + p_3^{-\alpha}p_4^{2-\alpha}(I-y_{12})^2 \quad \text{...(B)}$$

If you sketch the graph of $f(y_{12})$ for $0 \leq y_{12} \leq I$, you will see see that it can attain max only at $y_{12} = 0$ or $y_{12} = I$ (or both, if the value of $f$ at these two points are equal).

Then, as $f(0) = p_3^{-\alpha}p_4^{2-\alpha}I^2$ and $f(I) = p_1^{-\alpha}p_2^{2-\alpha}I^2$, $f$ is maximized at :

- $y_{12} = 0$ if $f(0) \geq f(I) \Leftrightarrow p_3^{-\alpha}p_4^{2-\alpha} \geq p_1^{-\alpha}p_2^{2-\alpha}$
- $y_{12} = I$ if $f(0) \leq f(I) \Leftrightarrow p_3^{-\alpha}p_4^{2-\alpha} \leq p_1^{-\alpha}p_2^{2-\alpha}$

Bev consumes only the first two commodities if the latter condition holds with strict inequality. So the condition we are asked is, $p_3^{-\alpha}p_4^{2-\alpha} < p_1^{-\alpha}p_2^{2-\alpha}$, or $p_3^2p_4^{2-\alpha} > p_1^2p_2^{2-\alpha}$

3. Modified Cobb-Douglas Preferences

This is a good practice for applying the Kuhn-Tucker condition. Notice the difference in result from the standard Cobb-Douglas preference $U(x) = \alpha_1 \ln x_1 + \alpha_2 \ln x_2$ case.

(a) The consumer solves,

$$\max_{x_1, x_2} \alpha_1 \log(1+x_1) + \alpha_2 \log(1+x_2)$$

s.t. $p_1 x_1 + p_2 x_2 \leq I, \quad x_1, x_2 \geq 0$

Setting up the lagrangian,

$$L(x_1, x_2, \lambda) = \alpha_1 \log(1+x_1) + \alpha_2 \log(1+x_2) + \lambda(I-p_1 x_1 - p_2 x_2)$$

Since the objective function is concave (as it is sum of concave functions) by inspection and the constraint is linear, FOCs guarantee the maximum.

(FOCs)

$$\begin{align*}
(x_1) & \quad \frac{\alpha_1}{1+x_1} - \lambda p_1 \leq 0, \text{ with equality if } x_1 > 0 \quad \text{...(1)} \\
(x_2) & \quad \frac{\alpha_2}{1+x_2} - \lambda p_2 \leq 0, \text{ with equality if } x_2 > 0 \quad \text{...(2)} \\
(\lambda) & \quad I - p_1 x_1 - p_2 x_2 \geq 0, \text{ with equality if } \lambda > 0 \quad \text{...(3)}
\end{align*}$$

Again, since the utility is monotonic, the budget constraint ((3)) must hold with equality at the maximum. This implies that $x_1 = x_2 = 0$ can’t be a solution, so there are 3 cases to consider.

<Case 1> $x_1 > 0$ and $x_2 > 0$

In this case, (1) and (2) both hold with equality.

Therefore,

$$\lambda = \frac{\alpha_1}{p_1(1+x_1)} = \frac{\alpha_2}{p_2(1+x_2)} = \frac{\alpha_1 + \alpha_2}{p_1(1+x_1) + p_2(1+x_2)} = \frac{\alpha_1 + \alpha_2}{p_1 + p_2 + I} \quad \text{...(4)}$$

Solving for $x_1$ and $x_2$, we get

$$x_1 = \frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{p_1 + p_2 + I}{p_1} - 1 \quad \text{and} \quad x_2 = \frac{\alpha_2}{\alpha_1 + \alpha_2} \frac{p_1 + p_2 + I}{p_2} - 1$$
Now, we need to make sure that \( x_1 > 0 \) and \( x_2 > 0 \).

It requires that
\[
\begin{align*}
x_1 &= \frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{p_1 + p_2 + I}{p_1} - 1 > 0 \iff I > \frac{\alpha_1}{\alpha_2} p_1 - p_2 \\
x_2 &= \frac{\alpha_2}{\alpha_1 + \alpha_2} \frac{p_1 + p_2 + I}{p_2} - 1 > 0 \iff I > \frac{\alpha_1}{\alpha_2} p_2 - p_1
\end{align*}
\]

Case 2 > \( x_1 > 0 \) and \( x_2 = 0 \)

In this case, (1) holds with equality but (2) does not need to.

By substituting \( x_2 = 0 \) in the budget constraint (3) and solving for \( x_1 \),
\[
x_1 = \frac{I}{p_1}
\]

Again, we need to make sure that all FOCs are satisfied.

From (1) and (3),
\[
\lambda = \frac{\alpha_1}{p_1(1+x_1)} = \frac{\alpha_1}{p_1+I}
\]

Substituting \( x_2 = 0 \) and \( \lambda \) into (2),
\[
\alpha_2 \leq \lambda p_2 = \frac{\alpha_1 p_2}{p_1+I} \iff I \leq \frac{\alpha_1}{\alpha_2} p_2 - p_1
\]

Case 3 > \( x_1 = 0 \) and \( x_2 > 0 \)

By similar argument as in Case 2, we get
\[
x_2 = \frac{I}{p_2} \text{ and } I \leq \frac{\alpha_1}{\alpha_2} p_1 - p_2
\]

Summarizing the 3 cases, we obtain

1) If \( I > \frac{\alpha_1}{\alpha_2} p_1 - p_2 \) and \( I > \frac{\alpha_1}{\alpha_2} p_2 - p_1 \),
\[
x_1 = \frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{p_1 + p_2 + I}{p_1} - 1 \text{ and } x_2 = \frac{\alpha_2}{\alpha_1 + \alpha_2} \frac{p_1 + p_2 + I}{p_2} - 1
\]

2) If \( I \leq \frac{\alpha_1}{\alpha_2} p_1 - p_2 \), \( x_1 = \frac{I}{p_1} \) and \( x_2 = 0 \)

3) If \( I \leq \frac{\alpha_1}{\alpha_2} p_2 - p_1 \), \( x_1 = 0 \) and \( x_2 = \frac{I}{p_2} \)

(b) From the result in (a), only commodity 1 is consumed if
\[
I \leq \frac{\alpha_1}{\alpha_2} p_2 - p_1
\]

4. Elasticity of consumption ratios

(a) Consumer solves,
\[
\max_x U(x) \text{ s.t. } px \leq I \text{ and } x \geq 0
\]

As we have seen in 1(c), since he has CES preference, there exists his utility representation \( U(x) \) such that
\[
U(x) = \sum_{j=1}^{n} u_j(x_j) \text{ and } u_j'(x_j) = \frac{\alpha_j}{x_j}
\]

Setting up the Lagrangian,
\[
L(x, \lambda) = U(x) + \lambda (I - px) = \sum_{j=1}^{n} u_j(x_j) + \lambda (I - px)
\]

FOCs are
\[
(x_j) \quad u_j'(x_j) - \lambda p_j \leq 0, \text{ with equality if } x_j > 0
\]

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\[ \iff \frac{\alpha_j}{x_j} - \lambda p_j \leq 0, \text{ with equality if } x_j > 0 \ (j = 1, 2, \ldots, n) \ldots (1) \]

\[ (\lambda) \quad I - px \geq 0, \text{ with equality if } \lambda > 0 \ldots (2) \]

Note that \( \frac{\alpha_j}{x_j} \to \infty \) as \( x_j \downarrow 0 \). Therefore, to satisfy (1) we need \( x_j > 0 \) for all \( j \).

Hence, (1) holds with equality for all \( j = 1, 2, \ldots, n \).

Therefore, \( \lambda = \frac{\alpha_i}{p_i x_i} = \frac{\alpha_j}{p_j x_j} \iff \frac{x_i}{x_j} = \left( \frac{p_j}{p_i} \right)^{\frac{1}{\sigma}} \).

(b) The optimal ratio of demand \( \frac{x_i}{x_j} \) we obtained in (a) does not include income \( I \) in its expression. Therefore, this ratio does not vary with income.

Note:
This is an important property of CES utility. Due to this property, "the Income Expansion Path" becomes a ray from the origin.

(c) Taking log of the ratio we obtained in (a), we obtain
\[ \ln \frac{x_i}{x_j} = \frac{1}{\sigma} \left( \ln p_i - \ln p_j + \ln \frac{\alpha_j}{\alpha_i} \right) \]

Therefore, the elasticities are;
\[ E(\frac{x_i}{x_j}, p_i) = \frac{\partial \ln \frac{x_i}{x_j}}{\partial \ln p_i} = \frac{1}{\sigma} \]
and
\[ E(\frac{x_i}{x_j}, p_j) = \frac{\partial \ln \frac{x_i}{x_j}}{\partial \ln p_j} = -\frac{1}{\sigma} \]

Note that \( E(\frac{x_i}{x_j}, p_i) \) is the elasticity of substitution, and it is constant \( (\frac{1}{\sigma}) \) here - hence the name CES (constant elasticity of substitution)!