1. Screening without Transfers

A principal employs an agent who privately observes the state of the world $\theta \in [\underline{\theta}, \bar{\theta}]$ which is distributed with density $f(\theta)$. The principal first makes a report to the principal who chooses an action $q \in \{1, 2\}$. Consider the following direct–revelation mechanism:

1. The principal commits to a mechanism $q(\hat{\theta}) \in \{1, 2\}$.
2. The agent observes the state $\theta$.
3. The agent then sends a message to the principal $\hat{\theta}$.
4. The principal receives payoff $v(\theta, q)$ and the agent receive payoff $u(\theta, q)$.

(a) Suppose $u(\theta, q)$ is supermodular in that

$$u(\theta_H, q_H) + u(\theta_L, q_L) > u(\theta_L, q_H) + u(\theta_H, q_L)$$

for $\theta_H > \theta_L$ and $q_H > q_L$. Show incentive compatibility implies that $q(\theta)$ is increasing.

(b) Characterise the mechanism, $q(\cdot)$, that maximises the principal’s expected payoff.

(c) Intuitively, what happens to the optimal mechanism as the principal’s preferences converge to those of the agent’s? That is, $v(\theta, q) \to u(\theta, q)$ in $L^1$. 
2. Holdup

Consider the following holdup game where the quantity traded is \( q \in \{0, 1\} \). Suppose the agents sign a contract that gives the seller the option to sell \( q = 0 \) at price \( p_0 \) or \( q = 1 \) at price \( p_1 = p_0 + k \). The game works as follows.

1. Investments are made simultaneously. The buyer invests \( b \in \mathbb{R} \) and the seller invests \( s \in \mathbb{R} \).
2. The state of nature \( \theta \) is revealed.
3. The seller has the option to supply \( q = 0 \) at price \( p_0 \) or \( q = 1 \) at price \( p_1 = p_0 + k \). The buyer makes a TIOLI renegotiation offer to the seller.
4. Payoffs are \( v(b, \theta)q - p - b \) for the buyer and \( p - c(s, \theta)q - s \) for the seller, where \( p \) is the traded price.

Assumptions:

- \( v(b, \theta) \) is concave in \( b \). \( c(s, \theta) \) is convex in \( s \).
- \( v, u, s, b, \theta \) are observable but not verifiable.
- There exists states \( \theta \) such that \( c(s, \theta) > v(b, \theta) \) and \( c(s, \theta) < v(b, \theta) \).

(a) Define the first–best investment for the buyer and seller.

(b) What are the seller’s payoffs after renegotiation? [Note, this will depend on whether or not \( c(s, \theta) > k \)].

(c) Write down the seller’s investment problem.

(d) Show that there exists a choice of \( k \) such that the seller chooses the first best investment.

(e) Show that under the optimal \( k \) the buyer also chooses the first–best investment level.
3. Holdup and Private Information

Suppose a buyer invests $b$ at cost $c(b)$, where $c(\cdot)$ is increasing and convex. Investment $b$ induces a stochastic valuation $v$ for one unit of a good. The valuation is observed by the buyer and is distributed according to $f(v|b)$.

The seller then makes a TIOLI offer to the buyer of a price $p$. The buyer accepts or rejects.

(a) First suppose the seller observes $v$. How much will the buyer invest?

For the rest of the question, suppose that the seller observes neither $b$ nor $v$. Assume that buyer’s and seller’s optimisation problems are concave.

(b) Assume $f(v|b)$ satisfies the hazard rate order in that

$$\frac{f(v|b)}{1 - F(v|b)} \text{ decreases in } b$$

(HR)

Derive the seller’s optimal price. How does the optimal price vary with $b$?

(c) Derive the buyer’s optimal investment choice. Notice that (HR) implies that $F(v|b)$ decreases in $b$. How does the optimal investment vary with the expected price, $p$?

(d) Argue that there will be a unique Nash equilibrium in $(b, p)$ space.

(e) How does the level of investment differ from part (a)? Why?
4. Moral Hazard and Asymmetric Information

[25 points] A firm employs an agent who is risk–neutral, but has limited liability (i.e. they cannot be paid a negative wage). There is no individual rationality constraint. The agent can choose action \( a \in \{L, H\} \) at cost \( \{0, c\} \). There are two possible outputs \( \{q_L, q_H\} \). The high output occurs with probability \( p_L \) or \( p_H \) if the agent takes action \( L \) or \( H \), respectively. The agent’s payoff is

\[
w - c(a)
\]

where \( w \) is the wage and \( c(a) \) the cost of the action. The principal’s payoff is

\[
q - w
\]

where \( q \) is the output and \( w \) is the wage.

(a) Characterise the optimal wages and action.

Suppose there are two types of agents, \( i \in \{1, 2\} \). The principal cannot observe an agent’s type but believes the probability of either type is 1/2. The agents are identical except for their cost of taking the action: for agent \( i \in \{1, 2\} \) the cost of \( a \in \{L, H\} \) is \( \{0, c^i\} \), where \( c^2 > c^1 \).

(b) What are the optimal wages if the principal wishes to implement \( \{a^1, a^2\} = \{L, L\} \)?

(c) What are the optimal wages if the principal wishes to implement \( \{a^1, a^2\} = \{H, H\} \)?

(d) What are the optimal wages if the principal wishes to implement \( \{a^1, a^2\} = \{L, H\} \)?

(e) What are the optimal wages if the principal wishes to implement \( \{a^1, a^2\} = \{H, L\} \)?