Homework 3
November 23, 2011

1. Nonlinear Pricing with Two Types

Suppose a seller of wine faces two types of customers, $\theta_1$ and $\theta_2$, where $\theta_2 > \theta_1$. The proportion of type $\theta_1$ agents is $\pi \in [0, 1]$. Let $q$ be the quality of the wine and $t$ the price.

Let type $\theta_1$ buy contract $(q_1, t_1)$ and type $\theta_2$ buy $(q_2, t_2)$. The cost of production is zero, $c(q) = 0$, and the seller maximises profit $\pi t_1 + (1 - \pi) t_2$

(a) Suppose agent $\theta_i$ has utility

$$u(\theta_i) = \theta_i q - \frac{1}{2} q^2 - t$$

Derive the first–best and profit–maximising qualities.

(b) Suppose agent $\theta_i$ has utility

$$u(\theta_i) = \theta_i (q - \frac{1}{2} q^2) - t$$

Derive the first–best and profit–maximising qualities.

2. Dynamic Mechanism Design

A firm sells to a customer over $T = 2$ periods. There is no discounting.

The consumer’s per-period utility is

$$u = \theta q - p$$

where $q \in \mathbb{R}$ is the quantity of the good, and $p$ is the price. The agent’s type $\theta \in \{\theta_L, \theta_H\}$ is privately known. In period 1, $\Pr(\theta = \theta_H) = \mu$. In period 2, the agent’s type may change. With probability $\alpha > 1/2$, her type remains the same; with probability $1 - \alpha$ her type switches (so a high type becomes a low type, or a low type becomes a high type).

The firm chooses a mechanism to maximise the sum of its profits. The per-period profit is given
by
\[ \pi = p - \frac{1}{2}q^2. \]

A mechanism consists of period 1 allocations \( \langle q_L, q_H \rangle \), period 2 allocations \( \langle q_{LL}, q_{LH}, q_{HL}, q_{HH} \rangle \), and corresponding prices, where \( q_{LH} \) is the quantity allocated to an agent who declares \( L \) in period 1 and \( H \) in period 2.

(a) Consider period \( t = 2 \). Fix the first period type, \( \theta \). Assume in period 2 that the low-type’s (IR) constraint binds, the high type’s (IC) constraint binds and we can ignore the other constraints. Characterise the second period rents obtained by the agents, \( U_{\theta L} \) and \( U_{\theta H} \), as a function of \( \{q_{LL}, q_{LH}, q_{HL}, q_{HH}\} \).

(b) Consider period \( t = 1 \). Assume the low-type’s (IR) constraint binds, the high type’s (IC) constraint binds and we can ignore the other constraints. Derive the lifetime rents obtained by the agents, \( U_L \) and \( U_H \), as a function of \( \{q_L, q_H, q_{LL}, q_{LH}, q_{HL}, q_{HH}\} \).

(c) Derive the firm’s total expected profits.

(d) Assume the firm does not want to exclude, i.e. that \( \Delta := \theta_H - \theta_L \) is sufficiently small. Derive the profit-maximising allocations \( \{q_L, q_H, q_{LL}, q_{LH}, q_{HL}, q_{HH}\} \). In particular, show that \( q_{HL} \) is first-best. Can you provide an intuition for this result?

(Bonus) Suppose \( T \) is arbitrary. Can you derive the form of the optimal mechanism?

3. Public Goods Provision

A firm is considering building a public good (e.g. a swimming pool). There are \( n \) agents in the economy, each with IID private value \( \theta_i \in [0, 1] \). Agents’ valuations have density \( f(\theta) \) and distribution \( F(\theta) \). Assume that

\[ MR(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)} \]

is increasing in \( \theta \). The cost of the swimming pool is \( cn \), where \( c > 0 \).

First suppose the government passes a law that says the firm cannot exclude people from entering the swimming pool. A mechanism thus consists of a build decision \( P(\theta_1, \ldots, \theta_n) \in [0, 1] \) and a payment by each agent \( t_i(\theta_1, \ldots, \theta_n) \in \mathbb{R} \). The mechanism must be individually
rational and incentive compatible. [Note: When showing familiar results your derivation can be heuristic.]

(a) Consider an agent with type $\theta_i$, whose utility is given by

$$\theta_i P - t_i$$

Derive her utility in a Bayesian incentive compatible mechanism.

(b) Given a build decision $P(\cdot)$, derive the firm’s profits.

(c) What is the firm’s optimal build decision?

(d) Show that $E[MR(\theta)] = 0$.

(e) Show that as $n \to \infty$, so the probability of provision goes to zero. [You might wish to use the Chebyshev inequality, which says that $Pr(|Z - E[Z]| \geq \alpha) \leq \frac{\text{Var}(Z)}{\alpha^2}$ for a random variable $Z$.]

Next, suppose the firm can exclude agents. A mechanism now consists of a build decision $P(\theta_1, \ldots, \theta_n) \in [0,1]$, a participation decision for each agent $x_i(\theta_1, \ldots, \theta_n) \in [0,1]$ and a payment $t_i(\theta_1, \ldots, \theta_n) \in \mathbb{R}$. Agent $i$’s utility is now given by

$$\theta_i x_i P - t_i$$

The cost is still given by $cn$, where $n$ is the number of agents in the population.

(f) Solve for the firm’s optimal build decision $P(\cdot)$ and participation rule $x_i(\cdot)$.

(g) Suppose $n \to \infty$. Show there exists a cutoff $c^*$ such that the firm provides the pool with probability one if $c < c^*$, and with probability zero if $c > c^*$.

4. Costly State Verification

There is a risk–neutral entrepreneur $E$ who has a project with privately observed return $y$ with density $f(y)$ on $[0,Y]$. The project requires investment $I < E[y]$ from an outside creditor $C$.

A contract is defined by a pair $(s(y), B(y))$ consisting of payment and verification decision. If
an agent reports $y$ they pay $s(y) \leq y$ and are verified if $B(y) = 1$ and not verified if $B(y) = 0$. If the creditor verifies $E$ they pay cost $c(y)$ and get to observe $E$’s type.

The game is as follows:

- $E$ chooses $(s(y), B(y))$ to raise $I$ from a competitive financial market.
- Output $y$ is realised.
- $E$ claims the project yields $\hat{y}$. If $B(\hat{y}) = 0$ then $E$ pays $s(\hat{y})$ and is not verified. If $B(\hat{y}) = 1$ then $C$ pays $c(y)$ and observes $E$’s true type. If they are telling the truth they pay $s(y)$; if not, then $C$ can take everything.
- Payoffs. $E$ gets $y - s(y)$, while $C$ gets $s(y) - c(y)B(y) - I$.

(a) Show that a contract is incentive compatible if and only if there exists a $D$ such that $s(y) = D$ when $B(y) = 0$ and $s(y) \leq D$ when $B(y) = 1$.

Consider $E$’s problem:

\[
\max_{s(y), B(y)} E[y - s(y)]
\]
\[
\text{s.t. } s(y) \leq y \quad (MAX)
\]
\[
E[s(y) - c(y)B(y) - I] \geq 0 \quad (IR)
\]
\[
s(y) \leq D \quad \forall y \in B^V \quad (IC1)
\]
\[
s(y) = D \quad \forall y \notin B^V \quad (IC2)
\]

where $B^V$ is the verification region (where $B(y) = 1$).

(b) Show that constraint (IR) must bind at the optimum. [Hint: Proof by contradiction.]

Now $E$’s problem becomes

\[
\min_{s(y), B(y)} E[c(y)B(y)]
\]
\[
\text{s.t. } (MAX), (IC1), (IC2)
\]
\[
E[s(y) - c(y)B(y) - I] = 0 \quad (IR)
\]
(c) Show that any optimal contract \((s(y), B(y))\) has a verification range of the form \(B^V = [0, D]\) for some \(D\). [Hint: Proof by contradiction.]

(d) Show that any optimal contract \((s(y), B(y))\) sets \(s(y) = y\) when \(B(y) = 1\). [Hint: Proof by contradiction.]

(e) A contract is thus characterised by \(D\). Which \(D\) maximises \(E\)'s utility? Can you give a financial interpretation to this contract?

5. Ironing

Consider the continuous–type price discrimination problem from class, where the principal chooses \(q(\theta)\) to maximise

\[
E[q(\theta)MR(\theta) - c(q(\theta))]
\]

subject to \(q(\theta)\) increasing in \(\theta\).

For \(v \in [0, 1]\), let

\[
H(v) = \int_0^v MR(F^{-1}(x))dx
\]

be the expected marginal revenue up to \(\theta = F^{-1}(v)\). Let \(\overline{H}(v)\) be the highest convex function under \(H(v)\). Then define \(\overline{MR}(\theta)\) by

\[
\overline{H}(v) = \int_0^v \overline{MR}(F^{-1}(x))dx
\]

Finally, let \(\Delta(\theta) = H(F(\theta)) - \overline{H}(F(\theta)).\)

(a) Argue that \(\Delta(\theta) > 0\) implies \(\overline{MR}(\theta)\) is flat. Also argue that \(\Delta(\theta) = \Delta(\overline{\theta}) = 0\).

(b) Since \(q(\theta)\) is an increasing function, show that

\[
E[q(\theta)MR(\theta) - c(q(\theta))] = E[q(\theta)\overline{MR}(\theta) - c(q(\theta))] - \int_0^\theta \Delta(\theta)dq(\theta)
\]

(c) Derive the profit–maximising allocation \(q(\theta)\).

\footnote{Note, it is important that we take the convex hull in quantile space. If we use \(\theta\)-space, then \(\Delta(\theta) > 0\) implies \(\overline{MR}(\theta)f(\theta)\) is flat, which is not particularly useful.}
6. Negotiations and Auctions

Assume all bidders have IID private valuations \( v_i \sim F(v) \) with support \([V, \bar{V}]\). Define marginal revenue as

\[
MR(v) = v - \frac{1 - F(v)}{f(v)}
\]

(a) Show that \( E[MR(v)] = V \).

(b) In terms of marginal revenues, what is the revenue from 2 bidders with no reservation price?

(c) Let the sellers valuation be \( v_0 \). In terms of marginal revenue, what is the revenue from 1 bidder and a reservation price?

(d) Assume \( V \geq v_0 \), i.e. all bidders are “serious”. How is revenue affected if one bidder is swapped for a reservation price?