Auction Choice part B - APPENDICES

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Equations referred to in part a

1.
2. (2.1) (2.2) (2.3) (2.4) (2.5) (2.6) (2.7)
3.
4.
5. (4.1) (4.2) (4.3)(4.4)(4.5)
6. (6.1)
7. (7.1)
8.
9.

(9.1)

footnotes (11

FN1 \(^1\)  FN2 \(^2\)  FN3 \(^3\)  FN4 \(^4\)  FN5 \(^5\)  FN6 \(^6\)  FN7 \(^7\)  FN8 \(^8\)  FN9 \(^9\)  FN\(^{10}\)  FN\(^{11}\)

==============================================================================

\(^1\) FN1
\(^2\) FN2
\(^3\) FN3
\(^4\) FN4
\(^5\) fn5
\(^6\) fn6
\(^7\) fn7
\(^8\) fn8
\(^9\) fn9
\(^{10}\) fn10
\(^{11}\) fn11
10. APPENDICES

APPENDIX A

Lemma 2: Ordering maximum bids

If Assumption 1 holds then maximum bids satisfy \( b_1^* = b_2^* \geq \ldots \geq b_n^* \). Moreover, if \( b_i^* > b_{i+1}^* \), \( \beta_i > \beta_{i+1} \)

Proof: If all bidders but bidder \( j \) bid \( b \) or less with probability 1, bidder \( j \) has no incentive to bid higher than \( b \) either. Thus the maximum bid of at least two bidders must be the maximum of all the bids. Suppose that the \( (m+1) \)th highest maximum bid \( b_j^* \) is strictly less than \( b^* \) the smallest of the \( m \) highest maximum bids. Let \( B_m \) be the set of \( m \) bidders with maximum bids of at least \( b^* \). Let bidder \( i \) be a bidder in \( B_m \). Then \( b_j^* \leq b^* \).

Since \( b_j^* \) is buyer \( i \)'s maximum bid and equilibrium bidding is monotonic, it is a best response for \( \beta_i \). Hence

\[
\times_{j\neq i} p_j(b)u(b, \beta_i) \leq \times_{j\neq i} p_j(b_i^*)u(b_i^*, \beta_i), \quad \text{for all } b < b_i^*.
\]

Moreover, \( p_i(b) < 1 \) for all \( b < b_i^* \), hence

\[
\times_{j\neq i} p_j(b_j^*)u(b_j^*, \beta_i) \leq \times_{j\neq i} p_j(b_i^*)u(b_i^*, \beta_i),
\]

Since buyer \( j \) bids \( b_j^* \) or less with probability 1, it follows that

\[
\times_{k \neq j} p_k(b_j^*)u(b_j^*, \beta_i) \leq \times_{k \neq j} p_k(b_i^*)u(b_i^*, \beta_i).
\]

Appealing to Assumption 1, it follows that

\[
\times_{k \neq j} p_k(b_j^*)u(b_j^*, v_j) \leq \times_{k \neq j} p_k(b_i^*)u(b_i^*, v_j), \quad \forall v_j > \beta_i.
\]

Hence buyer \( j \) is strictly better off bidding \( b_i^* \) when his type is \( \beta_j \) unless \( \beta_j < \beta_i \), that is, unless \( j > i \).

Q.E.D.
**Lemma 4:** Given Assumption 1, the equilibrium bid distributions of bidder $i$ is an interval $[r, b^*_i]$.

**Proof:** Suppose that buyer $i$'s equilibrium bid distribution has a "gap" $[b^o, b^{oo}]$. That is, $b^o$ and $b^{oo}$ are both best replies for buyer $i$ of type $\phi_i(b^o)$. Buyer $i$ faces a distribution of equilibrium bids with c.d.f $\pi_i(b) = \times p_i(b)$. His expected utility is therefore

$$e_i(b, \phi_i(b^o)) = \pi_i(b)u(b, \phi_i(b^o)). \quad (10.1)$$

Suppose that $m$ buyers bid with strictly positive probability in all right neighborhoods of $b^o$. Without loss of generality, let these be buyers 1, through $m$. Then $i > m$. Arguing as in the proof of Proposition 3, the bid distributions for these $m$ bidders must satisfy

$$\sum_{k=1}^{m} \frac{d}{db} \ln p_j(b^o) = L(b^0, H_k(p_k(b^o))) = L(b^0, \phi_k(b^o)), \quad k = 1, \ldots, m$$

Summing over $k$,

$$\sum_{k=1}^{m} \sum_{j=1}^{m} \frac{d}{db} \ln p_j(b^o) = (m-1)\sum_{k=1}^{m} \frac{p_j}{p_k} = \sum_{k=1}^{m} L(b^0, \phi_k(b^o)), \quad (10.2)$$

From (10.1)

$$\frac{\partial}{\partial b} \ln e_i \bigg|_{b=b^o} = \frac{\partial}{\partial b} \sum_{j=1}^{m} \ln p_i + \frac{\partial}{\partial b} \ln u_i = \frac{\partial}{\partial b} \sum_{j=1}^{m} p_j \frac{p_j}{p_k} - L_i(b^o, \phi_i(b^o)).$$

Since buyer $i$ of type $\phi_i(b^o)$ is indifferent between bidding $b^o$ and $b^{oo}$ and at least weakly prefer these bids to any in between, it follows that the right hand side must be non-positive. Substituting from (10.2), we obtain the necessary condition,

$$\frac{\partial}{\partial b} \ln e_i \bigg|_{b=b^o} = \frac{1}{m-1} \sum_{k=1}^{m} L(b^o, \phi_k(b^o)) - L_i(b^o, \phi_i(b^o)) \leq 0. \quad (10.3)$$

Moreover, since $\phi_i(b^o)$ is indifferent between $b^o$ and $b^{oo}$, Lemma 1 implies that, for any type $v > \phi_i(b^o)$,

$$\prod_{j=1}^{n} p_j(b^o)u(b^o, v) < \prod_{j=1}^{n} p_j(b^{oo})u(b^{oo}, v).$$
Since buyer $i$ bids in the interval $[b^o, b^\infty]$ with zero probability and each buyer $k = 1, \ldots, m$ bids with strictly positive probability, it follows that

$$
\prod_{j=1}^{n} p_j(b^o)u(b^o, v) < \prod_{j=1}^{n} p_j(b^\infty)u(b^\infty, v), \; k = 1, \ldots, m.
$$

Then if, for some $k < m$, $\phi_k(b^o) > \phi_i(b^o)$, it follows that

$$
e_k(b^o, \phi_k(b^o)) < e_k(b^\infty, \phi_k(b^o)).$$

But this cannot the case since bidding $b^o$ is a best reply for $\phi_k(b^o)$. Thus we may conclude that $\phi_k(b^o) \leq \phi_i(b^o)$, $k = 1, \ldots, m$. Assumption 1 then implies that

$$L(b^o, \phi_k(b^o)) \geq L(b^o, \phi_i(b^o)), \; k = 1, \ldots, m.$$ 

It follows immediately that

$$\frac{1}{m-1} \sum_{k=1}^{m} L(b^o, \phi_k(b^o)) \geq \frac{m}{m-1} L(b^o, \phi_i(b^o)), \; k = 1, \ldots, m.$$ 

But this contradicts (10.3). Thus there can be no interval $[b^o, b^\infty]$ over which bidder $i$ bids with zero probability.

Q.E.D.

**Proposition 6:** Suppose Assumptions 1 and 2 hold. Then, for any $b^o > r$ such that $u(b^o, \beta^i) > 0$, $p(b)$ defined by (2.7) satisfies

$$p'(b) \geq 0, \; b < b^o \; \text{and} \; p'(b), \ldots, p_m'(b) > 0, \; b \in [b_{m+1}, b_m^o].$$

**Proof:** Suppose that $b_{k+1}^o < b_k^o = \ldots = b_i^o = b^o$. From (2.7), $e_k(b, \beta_k^i) = \prod_{j=1}^{k-1} p_j(b)u(b, \beta_k^i)$ has a local maximum at $b^o$, where $p(b)$ satisfies (2.3) with $m = k - 1$. It follows that

$$\frac{d}{db} \ln e_k = \sum_{j=1}^{k-1} \frac{d}{db} \ln p_j(b^o) - L(b^o, \beta_k^i) \geq 0. \quad (10.4)$$

From (2.4),
\[
\sum_{j=1}^{k-1} \frac{d}{db} \ln p_j(b^o) - L(b^o, \beta_j) = 0, \ i \leq k - 1
\]  
(10.5)

Summing over \(i, \ i = 1,...,k - 1\)

\[
(k - 2)\sum_{j=1}^{k-1} \frac{d}{db} \ln p_j(b^o) - \sum_{j=1}^{k-1} L(b^o, \beta_j) = 0
\]

Substituting this expression into (10.4)

\[
\sum_{j=1}^{k-1} L(b^o, \beta_j) - (k - 2)L(b^o, \beta_k) \geq 0
\]  
(10.6)

Over the interval \([b^o_{k+1}, b^o]\)

\[
\sum_{j=1}^{k} \frac{d}{db} \ln p_j(b) - L(b, \phi_i(b)) = 0, \ i \leq k
\]  
(10.7)

Inverting, it follows that

\[
p_i'(b) = \frac{1}{k - 1} \left( \sum_{j=1}^{k} L(b, \phi_j(b)) - (k - 2)L(b, \phi_1(b)) \right)
\]

\[
= \frac{1}{k - 1} \left( \sum_{j=1}^{k} L(b, \phi_j(b)) - (k - 1)L(b, \phi_1(b)) \right).
\]  
(10.8)

From (10.6) and (10.8) \(p_k'(b^o) \geq 0\). Since \(\beta_1 \geq \ldots \geq \beta_k\), \(L(b^o, \beta_1) \leq \ldots \leq L(b^o, \beta_k)\). Hence, from (10.8)

\[
p_1'(b^o) \geq \ldots \geq p_k'(b^o) \geq 0.
\]  
(10.9)

It follows that either \(p_1'(b), \ldots, p_k'(b) > 0\) on \([b^o_{k+1}, b^o]\) or there exists some \(i\) and some \(\hat{b} \in [b^o_{k+1}, b^o]\) such that \(p_i'(\hat{b}) = 0\) and \(p_1'(b), \ldots, p_k'(b) \geq 0\) on \([\hat{b}, b^o]\). Suppose the latter.

We will show that it must also be the case that \(p_i''(\hat{b}) < 0\). Then there can be no such \(\hat{b} < b^o\), since \(p_i'(\hat{b}) = 0\) and \(p_i'(b) \geq 0\), \(b > \hat{b} \Rightarrow p_i''(\hat{b}) \geq 0\). Then \(\hat{b} = b^o\), and \(p_i'(b^o) < 0\). It follows that \(p_1'(b), \ldots, p_k'(b) > 0\) \(b \in [b^o_{k+1}, b^o]\).

Suppose then that
\[ p_i'(\hat{b}) = 0 \text{ and } p_i''(\hat{b}) \geq 0. \]  

(10.10)

From (10.9) it follows that \( \phi_j(\hat{b}) \geq \phi_j(\hat{b}), \ j \leq k \). Hence, by Assumption 1, 

\[ L_j \equiv L(\hat{b}, \phi_j(\hat{b})) \leq L(\hat{b}, \phi_i(\hat{b})) = L_i, \ j \leq k. \]

Also, from (10.8), \( p_i'(\hat{b}) = 0 \) implies that 

\[ \sum_{j=1}^{k} L_j = (k-2)L_i \]  

(10.11)

Since \( L_j \leq L_i, \ j \leq k \) equation (10.11) implies that 

\[ L_j < L_i \text{ for some } j \neq i \]  

(10.12)

Differentiating (10.7),

\[ \sum_{j=1}^{k} \frac{d^2}{db^2} \ln p_j(\hat{b}) = \frac{\partial}{\partial b} L_m(\hat{b}, \phi_m) + \frac{\partial}{\partial \phi} L_m(\hat{b}, \phi_m) \frac{\partial \phi_m}{\partial b}, \ m \leq k. \]

Setting \( m = i \), (10.10) implies that

\[ \sum_{j=1}^{k} \frac{d^2}{db^2} \ln p_j(\hat{b}) = \frac{\partial L_i}{\partial b} \]  

(10.13)

Also, for all \( m \neq i \), \( p_m'(\hat{b}) = F_m'(\phi_m) \phi_m'(\hat{b}) \geq 0 \) implies that 

\[ \sum_{j=1}^{k} \frac{d^2}{db^2} \ln p_j(\hat{b}) \leq \frac{\partial L_m}{\partial b}. \]

Summing for \( m = 1, \ldots, i-1, i+1, \ldots, k \),

\[ \sum_{m=1}^{k} \sum_{j=1}^{k} \frac{d^2}{db^2} \ln p_j(\hat{b}) = (k-2) \sum_{m=1}^{k} \frac{d^2}{db^2} \ln p_j(\hat{b}) + (k-1) \frac{d^2}{db^2} \ln p_i(\hat{b}) \leq \sum_{m=1}^{k} \frac{\partial L_m}{\partial b} \]
Condition (10.10) also implies that \( \frac{d^2}{db^2} \ln p_i(\hat{b}) \geq 0 \). Hence

\[
(k - 2) \sum_{m \neq i}^k \frac{d^2}{db^2} \ln p_j(\hat{b}) \leq \sum_{m \neq i}^k \frac{\partial L_m}{\partial b}
\]  

(10.14)

Together, conditions (10.13) and (10.14) imply that

\[
\sum_{j=1 \atop j \neq i}^k \frac{\partial L_j}{\partial b} \geq (k - 2) \frac{\partial L_i}{\partial b}
\]  

(10.15)

Next note that \( \frac{\partial L_i}{\partial b} = \frac{\partial}{\partial b} - \frac{\partial u_j}{\partial b} = A_j(b, \phi_j) L_j + L_j^2 \).

Thus (10.15) can be rewritten as

\[
\sum_{j=1 \atop j \neq i}^k A_j L_j + L_j^2 \geq (k - 2)(A_i L_i + L_i^2)
\]  

(10.16)

Also, by Assumption 1 and 2 respectively, \( \phi_j \geq \phi_i \Rightarrow L_j \leq L_i \) and \( A_j \leq A_i \). From (10.11) and (10.12) we therefore obtain

\[
\sum_{j=1 \atop j \neq i}^k A_j L_j < (k - 2) A_i L_i
\]

Finally,

\[
\sum_{j=1 \atop j \neq i}^k L_j(\hat{b}, \phi_j)^2 \leq \text{Max} \left\{ \sum_{j=1 \atop j \neq i}^k L_j \mid 0 \leq L_j \leq L_i, \sum_{j=1 \atop j \neq i}^k L_j = (k - 2) L_i \right\} = (k - 2) L_i^2
\]

Combining these last two results, \( \sum_{j=1 \atop j \neq i}^k A_j L_j + L_j^2 < (k - 2)(A_i L_i + L_i^2) \). But this contradicts (10.16) therefore \( p_i''(\hat{b}) < 0 \) as claimed and so the Proposition is proved for \( b \in [b_{k+1}^\alpha, b_k^\alpha] \).

Exactly the same argument can be applied for \( b \in [b_m^\alpha, b_m^\alpha] \), for \( m > k \).
Lemma 7: Monotonicity Property

Consider two solutions $p(b)$ and $\pi(b)$ to the differential equation system (2.3) in the interior of $B = \{(b, p_1, \ldots, p_n) | b \geq r, \ 0 < p_i \leq 1, \ u(b, H_i(p_i)) > 0, \ i = 1, \ldots, n\}$. If $p(b_o) \geq \pi(b_o)$, then $p(b) > \pi(b)$, $b < b_o$ and

$$\prod_{i=1}^{n} p_i(b) > \prod_{i=1}^{n} \pi_i(b_o) \quad b < b_o.$$  \hspace{1cm} (10.17)

Proof: We provide a formal proof for the case of identical supports. However, the proof is readily generalized. Suppose that there is some $\hat{b}$ such that $p_j(\hat{b}) = \pi_j(\hat{b})$ and for all $b \in (\hat{b}, b_o]$, $p_i(b) > \pi_i(b)$, $i = 1, \ldots, n$. Since the solution to (2.7) through $(\hat{b}, p(\hat{b}))$ is unique, there must be some $k$ such that $p_k(\hat{b}) > \pi_k(\hat{b})$. Next we invert (2.3), and obtain:

$$\frac{p_j'}{p_j} = \frac{1}{n-1} \left( \sum_{i \neq j}^{n} L(b, H_i(p_i)) - (n-2)L(b, H_j(p_j)) \right)$$  \hspace{1cm} (10.18)

and

$$\frac{\pi_j'}{\pi_j} = \frac{1}{n-1} \left( \sum_{i \neq j}^{n} L(b, H_i(\pi_i)) - (n-2)L(b, H_j(\pi_j)) \right)$$

Therefore, at $\hat{b}$, since $p_j(\hat{b}) = \pi_j(\hat{b})$

$$\frac{p_j'}{p_j} - \frac{\pi_j'}{\pi_j} = \frac{1}{n-1} \left( \sum_{i \neq j}^{n} L(b, H_i(p_i)) - \sum_{i \neq j}^{n} L(b, H_i(\pi_i)) \right).$$

But, $p_i(\hat{b}) \geq \pi_i(\hat{b})$, $i = 1, \ldots, n$ and the inequality is strict for some $i \neq j$. Therefore

$$\frac{p_j'}{p_j} < \frac{\pi_j'}{\pi_j} \text{ at } b = \hat{b}.$$  

Since $p_j(\hat{b}) = \pi_j(\hat{b})$, this implies that in some right-neighborhood of $\hat{b}$, $p_j(b) < \pi_j(b)$, contradicting our earlier hypothesis. Thus there can be no such $\hat{b}$ and, as claimed, $p_i(b) > \pi_i(b)$, $b \leq b_o$, $i = 1, \ldots, n$
Next, summing (4.2) over $i$,

$$
\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{d}{db} \ln p_j(b) = (n-1) \sum_{i=1}^{n} \frac{d}{db} \ln p_i(b) = \sum_{i=1}^{n} L(b, H_i(p_i(b))) .
$$

(10.19)

Also

$$
(n-1) \sum_{i=1}^{n} \frac{d}{db} \ln \pi_i(b) = \sum_{i=1}^{n} L(b, H_i(\pi_i(b))) .
$$

(10.20)

Then, since $p(b) > \pi(b), b < b_o$, the right hand side of (10.10) is less than the right-hand side of (10.20). Thus

$$
\sum_{i=1}^{n} \frac{d}{db} \ln p_i(b) < \sum_{i=1}^{n} \frac{d}{db} \ln \pi_i(b).
$$

Integrating with respect to $b$ over the interval $[\hat{b}, b_o]$ yields (10.11).

Q.E.D.

**Lemma 15:** If $\frac{F_i''(v_i)}{F_i'(v_i)} \geq -\frac{2}{v_i}$, $L_i(b, H_i(p_i))^{-1}$ is convex in $p_i$.

**Proof:** From the definition of $H_i(\cdot)$,

$$
v_i(b) = F^{-1}(p_i(b)) = H_i(p_i) .
$$

Hence

$$
H_i'(p) = \frac{1}{F_i'(v)} \quad \text{and} \quad H_i''(p) = -\frac{F_i''}{(F_i')^3} .
$$

(10.21)

Define $M(b, p) = \frac{1}{L(b, H_i(p))} = \frac{1}{H_i(p) - b}$. Then

$$
\frac{\partial M}{\partial p_i} = \frac{-H_i'}{(H_i - b)^2} \quad \text{and so} \quad \ln \frac{\partial M}{\partial p_i} = \ln H_i' - 2 \ln(H_i - b) .
$$
Hence

\[ \frac{\partial^2 M}{\partial p_i^2} = \frac{H_i''}{H_i'} - \frac{2H_i'}{H_i - b}. \] (10.22)

If \( F_i'' > 0 \), it follows from (10.21) that \( H_i'' < 0 \) and so the right-hand side of (10.22) is negative. Then, since \( \frac{\partial M}{\partial p_i} < 0 \), \( \frac{\partial^2 M}{\partial p_i^2} > 0 \).

If \( F_i'' < 0 \), so that \( H_i'' > 0 \)

\[ \frac{\partial^2 M}{\partial p_i^2} = \frac{\partial M}{\partial p_i} \left( \frac{H_i''}{H_i'} - \frac{2H_i'}{H_i - b} \right) > \frac{\partial M}{\partial p_i} \left( \frac{H_i''}{H_i'} - \frac{2H_i'}{H_i} \right) = -\frac{\partial M}{\partial p_i} H_i' \left( \frac{2}{H_i} - \frac{H_i''}{(H_i')^2} \right) \]

\[ = -\frac{\partial M}{\partial p_i} H_i' \left( \frac{2}{v_i} - \frac{F_i''}{F_i'} \right) > 0, \text{ if } \frac{F_i''}{F_i'} > \frac{2}{v_i}. \]

Q.E.D.
Appendix B: Computational Issues

The numerical computation in this paper involves essentially two stages. First, we solve the simultaneous differential equation system (4.2), subject to a set of boundary conditions, the so called two point boundary value problem. Second, once the inverse bid functions are found, we compute various values of interest, such as the expected revenues in a high-bid auction.

The two bidder case is illustrated in Figure 10.1 which depicts the two inverse bid functions \( \phi_1(b) \) and \( \phi_2(b) \) (transformed by \( v_1(\cdot) \) and \( v_2(\cdot) \)). As discussed in the paper they are monotonically increasing and satisfy the boundary conditions \( v_1(\phi_1(b^*), \phi_2(b^*)) = v_1(s^*) \), \( v_2(\phi_1(b^*), \phi_2(b^*)) = v_2(s^*) \), and \( v_2(\phi_1(b_s), \phi_2(b_s)) = v_2(s_s) \). The lower bound \( b_s \) is either known or can be computed using Proposition 1 in the main text, but the upper bound \( b^* \) has to be searched. In the first round of search, we start with an interval \([x_{low}, x_{high}]\) that is large enough to trap the highest bid \( b^* \), and set \( x_0 = (x_{low} + x_{high}) / 2 \) as the first estimate of \( b^* \). We then compute \( \phi_1(b_s) \) and \( \phi_2(b_s) \) by solving (4.2). If \( v_2(\phi_1(b_s), \phi_2(b_s)) > b_s \), then the \( x_0 \) we picked is smaller than \( b^* \). In this case we set \( x_{low} = x_o \), otherwise we set \( x_{high} = x_0 \) and repeat the search. For any given \( \varepsilon > 0 \), repeating this process sufficient number of times will
produce $x_{\text{high}} - x_{\text{low}} < \varepsilon$. Given that $x_0$ and $b^*$ are trapped between $[x_{\text{low}}, x_{\text{high}}]$, we have $|x_0 - b^*| < \varepsilon$, thus $x_0$ is an estimate of the highest bid $b^*$ with an error-bound (accuracy) $\varepsilon$.

This procedure is basically the so called shooting method. One slight complication is the potential singularity when integrating from $b_1$ to $b^*$. Note that in (4.2) and both $v_1 - b$ and $v_2 - b$ are denominators, therefore the bids must be kept below the respective valuations at all times, or singularities can happen when the two are equal, that is when one of the bid functions touches the 45º line. As a result the actual convergence has to come from the left of the 45º line.

This sort of trapping will always produce a $x_0$ with a given theoretical accuracy $\varepsilon$, but two other types of errors can compound to $\varepsilon$ and have to be carefully monitored. The first type of error $\varepsilon_1$ is associated with the numerical integration of functions when their analytic values are not available, such as the cumulative distribution function of a normal distribution. The second type of error, $\varepsilon_2$, is from solving (4.2).

Our basic integration routine is the “extended trapezoidal rule.” Accuracy $\varepsilon_1$ is controlled by successive calling of the basic routine over smaller and smaller intervals. Two or three-dimensional integrations are also used in the program. These higher-dimensional integrations are computed by creating identical copies of one-dimensional routines and using it two or three times.

Our routine for solving the differential equations is Richardson extrapolation as used in the Bulirsch-Stoer method. Accuracy $\varepsilon_2$ is controlled by applying the approach to a sequence of finer and finer substeps to match a prescribed error bound. Because bid functions $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are smooth and there are no singular points between the interval $[b_1, b^*]$, the Bulirsch-Stoer method is theoretically much more accurate and efficient than other more conventional methods, such as the Runge-Kutta method. Our experiments confirm this is indeed the case.

Once the bid functions are solved, many indicators can be computed, for example the expected revenues for the buyers and seller in a high-bid auction. Each time a value of the bid function is needed, we have to solve the differential equations (4.2) once, which is a

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12 Several subroutines in this program are adapted from sample programs in Press, Vetterling and Flannery (Numerical Recipes in FORTRAN; the art of scientific computing, Cambridge University Press, 2nd edition, 1992). Names of the subroutines and numerical methods referred to in this appendix also follow the convention set in this book, from which more details and subsequent references can be found.

13 An alternative approach for higher-dimensional integration is the Monte Carlo method. But it is generally less accurate because asymptotic convergence is slow.

14 The basic reason is that the Bulirsch-Stoer method uses a fitting function for extrapolation, and it is more efficient for a smooth function such as ours.
computationally intensive task. In some cases, we need to use the bid function as integrand for three-dimensional integrals. Suppose the integrand has to be computed $10^2$ times to estimate a one-dimensional integral, it then has to be computed $10^6$ times for a three-dimensional integral. Estimating conservatively, if each computation of $\phi_1(b)$ and $\phi_2(b)$ takes one second, computing this three-dimensional integral will take $10^6$ seconds = 12 days.

To overcome this difficulty, we use a functional approximation procedure. After $b^*$ is estimated, we solve the differential equations one more time, and store the values of the bid functions at 100 even intervals between $[b_*, b^*]$. That is, we store $\phi_i(x_i)$ and $\phi_j(x_i)$ for $x_i = b_* + i \cdot (b^* - b_*) / 100, \ i = 0, 1, \cdots, 100$. With these values, $\phi_1'(b^*), \phi_2'(b^*), \phi_1'(b_*), \phi_2'(b_*)$, plus the fact $\phi_1'(b^*) = \phi_2'(b^*) = \infty$ (set at a very large number, $10^{30}$), we are able to interpolate $\phi_1(b)$ and $\phi_2(b)$ with cubic-spline functions over $[b_*, b^*]$. Because the bid functions are generally smooth, and cubic-spline functions are smooth in the first derivative, continuous in the second derivative, our functional approximation is for all practical purposes perfect. A cubic-spline function is just a piecewise cubic polynomial, computing its value is trivial. This way a three-dimensional integral can be computed in a split second. This procedure allows us to achieve demanding accuracy for computations based on the bid functions quite efficiently.

One last notable trick in the program is the computation of expected revenue in a sealed high-bid auction. Since an important goal of this paper is to compare the revenues of high-bid and open auctions, the accuracy of this estimate is crucial. As presented in the paper, the expected revenue in the high-bid auction is

\[
R_0 = \int_{b_*}^{b^*} bdF(\phi_1(b), \phi_2(b)) = \int_{b_*}^{b^*} b \left[ \int_{a_1}^{\phi_1(b)} \int_{a_2}^{\phi_2(b)} f(s_1, s_2) ds_1 ds_2 \right] db.
\]

This is a single or double integral depending on if the cumulative distribution functions are known, therefore is relatively quick to compute. The real difficulty is that $\phi_1'(b)$ and $\phi_2'(b)$ approach infinity near $b_*$, and this makes the numerical integral grossly biased. An alternative approach is to rewrite

\[
R_0 = b F(\phi_1(b), \phi_2(b))|_{b_*}^{b^*} - \int_{b_*}^{b^*} F(\phi_1(b), \phi_2(b)) db
\]

This is a double or even triple integral depending on if the cumulative distribution functions are known. So the computation takes longer, but since it has no singularity points, the error is controlled. The cubic-spline procedure introduced earlier makes computing (10.24) practical, and this is the adopted approach in the program.

\[15\] We check this by comparing the integrals of the bid functions and the integrals of the cubic-splines. The discrepancies are orders smaller than any required error-bound in this paper.
Overall, this computational procedure allows us to investigate the issues in the paper efficiently with a controlled error-bound. A notable alternative program for solving first price auction problems is proposed by Marshall, Meurer, Richard, and Stromquist (1994). In comparison our procedure is different and in many aspects represents an improvement.

We start with a benchmark case in Marshall et al. (1994), two bidders with valuations $\tilde{v}_1, \tilde{v}_2$ uniformly distributed in $[0, 1]$. The analytical solution is $b^* = 0.5$, $\phi_1(0) = \phi_2(0) = 0$. Our numerical solution is $b^* = 0.5 - 1.03 \times 10^{-13}$, $\phi_1(0) = \phi_2(0) = 4.5 \times 10^{-6}$. By shooting a target sufficiently close to but above the 45° line, singularity at the end-point can be avoided, and yet the distortion of this error is so small that a high degree of accuracy is still achieved. Clearly in this linear case where bid functions have constant slopes, the steeper the slope, the smaller the distortion. With nonlinear functions, which have infinite slopes at the lower end-point, distortion of this sort is even smaller.

To follow up on this case, suppose there are two type-1 bidders and three type-2 bidders, then it is a problem computationally similar to those in Marshall et al. Theoretical results are $b^* = 0.8$ and revenue equivalence from the high-bid and open auctions. Our estimates are $b^* = 0.7998$ and a 0.01% revenue difference from the two auctions, therefore it is reasonably accurate. A single run from $b^*$ to $b$, takes an average of 5/100 second on a Pentium 133, and to trap $b^*$ within a $10^{-13}$ interval takes 2.52 seconds, whereas Marshall et. al reported a single run time of the order of 5 seconds. The reason is that Marshall et al. always divide $[0, 1]$ into 10,000 equal subintervals to operate on, whereas our program controls the number of steps and each step-length by ensuring the allowed error-bound is not exceeded. For an example of a linear case, one step is all that is needed for a single run. Furthermore, arbitrarily small steps can be taken in our program if needed, and if error-bound cannot be met, no results will be produced.
Appendix C

We present here some further tables paralleling the results presented in section 8. In each case, despite the fact that the distributions are very different, the revenue differences are remarkably similar.

**TABLE 8.1B: Single Strong Bidder (Uniform case)**

\[ \text{supp}\{\bar{v}_A\} = [\mu_A - 20, \mu_A + 20], \text{supp}\{\bar{v}_B\} = [80,120]. \]

<table>
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<tr>
<th># of B bidders</th>
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<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1.5</td>
<td>3.8</td>
<td>7.4</td>
<td>11.2</td>
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<tr>
<td>2</td>
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<td>0.6</td>
<td>2.3</td>
<td>5.6</td>
<td>9.3</td>
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<td>0.5</td>
<td>1.6</td>
<td>4.0</td>
<td>7.7</td>
</tr>
<tr>
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<td>0.3</td>
<td>1.2</td>
<td>3.6</td>
<td>6.5</td>
</tr>
</tbody>
</table>

**TABLE 8.2B: Competitive Fringe (Uniform Distributions)**

\[ \text{supp}\{\bar{v}_A\} = [\mu_A - 20, \mu_A + 20] \]

\[ \text{supp}\{\bar{v}_B\} = [80,120] \]

\[ m_A = 2 \]

<table>
<thead>
<tr>
<th># of Type B bidders</th>
<th>(R_H)</th>
<th>(R_S)</th>
<th>(\Delta R)</th>
<th>(R_H)</th>
<th>(R_S)</th>
<th>(\Delta R)</th>
<th>(R_H)</th>
<th>(R_S)</th>
<th>(\Delta R)</th>
</tr>
</thead>
<tbody>
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<td>103.4</td>
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<td>113.3</td>
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<td>123.3</td>
<td>0.0</td>
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<td>114.8</td>
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<td>0.3</td>
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<tr>
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<td>114.6</td>
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<td>123.9</td>
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