5.4 RETURNS TO SCALE

In the previous sections we have examined the implications of price-taking behavior by firms and the use of prices to achieve production efficiency. The plausibility of the price-taking hypothesis depends, to a great extent, on whether it is technologically advantageous for a firm to be large. If, when inputs are scaled up, outputs are scaled up less than proportionally, the technology exhibits decreasing returns to scale. If outputs are scaled up more than proportionally, the technology exhibits increasing returns to scale. When there are decreasing returns to scale, it is hard for a large firm to compete with smaller firms and so the number of firms in the market is likely to be relatively large. It is in such situations that a firm is most likely to lose much of its sales, if it raises prices and so the price-taking assumption is most plausible.

Definition: Returns to Scale[1]

The production set $\mathcal{Y} \subset \mathbb{R}^n$ exhibits constant returns to scale if for all $y \in \mathcal{Y}$, and any $\lambda > 0$, $\lambda y \in \mathcal{Y}$. The production set exhibits increasing returns to scale if, for $y \in \mathcal{Y}$, such that $y_j \neq 0,\ j = 1,...,n$ and any $\lambda > 1$. $\lambda y \in \text{int}\,\mathcal{Y}$. The production set exhibits decreasing returns to scale if, for any $y \in \mathcal{Y}$ such that $y_j \neq 0,\ j = 1,...,n$ and any $\mu \in (0,1)$, $\mu y \in \text{int}\,\mathcal{Y}$

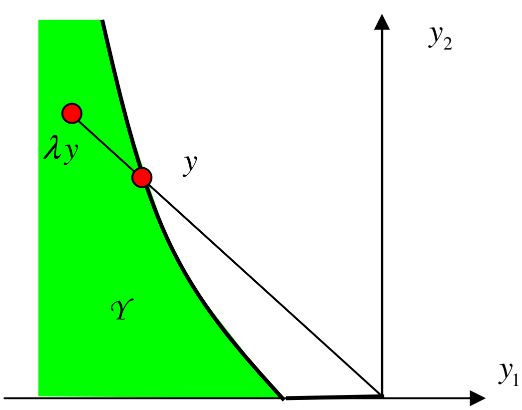

Fig 5.4-1(a): Increasing returns to scale

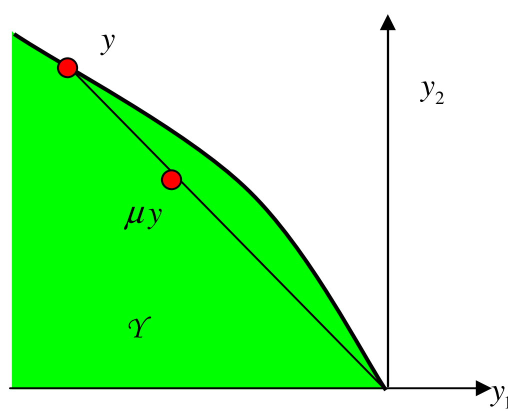

Fig 5.4-1(b): Decreasing returns to scale

[1] If $y \in \mathcal{Y}$ and y is not a boundary point, then it is called an interior point and we write $y \in \text{int}\,\mathcal{Y}$

We begin by showing that this definition yields the familiar definitions for firms producing a single output.

Constant Returns to Scale Production Function

Consider a firm with a strictly increasing production function $F(\cdot)$ and suppose that the firm exhibits constant returns to scale (CRS). With input vector z the maximum feasible output is $F(z)$. Since $(z, F(z))$ is feasible the CRS assumption implies that $(\lambda z, \lambda F(z))$ is a feasible plan. With input vector λz the maximum output is $F(\lambda z)$. Thus

$$F(\lambda z) \geq \lambda F(z).$$

Moreover, since $(\lambda z, F(\lambda z))$ is a feasible plan and production plans exhibit CRS, $\frac{1}{\lambda}(\lambda z, F(\lambda z)) = (z, \frac{1}{\lambda}F(\lambda z))$ is a feasible plan. But with input vector z, $F(z)$ is maximum output. Therefore

$$\frac{1}{\lambda}F(\lambda z) \leq F(z).$$

Combining the two inequalities it follows that $F(\lambda z) = \lambda F(z)$.

Increasing Returns to Scale Production Function

Consider a firm with a strictly increasing production function $F(\cdot)$ and suppose that the firm exhibits increasing returns to scale. With input vector z the maximum feasible output is $F(z)$. With input vector λz the maximum output is $F(\lambda z)$. From the definition of IRS, for any $\lambda > 1$ the input-output vector $(\lambda z, \lambda q) = (\lambda z, \lambda F(z))$ lies in the interior of the production set. Thus with the input λz the maximum output must be greater than $\lambda F(z)$. Thus for any $z > 0$,

$$\lambda > 1 \ \Rightarrow F(\lambda z) > \lambda F(z). \qquad (5.4\text{-}1)$$

Decreasing Returns to Scale Production Function

If the production function exhibits decreasing returns to scale, then, for any $\mu \in (0,1)$, $(\mu z, \mu F(z))$ is in the interior of the production set. Therefore with the input μz the maximum output $F(\mu z)$ is greater. Thus

$$\mu \in (0,1) \;\Rightarrow\; F(\mu z) > \mu F(z),\ \forall z > 0 . \tag{5.4-2}$$

This holds for all $z > 0$ hence for any $\lambda > 1$ it holds for λz. That is

$$\mu \in (0,1) \;\Rightarrow\; F(\mu \lambda z) > \mu F(\lambda z),\ \lambda > 1 .$$

Choose $\mu = 1/\lambda$. Then $F(z) > \frac{1}{\lambda} F(\lambda z),\ \lambda > 1$.

Rearranging this inequality we have the standard definition for Decreasing Returns to Scale.

$$\lambda > 1 \;\Rightarrow F(\lambda z) < \lambda F(z),\ \forall z > 0 . \tag{5.4-3}$$

Returns to scale and the scale elasticity of output

The proportional increases in of output as the scale parameter rises from 1 to λ is

$$\frac{1}{q(1)} \frac{q(\lambda) - q(1)}{\lambda - 1} = \frac{1}{F(z)} \frac{F(\lambda z) - F(z)}{\lambda - 1} .$$

Taking the limit yields the point scale elasticity of output.

$$\mathcal{E}(F(\lambda z), \lambda)\Big|_{\lambda=1} = \frac{\lambda}{F(z)} \frac{\partial}{\partial \lambda} F(\lambda z)\Bigg|_{\lambda=1} .$$

With decreasing returns to scale, it follows from inequalities (5.4-2) and (5.4-3), that for all $\lambda > 0$, $\lambda \neq 1$,

$$\frac{1}{F(z)} \frac{F(\lambda z) - F(z)}{\lambda - 1} < \frac{1}{F(z)} \frac{\lambda F(z) - F(z)}{\lambda - 1} = 1$$

Taking the limit as $\lambda \to 1$ yields the following result.

Decreasing Returns to Scale $\Rightarrow$ $\mathcal{E}(F(\lambda z), \lambda)\big|_{\lambda=1} \leq 1$.

An almost identical argument establishes that

Increasing Returns to Scale $\Rightarrow$ $\mathcal{E}(F(\lambda z), \lambda)\big|_{\lambda=1} \geq 1$.

Local returns to scale

The assumption of (global) increasing or decreasing returns is a very strong one. Typically firms exhibit increasing returns at low output levels because of indivisibilities in entrepreneurial setup and monitoring costs. As output grows large, the costs of monitoring a large managerial work force and providing appropriate work incentives typically rise more rapidly than output. These cost increases can more than offset any purely technological advantages to greater scale.

It is therefore helpful to consider returns to scale locally. Local returns are increasing at the input vector z if a small proportional increase in z leads to a higher proportional increase in output. Local returns are decreasing if the proportional increase in inputs leads to a lower proportional increase in output.

Since the point elasticity $\mathcal{E}(y,x) \equiv \frac{x}{y}\frac{\partial y}{\partial x}$ it follows that

$$\mathcal{E}(F(\lambda z),\lambda) = \frac{\lambda}{F(\lambda z)}\frac{\partial}{\partial \lambda}F(\lambda z) = \frac{\lambda z \cdot \frac{\partial F}{\partial z}(\lambda z)}{F(\lambda z)}.$$

Hence the scale elasticity at z can be expressed as follows.

$$\mathcal{E}(F(\lambda z),\lambda)\Big|_{\lambda=1} = \frac{z \cdot \frac{\partial F}{\partial z}(z)}{F(z)}$$

Definition: Local Returns to scale

The production function $F(z)$ exhibits local

increasing returns at z if the scale elasticity, $\mathcal{E}(F(\lambda z),\lambda)\Big|_{\lambda=1} = z \cdot \frac{\partial F}{\partial z} / F(z) > 1$

and decreasing returns if the scale elasticity $\mathcal{E}(F(\lambda z),\lambda)\Big|_{\lambda=1} = z \cdot \frac{\partial F}{\partial z} / F(z) < 1$.

It is left as an exercise to establish that if a production function exhibits local increasing returns to scale everywhere, it exhibits (global) increasing returns.

We next show that average cost exceeds marginal cost if and only if local returns are increasing.

Proposition 5.4-1: Average and Marginal Cost

If z is cost minimizing for output *q*, then

$$\frac{AC(q)}{MC(q)} = \frac{z \cdot \frac{\partial F}{\partial z}}{F(z)} = \mathcal{E}(F(\lambda z), \lambda)\Big|_{\lambda=1} \tag{5.4-4}$$

<u>Proof</u>: Given the input price vector *r*, the cost function is

$$C(q,r) = \underset{z}{Min}\{r \cdot z \mid q \le F(z)\}$$

Converting this to a maximization problem, the associated Lagrangian is

$$\mathfrak{L} = -r \cdot z + \lambda(F(z) - q)$$

By the Envelope Theorem,

$$MC(q) = \frac{\partial C}{\partial q} = \lambda$$

Moreover the necessary conditions for an interior maximum are

$$\frac{\partial \mathfrak{L}}{\partial z_i} = -r_i + \lambda \frac{\partial F}{\partial z_i} \le 0, \qquad i = 1,\ldots,n \text{ with equality if } z_i > 0\,.$$

Multiplying this inequality by z_i, it follows that $-r_i z_i + \lambda z_i \dfrac{\partial F}{\partial z_i} = 0$

Then total cost,

$$C(q,r) = r \cdot z = \lambda z \cdot \frac{\partial F}{\partial z} \tag{5.4-5}$$

Thus the average cost of production,

$$AC(q) = \frac{C(q)}{q} = \lambda \frac{z \cdot \frac{\partial F}{\partial z}}{F(z)} = MC(q) \frac{z \cdot \frac{\partial F}{\partial z}}{F(z)}$$

QED

Note next that since $C(q) = qAC(q)$,

$$MC(q) = \frac{\partial C}{\partial q} = AC(q) + q\frac{\partial AC}{\partial q}.$$

With local increasing returns, average cost is greater than marginal cost therefore $AC(q)$ is a decreasing function of q and with local decreasing returns, $AC(q)$ is an increasing function.

E***xercise 5.4-1: Increasing Returns to Scale***

If the strictly increasing production function $F(\cdot)$ exhibits increasing returns to scale show that for all $z \neq 0$ and $\mu \in (0,1)$, $F(\mu z) < \mu F(z)$.

Exercise 5.4-2: Returns to Scale and Average cost

Prove that if a firm exhibits increasing/decreasing returns to scale then average cost must decrease/increase with output.

Exercise 5.4-3: Modified Cobb-Douglas Production Function

The production function of a firm is defined implicitly as follows.

$$q = K^{\frac{\alpha}{q}} L^{\frac{\beta}{q}},\ \alpha,\beta > 0.$$

(a) Given input prices (r,w), show that the cost minimizing input demands satisfy

$$\frac{\alpha}{rK} = \frac{\beta}{wl} = \frac{\alpha+\beta}{C(q)}$$

(b) Hence or otherwise obtain an expression for the firm's cost function.

(c) If $\alpha + \beta = 1$, show that the Average cost function is U-shaped, with a minimum at $q = 1$..

(d) Does a change in an input price have any effect on the cost minimizing output?

Exercise 5.4-4: Local and Global Returns to Scale

(a) Show that if a production function $F(z)$ exhibits local increasing returns to scale everywhere, then the scale elasticity $E[F(\mu z),\mu] > 1$ for all z and all $\mu > 0$.

(b) Hence show that $\frac{\partial}{\partial \mu}\ln F(\mu z) > \frac{1}{\mu}$.

(c) Show that, for any $\lambda > 1$,

$$\ln \frac{F(\lambda z)}{F(z)} = \int_1^{\lambda} \frac{\partial}{\partial \mu} \ln F(\mu z) d\mu \ .$$

Then appeal to part (b) to establish that the production function exhibits (global) increasing returns to scale.