Instructor: John Riley

Ec 201A Microeconomic Theory
Midterm Friday, October 28, 1994

Time allowed: 1¾ hr plus 10 minutes of reading only. Attempt three (3) questions only.

1) Utility maximization under tight and loose budget constraints

A consumer has the following utility function, that depends on his consumption of two goods ("x" and "y"):

\[ u = (100-0.5x)x + (50-y)y; \]

subject to the constraint that he cannot spend more than his income (I) in buying such goods. Assume that the price of both goods is equal to one (i.e., \( p_x = p_y = 1 \)), and find the quantities of "x" and "y" that maximize the consumer's utility subject to his budget constraint in the three following cases:
(a) \( I = 30 \), (b) \( I = 80 \), (c) \( I = 130 \).

2) Profit maximization with two capacity constraints

A monopolist has to decide his level of production (q) and capacity (x) for the following year. He already has a certain level of current capacity \( (x_o) \) and faces an operating cost of $10 per unit produced and a capacity cost of $20 per unit of additional capacity installed. He knows that his new capacity (current plus additional) must be at least as great as his current capacity (i.e., \( x \geq x_o \)), but also that his production cannot be greater than such new capacity (i.e., \( q \leq x \)).

Our monopolist faces the following demand function for his production:

\[ q = 100 - p \]

where "p" is the price. Find the profit-maximizing levels of "q", "x" and "p" in the follow three situations: (a) \( x_o = 30 \), (b) \( x_o = 40 \), (c) \( x_o = 50 \).

3) Consumer Choice with rationing

The Clinton Oil Crisis has led to rationing. An individual with utility function \( U(x) \) and income I is issued B ration points. He faces not only a vector of prices p but also a vector of ration point prices r. That is, if the individual purchases \( x_i \) units of commodity i he pays \( p_i x_i \) dollars and \( r_i x_i \) ration points.

a) Write down the 2 constraints and then form a Lagrangian to solve for the consumer's optimum. Let \( \lambda \) be the shadow price for the budget constraint and \( \mu \) be the shadow price for the ration point constraint.
Time allowed: three (3) hours. Attempt four (4) questions only. Credit is weighted more to the earlier parts of each question so do not linger on a final part, unless you really have the extra time to go for a few extra points.

1. Demand and Supply
A firm produces two products $x_1$ and $x_2$ and uses inputs $z = (z_1, \ldots, z_n)$. The firm is a price taker on input markets and in the market for commodity two. However, it has monopoly power over $p_1(x_1)$, the price of commodity one.
(a) Show that the first law of input demand holds in this case.
(b) Show also that the first law of supply holds for commodity 2.
(c) Suppose $p_1(x_1) = g(x_1) + \alpha$. What, if anything, can be said about the effect of an increase in $\alpha$ on the firm's choice of $x_1$?
(d) If the price of input 1 rises, will the firm necessarily produce less of at least one commodity?

2. Roberto Caruso
Roberto Caruso produces output per day $Y$ using labor per day $L$. The production plan $(L,Y)$ must lie in the production set
$$S = \{(L,y) \mid y \leq 6(L-o)^{1/2}\}.$$ Unless you are explicitly told otherwise, assume $\alpha = 2$. Roberto is the only person living on this South-Atlantic island.
(a) His utility function is $U = \frac{H}{T} Y^H$, where leisure time $H = 24-L$. Show that his optimal output is 18. What is the optimal supply of labor?
(b) If Roberto, as manager of the firm, acts as a price taker, solve for his optimal production plan given an output price, $p$ and price of labor, $w$. Hence show that the optimal production plan will be produced if $w = p = 1$.
(c) At these prices (of output and labor) show that maximized profit is zero. At night Caruso decides on his optimal consumption decision. Show that his night-time decision implies that these prices are Walrasian equilibrium prices?
(d) Depict your answers to parts (a) - (c) in a neat figure with $Y$ on the vertical axis and $-L$ on the horizontal axis.
(e) Would price taking make sense if there were a large number of identical firms, each owned by a single individual like Roberto Caruso?
(f) Discuss the effect on the optimum and the Walrasian equilibrium of an
earthquake which has the effect of increasing \( \alpha \) for EVERY firm by a small amount.

3. Walrasian equilibrium

Alex has a utility function \( U_A(x, y) = xy \). Bev has a utility function \( U_B(x, y) = \ln(10+x) + \ln y \). The aggregate endowment is 10 units of \( x \) and 20 units of \( y \).

(a) Show that any point on the line \( y_A = x_A \) in the Edgeworth Box (with the origin for Alex in the bottom left corner and \( x \) on the horizontal axis) is Pareto efficient.

(b) Suppose Alex has an endowment \( (x_A, y_A) = (7, 7) \). What will be the Walrasian equilibrium price ratio \( p_1/p_2 \)? What trade will take place in equilibrium.

(c) If Alex's endowment of \( x \) declines (and the aggregate endowment remains constant), what will be the effect on the equilibrium price ratio?

(d) Again holding constant the aggregate endowment, determine the range of potential equilibrium price ratios associated with different endowments for Alex.

\[
\rho = \frac{A}{A+\delta} \quad (\rho = 1)
\]

4. Life cycle consumption

\[ (1+r)\delta < 1 \]

\[ |r| < \frac{1}{\delta} \]

An individual has a utility function \( U(c_1, c_2, \ldots, c_T) = \sum_{t=1}^{T} \delta^{t-1} u(c_t) \), where \( u \) is concave. The interest rate (for borrowing or lending) is \( r \), where

\[
\frac{1}{1+r} > \delta. \quad \text{The individual has an initial asset level} \ K_0 \quad \text{and no source of income other than interest income.}
\]

(a) Determine as fully as you can the nature of his consumption path over the \( T \) periods. (Draw a neat phase diagram.)

(b) At age 30 a new technology is announced that will allow him to live longer. How will this affect his consumption plan?

(c) Contrast your answer to part (a) with the consumption plan of an individual who has the same initial asset level but earns a wage of \( w \) in each period.

(d) In this case, if the time horizon becomes very long, is there any bound on (i) his first period income and (ii) his debt position? Explain carefully.

\[
\frac{(1+r)}{\delta} > 1
\]

\[
\delta(1+r) < 1
\]

\[
\delta + r \delta < 1
\]

\[
\delta < (1+r)\delta
\]

\[
\frac{r}{\delta} < 1 - \frac{\delta}{\delta}
\]

\[
r < 1 - \frac{\delta}{\delta}
\]

\( 0.9 \)
5. Valuing assets.
There are two assets (firms) in an economy. There are 2 states s = 1, 2. The returns on the two assets are $z_1 = (200, 500)$ and $z_2 = (100, 100)$. The probability vector $(x_1, x_2) = (\frac{1}{2}, \frac{1}{2})$. All individuals have the same utility function $v(c_s) = \ln c_s$. For simplicity you may assume that there are only two individuals in the economy. There are no state claims markets but there is a market for each of the assets.
(a) Solve for the Pareto efficient allocations of state claims. Hence show that for all Pareto efficient allocations, the marginal rate of substitution is 4 for each individual.
(b) If each of the two individuals is endowed with 50% of each asset, explain why there will be no trade.
(c) Suppose that each individual initially is undiversified. That is his holdings are either of asset 1 or asset 2. Will there be any trade? If so, what will be the price of firm 1 if the price of firm 2 is $1,000?
(d) The owners of firm 1 sell 100 units of a riskless bond which pays off 1 unit in each state. What will be the market value of these bonds. What will be the value of firm 1 after it is saddled with the new debt?

6. The constant returns to scale economy
Commodity 1 is produced according to the production function

$$F(K, L) = 8\sqrt{KL}$$

Input prices of capital and labor, $(r, w) = (2, 8)$.
(a) How much can be produced at a total cost of $1? What is the ratio of demands, $K_1/L_1$. If commodity 1 is produced, what must be the price of this commodity?
(b) Commodity 2 has production function $G(K, L) = \min (K, L)$. If commodity 2 is produced, what must be its price?
(c) Suppose that the output prices that you have derived are fixed world prices and that the economy has an endowment $(\overline{K}, \overline{L}) = (100, 100)$. How much of each commodity will be produced? (Hint: Draw the $1 isoquants.)
(d) Explain briefly what would happen to the wage if $\overline{L}$ (i) falls slightly (ii) rises.
 Attempt all three questions.

1. Consumer choice

An individual consumes only two goods \((c_1 \text{ and } c_2)\), whose prices are \(p_1 = 1\) and \(p_2\). He has a certain income \(Y\) that defines his budget constraint, and his utility function is:

\[
U = (100c_1 - c_1^2) + \beta(100c_2 - c_2^2);
\]

where \(\beta\) is a parameter that represents the relative importance of the utility that "c_2" generates with respect to the utility that \(c_1\) does.

a) Assume that \(Y = 20\) and \(p_2 = 1\). What is the minimum value of \(\beta\) that makes the consumer willing to buy some \(c_2\)?

b) Now assume that \(\beta = 0.6\) and \(p_2 = 1\). Find an expression for the optimal level of \(c_1\) as a function of income. Draw a diagram for values of \(Y\) between 0 and 30.

c) Now assume that \(\beta = 0.5\) and \(Y = 20\). Find an expression for the optimal level of \(c_2\) as a function of its price.

d) What happens with \(c_1\) and \(c_2\) if \(p_2\) is equal to 1 and \(Y\) is equal to 110?

2. First and second laws of supply

(a) Prove the first law of supply making as few assumptions as you can. In particular, comment on assumptions that you make about whether or not the firm is a price taker in input markets.

(b) Explain carefully why own price elasticity can never be larger in the short-run than in the long run.
Attempt any four (4) questions. You have 10 minutes reading time and 2:50 writing time.

1. Electricity Production

(a) A monopoly with capacity $K$ can produce $q_1 \leq K$ units in the daylight hours and $q_2 \leq K$ at night. The marginal cost of operating the plant is $c = 20$ per unit. The day-time demand price is $p_1 = 100 - q_1$ and night-time demand price is $p_2 = 60 - q_2$. What will be the monopoly's price and output levels during the day and during the night? (You should provide an answer for all $K$ in the range $0 \leq K \leq 100$.) Confirm that the monopolist's revenue is a concave function of $K$.

(b) Let $R(K)$ be the monopolist's revenue in period $t$. Suppose that the cost at time $t$ of adding $z_t$ to capacity is $C(z_t) = \alpha z_t + \beta z_t^2$. Suppose also that the capacity depreciates at the rate $\theta$. The firm discounts the future using the interest rate $r$. Other data is as given in part (a).

Analyze the time path of capacity starting from zero capacity.

(c) Show that the profit maximizing day-time and night-time prices will initially decline. Will the decline in day and night prices continue for all $t$?

2. Equilibrium under uncertainty.

There are 2 individuals, Alex and Bev (or if you prefer, 1000 of each type.) They both have the same beliefs and expected utility function $U(c_1, c_2) = \pi_1 \frac{1}{1-\rho} c_1^{-\rho} + \pi_2 \frac{1}{1-\rho} c_2^{-\rho}$ where $\rho > 0$, and $\rho \neq 1$. Alex has an endowment of 0 units in state 1 and 4 units in state 2. Bev has an endowment of 8 units in state 1 and 0 in state 2.

(a) Characterize the Pareto efficient allocations.

(b) Show that the Walrasian equilibrium state claims price ratio is equal to the odds multiplied by a factor which varies with $\rho$.

(c) Suppose the "endowments" are actually farms. Solve for the equilibrium prices of the two farms. What happens as $\rho$ gets large? Give an explanation for this result.

(d) Suppose instead that both Alex and Bev are infinitely risk averse. Again characterize the Pareto efficient allocations and any Walrasian state claims equilibrium.
3. Choice over time

An individual has a wage \( w \) in each of the \( T \) periods of her life. The interest rate on her savings is \( r \). Her utility function is \( U = \sum_{t=1}^{T} \delta^{t-1} \ln(c_t) \). She has no other assets at \( t=1 \).

(a) Assume that \( \delta(1+r) > 1 \). Show that her consumption will increase over time.

(b) Define \( S_r(\alpha) = \sum_{t=1}^{T} \alpha^{t-1} = \frac{1-\alpha^T}{1-\alpha} \). Solve for her optimal first period consumption as a function of \( S_r\left(\frac{1}{1+r}\right) \) and the other parameters of the problem. Solve also for her first period saving.

(c) Let \( \rho \) be the interest rate at which she can borrow. Suppose \( (1+r)\delta < 1 < (1+\rho)\delta \). What will her first period consumption be?

(d) How would you answer to (b) change if the individual were to live for a further \( R \) periods in retirement with no additional wage income.

\( 4 \)

Monopoly demand and pricing

A monopolist sells a single output \( q \) using inputs \( z \). He is a price taker on input markets.

(a) Show that input demand is non-increasing in input price.

(b) Show that demand is more elastic in the long run.

(c) Suppose \( r(\beta) = (\beta z_1, \beta z_2, \rho_3, ..., \rho_n) \). As \( \beta \) increases, show that \( r_i z_1 + r_j z_2 \) is non-increasing.

(d) Establish conditions (if any) under which an increase in an input price will lead to a decrease in the price of the output.

\( 5 \)

The 2 x 2 model with no international trade

There are 2 inputs, \( L \) and \( K \). Commodity 1 is produced using the fixed-input technology \( x_1 = \text{Min}(L_1, K_1) \). Commodity 2 is produced according to the constant returns to scale production function \( x_2 = K_2^\gamma L_2^{\gamma/\gamma} \).

(a) If the input prices are \( w \) and \( r \), and both goods are produced, solve for the output prices.

(b) Consider the special case in which the aggregate endowment of \( L \) and \( K \) are equal. Both commodities are produced. What can you say (i) about the equilibrium input price ratio, and (ii) the equilibrium output price ratio?
6. The Slutsky Equation

a) Derive the Slutsky equation

\[ \frac{\partial x_i}{\partial p_i} = \frac{\partial x_i}{\partial p_j} \cdot \frac{\partial p_j}{\partial \mu} \]

b) Suppose that the individual has no money income but has an endowment \( \omega \) of the \( n \) goods. Derive his modified Slutsky equation.

c) Consider a pure endowment economy in which individual \( h \), \( h=1, \ldots, H \), has an endowment \( \omega^h \). Suppose that commodity \( i \) is an inferior good and that \( \frac{\partial x_i^h}{\partial \mu} \) is the same for all individuals. What can be said about the slope of the aggregate demand for commodity \( i \)?

\[
\begin{align*}
\chi^d(p, \bar{p}, \theta) &= \chi^h(p, u) \\
\frac{\partial \chi^d}{\partial \theta} + \frac{\partial \chi^d}{\partial \chi^d} \cdot \omega^h &= \frac{\partial \chi^h}{\partial \mu} \\
E &= \min \left\{ \max \left\{ \left| p_1(x_i - \omega_i) + p_2(x_2 - \omega_2) \right| \right\} \right\}
\end{align*}
\]